CONGRUENCES OF CURVES THROUGH POINTS OF A SUBSPACE AND HYPERSURFACE OF A FINSLER SPACE

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R. S. MISHRA [2] has obtained an expression for the v_i, i. e. $a_{ab}\, N^a\, \lambda^b\, ;\, i$, where N^a are coatravariant components of the unit normal to the hypersurface and λ^b are the contravariant components of a unit vector in the direction of a curve of a congruence of curves, one curve of which passes through each point of the hypersurface V_n. The object of this paper is to obtain the generalised expressions in the case of a subspace and a hypersurface of a FINSLER space. Some particular cases have been studied.

1. Subspace. If the coordinates $x^i\,(i=1, 2, 3, \ldots, n)$ of a FINSLER space F_n can he represented parametrically as

$$x^i = x^i\,(u^a) \tag{1.1}$$

such that $u^a\,(a=1, 2, 3, \ldots, m,\ m<n)$ form a coordinate system of F_m, then the space F_m is called the m-dimensional subspace of F_n. Let $g_{ab}\ (u,\ u')$ and $g_{ij}\,(x,\ x')$ be the metric tensors of F_m and F_n. They are related by the equation

$$g_{ab}\,(a,\ u') = g_{ij}\,(x,\ x')\, X^i_a\, X^j_b \qquad \text{where} \qquad X^i_a = \frac{\partial x^i}{\partial u^a}. \tag{1.2}$$

Thronghont this paper letters $(i,\ j,\ h, \ldots)$ vary from 1 to n, letters $(a,\ b,\ c, \ldots)$ vary from 1 to m, and $(p,\ q,\ r, \ldots)$ vary from $m+1$ to n in the case of a subspace. In the case of a hypersurface $(a,\ b,\ c, \ldots)$ vary from 1 to $n-1$,and $(p,\ q,\ r, \ldots)$ take a single value.

As the rank of $\| X^i_a \|$ is m, there exist $(n-m)$ independent solutions of $Y_i\, X^i_a = 0$, which form a system of normals at P. The $(n-m)$ 'normals which do not depend on the direction element x^i are given by the solution of

$$n_{(q)i}\, X^i_a \equiv g_{ij}\,(x,\ n_{(q)})\, n^j_{(q)}\, X^i_a = 0. \tag{1.3}$$

They are normalised by the relations

$$g_{ij}(x, n_{(q)})\, n^i_{(q)}\, n^j_{(q)} = n_{(q)j}\, n^j_{(q)} = 1 \tag{1.4}$$

and

$$g_{ij}(x, n_{(q)})\, n^i_{(q)}\, n^j_{(r)} = \cos(n_{(q)}, n_{(r)}) = a_{(q)(r)}. \tag{1.5}$$

The $(n-m)$ normals depending on the line element (x, x') are given by the solutions of

$$n^*_{(q)i}\, X^i_a = g_{ij}(x, x')\, n^{*j}_{(q)}\, X^i_a = 0. \tag{1.6}$$

They satisfy the relation

$$g_{ij}(x, n^*_{(q)})\, n^{*i}_{(q)}\, n^{*j}_{(q)} = 1. \tag{1.7}$$

It is to be noted that the covariaht vector $n^*_{(q)i} = g_{ij}(x, x')\, n^{*j}_{(q)}$ is not the covariant counterpart of $n^{*i}_{(q)}$ as given by (1.6), for we have

$$g_{ij}(x, x')\, n^{*i}_{(q)}\, n^{*j}_{(r)} = \psi_{(q)}\, \delta^q_r, \tag{1.8}$$

(no summation on q).

The relation (1.8) shows that the normal vectors $n^{*i}_{(q)}$ $(q = m+1 \ldots\ldots n)$ form a system of orthogonal vectors. These normals are called secondary normals. There exist $(n-m)$ symmetric tensors independent of direction. These are given by the relation

$$Y_{(q)ab} = g_{ij}(x, n_{(q)})\, X^i_a\, X^j_b. \tag{1.9}$$

In the case of a hypersurface also there exist two systems of normals. One system depends on the direction and other is independent of it. They are given by the solutions of the equations

$$n_i\, X^i_a = g_{ij}(x, n)\, n^j\, X^i_a = 0 \tag{1.10}$$

and

$$n^*_i\, X^i_a = g_{ij}(x, x')\, n^{*j}\, X^i_a = 0, \tag{1.11}$$

and are normalised respectively by the relations

$$g_{ij}(x, n)\, n^i\, n^j = n_j\, n^j = 1 \tag{1.12}$$

and

$$g_{ij}(x, n^*)\, n^{*i}\, n^{*j} = 1. \tag{1.13}$$

The equation (1.8) becomes

$$g_{ij}(x, x')\, n^{*i}\, n^{*j} = \psi\,. \tag{1.14}$$

In this case also there exists a symmetric tensor independent of direction, which is given by

$$Y_{ab} = g_{ij}\,(x, n)\, X^i{}_a\, X^j{}_b\,. \tag{1.15}$$

The vectors $n^i{}_{(q)}$ can be expressed linearly in terms of $n^{*i}{}_{(q)}$. Thus

$$n_{(q)i} = \sum_{(r)=m+1}^{n} A_{(q)(r)}\, n^*{}_{(r)i} \tag{1.16}$$

where

$$A_{(q)(r)} = \frac{\cos\,(n_{(q)}\,,\, n^*{}_{(r)})}{\psi_{(r)}}\,.$$

When $X^i{}_{ab}$ is regarded as a vector of the space F_n, it lies in the space spanned by the secondary normals $n^{*i}{}_{(q)}$. Thus

$$X^i{}_{ab} = \sum_{(q)} \Omega^*{}_{(q)ab}\;\; n^{*i}{}_{(q)}. \tag{1.17}$$

Multiplying this by $n_{(r)i}$, we have

$$\Omega_{(r)ab} = X^i{}_{ab}\, n_{(r)i} = \sum_{(q)} \cos\,(n_{(r)}\,,\, n^*{}_{(q)})\;\, \Omega^*{}_{(q)\,ab}. \tag{1.18}$$

$X^i{}_{ab}$ is also given by

$$X^i{}_{ab} = \sum_{(r)} B_{(r)\,ab}\; n^i{}_{(r)} + W^i{}_{ab} \tag{1.19}$$

where

$$n_{(r)i}\;\, W^i{}_{ab} = 0.$$

The quantities $\Omega_{(q)\,ab}$ and $\Omega^*{}_{(q)\,ab}$ are $(n-m)$ symmetric tensors of F_m. In the case of a hypersurface, n_i and $n^*{}_i$ are related by the equation

$$n_i = \frac{\cos\,(n,\, n^*)}{\psi}\;\, n^*{}_i\,. \tag{1.20}$$

$X^i{}_{ab}$ is given by the relation

$$X^i{}_{ab} = n^{*i}\;\, \Omega^*{}_{ab}\,. \tag{1.21}$$

and

(1.22) $$\Omega_{ab} = X^i_{ab} \; n_i = \cos(n, n^*) \; \Omega^*_{ah} .$$

X^i_{ab} can also be expressed as

(1.23) $$X^i_{ab} = \Omega_{ab} \; n^i + \omega^i_{ab} ,$$

where

(1.24) $$\omega^i_{ab} \; n_i = 0.$$

The covariant derivative of $n^j_{(q)}$ is given by

(1.25) $$n^j_{(q)\,;\,a} = A^b_{(q)\,a} \; X^j_b + \sum_{(k)} r^{(k)}_{(q)\,a} \; n^j_{(k)} .$$

where

(1.26) $$A^c_{(q)\,b} = -\gamma^{ac}_{(q)} \; \Omega_{(q)\,ab} - \gamma^{ac}_{(q)} \; E_{(q)\,ijk} \; X^k_b \; X^i_a \; n^j_{(q)}$$

$$[\, E_{(q)\,ijk} \;\; \underline{\underline{\text{def}}} \;\; g_{ij\,;\,k} \; (x, n_{(q)}) \,)$$

and

(1.27) $$n^j_{(r)\,;\,b} \; n_{(p)\,j} = \sum_{(k)} r^{(k)}_{(q)\,b} \;\; n^j_{(k)} \; n_{(p)\,j} = \sum_{(k)} r^{(k)}_{(q)b} \; a_{(k)\,(p)} .$$

The covariant derivative of $n^{*j}_{(q)}$ is given by

(1.28) $$n^{*j}_{(q)\,;\,b} = B^a_{(q)\,b} \; X^j_a + \sum_{(k)} N^{(k)}{}_{(q)\,b} \; n^{*j}{}_{(k)} .$$

where

(1.29) $$B^c_{(q)\,b} = -\psi_{(q)} \; \Omega^*_{(q)\,ab} \; g^{ac} - E^*_{ihk} \; (x, x') \; g^{ac} \; X^k_b \; X^i_a \; n^{*h}_{(q)}$$

$$[\, E^*_{ihk} \; (x, x') = g_{ih\,;\,k} \; (x, x') \,]$$

and

(1.30) $$N^{(\nu)}{}_{(q)\,b} \; \psi_{(r)} = n^{*j}_{(q)\,;\,b} \; n^*{}_{(r)\,j} .$$

In the case of a hypersurface, the covariant derivative of n^j is given by

(1.31) $$n^j{}_{;\,b} = A^c_b \; X^j_c + r_b \; n^j ,$$

where

(1.32) $$A^c_b = -\gamma^{ac} \; \Omega_{ab} - \gamma^{ac} \; E_{ijk} \; X^k_b \; X^i_a \; n^j ,$$

and

$$r_b = n^j{}_{;b}\, n_j . \tag{1.33}$$

The covariant derivative of n^{*j} is given by

$$n^{*j}{}_{;\beta} = B^d{}_b\, X^j{}_d + N_b\, n^{*j} \tag{1.34}$$

where

$$B^s{}_b = -\psi\, \Omega^*{}_{ab}\, g^{as} - E^*{}_{ibk}\, g^{as}\, X^k{}_b\, X^i{}_a\, n^{*b} \tag{1.35}$$

and

$$N_b\, \psi = n^{*j}{}_{;b}\, n^*{}_j . \tag{1.36}$$

2. The equations for $v_{(qr)a}$. Let $p^i{}_{(q)}$ be the contravariant components in the $x's$ of a unit vector in the direction of a curve of a congruence of curves, one curve of which passes through each point of F_m. Then the vector with components $p^i{}_{(q)}$ is, in general, given by

$$p^i{}_{(q)} = t^a{}_{(q)}\, X^i{}_a + \sum_{(r)} c_{(qr)}\, n^i{}_{(r)} \tag{2.1}$$

where $t^a{}_{(q)}$ and $c_{(qr)}$ are given by

$$c_{(r)ab}\, t^a{}_{(q)} = g_{ij}\,(x, n_{(r)})\, p^i{}_{(q)}\, X^j{}_b \tag{2.2}$$

and

$$\sum_{(r)} c_{(qr)} = g_{ij}\,(x, n_{(r)})\, p^i{}_{(q)}\, n^j{}_{(r)} . \tag{2.3}$$

Taking the covariant derivative of (2.1) with respect to u^b, we get

$$p^i{}_{(q);b} = t^a{}_{(q);b}\, X^i{}_a + t^a{}_{(q)}\, X^i{}_{ab} + \sum_{(r)} c_{(qr);b}\, n^i{}_{(r)} + \sum_{(r)} c_{(qr)}\, n^i{}_{(r);b} . \tag{2.4}$$

Substituting the values of $X^i{}_{ab}$ and $n^i{}_{(r);b}$ from equations (1.19) and (1.25), we get

$$p^i{}_{(q);b} = \Big(t^a{}_{(q);b} + \sum_{(r)} c_{(qr)}\, A^a{}_{(r)b}\Big) X^i{}_a + \sum_{(r)} n^i{}_{(r)}\, \Big(c_{(qr);b} + t^a{}_{(q)}\, B_{(r)ab} + \sum_{(\sigma)} c_{(q\sigma)}\, r^{(r)}{}_{\sigma(b)}\Big) + t^a{}_{(q)}\, W^i{}_{ab} . \tag{2.5}$$

Putting

$$q^a{}_{(q)b} = t^a{}_{(q);b} + \sum_{(r)} c_{(qr)}\, A^a{}_{(r)b} \tag{2.6}$$

and

$$v_{(qr)\,b} = c_{(qr)\,:\,b} + t^{a}{}_{(q)}\ B_{(r)\,ab} + \sum_{(\sigma)} c_{(q\sigma)}\ r^{(r)}{}_{(\sigma)\,b} \tag{2.7}$$

in (2.5), it becomes

$$p^{i}{}_{(q)\,;\,b} = q^{a}{}_{(\mu)\,b}\ X^{i}{}_{a} + \sum_{(r)} v_{(qr)b}\ n^{i}{}_{(r)} + t^{a}{}_{(q)}\ W^{i}{}_{ab}\,. \tag{2.8}$$

Taking the covariant derivative of (2.8) with respect to u^{c}, we get

$$p^{i}{}_{(q)\,;\,bc} = q^{a}{}_{(q)\,b\,;\,c}\ X^{i}{}_{a} + q^{a}{}_{(q)\,b}\ X^{i}{}_{ac} + \sum_{(r)} v_{(qr)\,b\,;\,c}\ n^{i}{}_{(r)}$$

$$+ \sum_{(r)} v_{(qr)\,b}\ n^{i}{}_{(r)\,;\,c} + t^{a}{}_{(q)\,;\,c}\ W^{i}{}_{ab} + t^{a}{}_{(q)}\ W^{i}{}_{ab\,;\,c}\,.$$

With the help of (1.19) and (1.25), it becomes

$$p^{i}{}_{(q)\,;\,br} = (v^{a}{}_{(q)\,b\,;\,r} + \sum_{r} v_{(qr)\,b}\ A^{a}{}_{(r)\,r})\ X^{i}{}_{a} + \tag{2.9}$$

$$+ \sum_{(r)} (v^{a}{}_{(q)\,b}\ B_{(r)\,ac} + v_{(qr)\,b\,;\,c} + \sum_{(x)} v_{(qx)\,b}\ r^{(r)}{}_{(x)\,c})\ n^{i}{}_{(r)}$$

$$+ v^{a}{}_{(q)\,b}\ W^{i}{}_{ac} + t^{a}{}_{(q)\,;\,c}\ W^{i}{}_{ab} + t^{a}{}_{(q)}\ W^{i}{}_{ab\,;\,c}\,.$$

From (2.9), we get

$$R^{i}{}_{jkl}\ p^{j}{}_{(q)}\ X^{k}{}_{b}\ X^{l}{}_{c} = 2\ p^{i}{}_{(q)\,;\,[bc]}\ {}^{1)} \tag{2.10}$$

$$= 2\,[\,\{\,v^{a}{}_{(q)\,[b\,;\,c]} + \sum_{(r)} v_{(qr)\,[b}\ A^{a}{}_{|(r)|\,c]}\,\}\ X^{i}{}_{a}$$

$$+ \sum_{(r)} \{\,v^{a}{}_{(q)\,[b}\ B_{|(r)\,a|\,c]} + v_{(qr)\,[b\,;\,c]}$$

$$+ \sum_{(x)} v_{(qx)\,[b}\ R^{(r)}{}_{|(x)|\,c]}\,\}\ n^{i}{}_{(r)} + v^{a}{}_{(q)\,[b}\ W^{i}{}_{|a|\,c]}$$

$$+ t^{a}{}_{(q)\,;\,[r}\ W^{i}{}_{|a|\,b]} + t^{a}{}_{(q)}\ W^{i}{}_{a\,[b\,;\,c]}\,]$$

where $R^{i}{}_{jkl}$ is the curvature tensor in F_{n} defined by [4]

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1) The square brackets denote the alternating part, *i.e.*

$$2\,T_{[\,i\,|\,j\,|\,k\,]} = T_{ijk} - T_{kji}\,.$$

$$R^i{}_{jkl} = 2\,[\,\partial_{[k}\, P^{*i}{}_{|j|\;\;]} + \partial'_h\, P^{*i}{}_{j[l}\,\partial_{k]}\;\; x'^h + P^{*i}{}_{h[k}\, P^{*h}{}_{|j|l]}.$$

It is obvious that this tensor is skew-symmetric in k, l, *i.e.*

$$R^i{}_{jkl} + R^i{}_{jlk} = 0.$$

Multiplying (2.10) by $g_{ih}\,(x, n_{(r)})\; n^h{}_{(r)}$ and summing with respect to i, we get

(2.11)
$$\begin{aligned} g_{ih}\,(x, n_{(r)})\; R^i{}_{jkl}\; p^j{}_{(q)}\; X^k{}_b\; X^l{}_c\; n^h{}_{(r)} \\ = 2 \sum_{(r)} \{\, v^a{}_{(q)\,[b}\; B_{|\,(r)\,a\,|\,c]} + v_{(qr)\,[b\,;\,c} \\ + \sum_{(x)} v_{(qx)\,[b}\; v^{(r)}{}_{|\,(x)\,|\,c]} \} \\ + 2\, t^a{}_{(q)}\; W^i{}_{a\,[b\,;\,c]}\; g_{ih}\,(x, n_{(r)})\; n^h{}_{(r)}. \end{aligned}$$

Similarly, if (2.10) is multiplied by $g_{ih}\;(x, n_{(r)})\; X^h{}_d$ and summation is performed with respect to i, we have

(2.12)
$$\begin{aligned} g_{ih}\,(x, n_{(r)})\; R^i{}_{jkl}\; p^j{}_{(q)}\; X^k{}_b\; X^l{}_c\; X^h{}_d \\ = 2\, c_{(r)\,ad}\; \{\, v^a{}_{(q)\;[b\,;\,c]} + \sum_{(r)} v_{(qr)\,[b}\; A^a{}_{|\,(r)\,|\,c]} \} \\ + 2\, g_{ih}\,(x, n_{(r)})\; X^h{}_d\; (v^a{}_{(q)\,[b}\; W^i{}_{|\,a\,|\,c]} \\ + t^a{}_{(q)\,;\,[c}\; W^i{}_{|\,a\,|\,b]} + t^a{}_{(q)}\; W^i{}_{a\,[b\,;\,c]}\,). \end{aligned}$$

The equations (2.11) and (2.12) are the fundamental equations which determine the subspace of a Finsler space F_n. These equations give the expression for

(2.13)
$$\sum_{(r)} v_{(qr)\,b} = g_{ih}\;(x, n_{(r)})\; n^h{}_{(r)}\; p^i{}_{(q)\,;\,b}$$

from (2.8).

In the case of a hypersurface, p^i is given by

(2.14)
$$p^i = t^a\; X^i{}_a + \Gamma\; n^i\,,$$

where t^a and Γ are given by

(2.15)
$$c_{ab}\; t^a = g_{ij}\;(x, n)\; p^i\; X^j{}_b$$

and

$$\Gamma = g_{ij}\,(x,n)\,p^i\,n^j = \cos\,(n,p). \tag{2.16}$$

The equations (2.6) and (2.7) become

$$v^a{}_b = t^a{}_{;b} + \Gamma\,A^a{}_b \tag{2.17}$$

and

$$v_b = \Gamma_{;b} + t^a\,\Omega_{ab} + \Gamma\,R_b. \tag{2.18}$$

In the case of a hypersurface, the equations (2.10) to (2.12) take the following forms respectively:

$$\begin{aligned} R^i{}_{jkl}\,p^j\,X^k{}_b\,X^l{}_c = 2\,[\,\{v^a{}_{[b;c]} + v_{[b}\,A^a{}_{c]}\}\,X^i{}_a \\ + \{v^a{}_{[b}\,\Omega_{|a|c]} + v_{[b;c]} + v_{[b}\,r_{c]}\}\,n^i \\ + v^a{}_{[b}\,w^i{}_{|a|c]} + w^i{}_{a[b}\,t^a{}_{;c]} + t^a\,w^i{}_{a[b;c]}\,], \end{aligned} \tag{2.19}$$

$$\begin{aligned} g_{ih}\,(x,n)\,R^i{}_{jkl}\,p^j\,X^k{}_b\,X^l{}_c\,n^h \\ = 2\,\{\Omega_{a[c}\,v^a{}_{b]} + v_{[b;c]} + v_{[b}\,r_{c]}\} + 2\,t^a\,w^i{}_{a[b;c]}\,g_{ih}\,(x,n)\,n^h. \end{aligned} \tag{2.20}$$

$$\begin{aligned} g_{ih}\,(x,n)\,R^i{}_{jkl}\,p^j\,X^k{}_b\,X^l{}_c\,X^h{}_d \\ = 2\,c_{ad}\,\{v^a{}_{[b;c]} + v_{[b}\,A^a{}_{c]}\} + 2\,g_{ih}\,(x,n)\,X^h{}_d \\ (v^a{}_{[b}\,w^i{}_{|a|c]} + w^i{}_{a[b}\,t^a{}_{;c]} + t^a\,w^i{}_{a[b;c]}). \end{aligned} \tag{2.21}$$

The equation (2.13), becomes

$$v_b = g_{ih}\,(x,n)\,n^h\,p^i{}_{;b}. \tag{2.22}$$

In this case also the equations (2.20) and (2.21) determine the hypersurface.

3. Particular case.

(*a*) Let us consider a congruence of curves which is such that the vector with the contravariant components $p^i{}_{(q)}$ in the direction of the curve of the congruence, is normal to F_m. Then

$$\bar{p}^i{}_{(q)} = \sum_{(r)} c_{(qr)}\,n^i{}_{(r)}. \tag{3.1}$$

Equations (2.6) to (2.8) become

$$\bar{v}^d{}_{(q)b} = \sum_{(r)} c_{(qr)}\,A^d{}_{(r)b}, \tag{3.2}$$

$$\bar{v}_{(qr)b} = c_{(qr)\,;\,b} + \sum_{(x)} c_{(qx)}\,r^{(r)}{}_{(x)b}, \tag{3.3}$$

and

(3.4) $$\bar{p}^i_{(q)\,;\,b} = q^d_{(q)b}\, X^i_d + \sum_{(r)} \bar{v}_{(qr)b}\, n^i_{(r)} .$$

Equations (2.10) to (2.12) reduce to

(3.5) $$R^i_{jkl}\, \bar{p}^j_{(q)}\, X^k_b\, X^l_c = 2\{ q^a_{(q)[b\,;\,c]} + \sum_{(r)} \bar{v}_{(qr)[b}\, A^a{}_{|(r)|c]} \} X^i_a$$
$$+ 2 \sum_{(r)} \{ q^a_{(q)[b}\, B_{|(r)a|c]} + \bar{v}_{(qr)[b\,;\,c]}$$
$$+ \sum_{(x)} \bar{v}_{(qx)[b}\, v^{(r)}{}_{|(x)|c]} \} n^i_{(r)} + 2\, q^a_{(q)[b}\, W^i{}_{|a|c]} ,$$

(3.6) $$g_{ih}(x, n_{(r)})\, R^i_{jkl}\, p^j_{(q)}\, X^k_b\, X^l_c\, n^h_{(r)}$$
$$= 2 \sum_{(r)} \{ q^a_{(q)[b}\, B_{|(r)a|c]} + \bar{v}_{(qr)[b\,;\,c]} + \sum_{(x)} \bar{v}_{(qx)[b}\, r^{(r)}{}_{|(x)|c]} \} ,$$

and

(3.7) $$g_{ih}(x, n_{(r)})\, R^i_{jkl}\, p^j_{(q)}\, X^k_b\, X^l_c\, X^h_d$$
$$= 2\, r_{ad} \{ q^a_{(q)[b\,;\,c]} + \sum_{(r)} \bar{v}_{(qr)[b}\, A^a{}_{|(r)|c]} \}$$
$$+ 2\, q^a_{(q)[b}\, W^i{}_{|a|c]}\, g_{ih}(x, n_{(r)})\, X^h_d .$$

For the case of a hypersurface, equations (3.5) to (3.7) reduce to

(3.8) $$R^i_{jkl}\, n^j\, X^k_b\, X^l_c = 2\{ A^a{}_{[b\,;\,c]} + r_{[b}\, A^a{}_{c]} \} X^i_a + 2\{ \Omega_a|_c\, A^a{}_{b]}$$
$$+ r_{[b\,;\,c]} \} n^i + 2\, W^i{}_{a[c}\, A^a{}_{b]} ,$$

(3.9) $$g_{ih}(x, n)\, R^i_{jkl}\, n^j\, X^k_b\, X^l_c\, n^h = 2\{ \Omega_a{}_{[c}\, A^a{}_{b]} + r_{[b\,;\,c]} \} ,$$

and

(3.10) $$g_{ih}(x, n)\, R^i_{jkl}\, n^j\, X^k_b\, X^l_c\, X^h_d$$
$$= 2\, C_{ad}(u) \{ A^a{}_{[b\,;\,c]} + r_{[b}\, A^a{}_{c]} \} + 2\, g_{ih}(x, n)\, X^h_d\, A^a{}_{[b}\, W^i{}_{|a|c} .$$

(*b*) If $p^i_{(q)} = t^a_{(q)}\, X^i_a$, then the equations (2.10) to (2.12) become

(3.11) $$R^i_{jkl}\, t^a_{(q)}\, X^j_a\, X^k_b\, X^l_c$$
$$= R^d{}_{abc}\, t^a_{(q)}\, X^i_d + 2\, t^a_{(q)} \sum_{(r)} B_{(r)a[b}\, A^d{}_{|(r)|c]}\, X^i_d$$
$$+ 2\, t^a_{(q)} \sum_{(r)} n^i_{(r)} \{ B_{(r)a[b\,;\,c]}$$
$$+ \sum_{(x)} B_{(x)a[b}\, r^{(r)}{}_{|(x)|c]} \} + 2\, t^a_{(q)}\, W^i{}_{a[h\,;\,c]} ,$$

(3.12)
$$g_{ih}\,(x, n_{(r)})\; R^i{}_{jkl}\; t^a{}_{(q)}\; X^j{}_a\; X^k{}_b\; X^l{}_c\; n^h{}_{(r)}$$
$$= 2\, t^a{}_{(q)} \sum_{(r)} \{ B_{(r)\,a\,[b\,;\,c]} + B_{(x)\,a\,[b}\, r^{(r)}{}_{|\,(x)\,|\,c]} \}$$
$$+ 2\; g_{ih}\,(x, n_{(r)})\; t^a{}_{(q)}\; n^h{}_{(r)}\; W^i{}_{a\,[b\,;\,c]}\,,$$

and

(3.13)
$$g_{ih}\,(x, n_{(r)})\; R^i{}_{jkl}\; t^a{}_{(q)}\; X^j{}_a\; X^k{}_b\; X^l{}_c\; X^h{}_e$$
$$= C_{(r)\,de}\; R^d{}_{abc}\; t^a{}_{(q)} + 2\, t^a{}_{(q)} \sum_{(r)} B_{(r)\,a\,[b}\; A^d{}_{|\,(r)\,|\,c]}\; C_{(r)\,de}$$
$$+ 2\, t^a{}_{(q)}\; g_{ih}\,(x, n_{(r)})\; X^h{}_e\; W^i{}_{a\,[b\,;\,c]}\,.$$

For the case of a hypersurface the equations (3.11) to (3.13) become

(3.14)
$$R^i{}_{jkl}\; t^a\; X^j{}_a\; X^k{}_b\; X^l{}_c$$
$$= R^d{}_{abc}\; t^a\; X^i{}_d + 2\, t^a\; \Omega_{a\,[b}\; A^d{}_{c]}\; X^i{}_d + 2\, t^a\; W^i{}_{a\,[b\,;\,c]}$$
$$+ 2\, t^a\; n^i\, \{ \Omega_{a\,[b\,;\,c]} + \Omega_{a\,[b}\, R_{c]} \}\,,$$

(3.15)
$$g_{ih}\,(x, n)\; R^i{}_{jkl}\; t^a\; X^j{}_a\; X^k{}_b\; X^l{}_c\; n^h$$
$$= 2\, t^a \{ \Omega_{a\,[b\,;\,c]} + \Omega_{a\,[b}\, R_{c]} \}$$
$$+ 2\; g_{ih}\,(x, n)\; t^a\; n^h\; w^i{}_{a\,[b\,;\,c]}\,,$$

and

(3.16)
$$g_{ih}\,(x, n)\; R^i{}_{jkl}\; t^a\; X^j{}_a\; X^k{}_b\; X^l{}_c\; X^h{}_e$$
$$= C_{de}\; R^d{}_{abc}\; t^a + 2\; t^a\; \Omega_{a\,[b}\; A^d{}_{c]}\; C_{de}$$
$$+ 2\, t^a\; g_{ih}\,(x, n)\; X^h{}_e\; w^i{}_{a\,[b\,;\,c]}.$$

(c) If $x'^i = u'^a\, X^i{}_a$ the equations (3.11) to (3.16) become

(3.17)
$$R^i{}_{jkl}\; X^j{}_a\; X^k{}_b\; X^l{}_c$$
$$= R^d{}_{abc}\; X^i{}_d + 2 \sum_{(r)} B_{(r)a\,[b}\; A^d{}_{|\,(r)\,|\,c]}\; X^i{}_d + 2\; W^i{}_{a\,[b\,;\,c]}$$
$$+ 2 \sum_{(r)} n^i{}_{(r)}\; \{ B_{(r)\,a\,[b\,;\,r]} + \sum_{(x)} B_{(x)\,a\,]b}\; R^{(r)}{}_{|\,(x)\,|\,c]} \}\,,$$

(3.18)
$$g_{ih}\,(x.\, n_{(r)})\; R^i{}_{jkl}\; X^j{}_a\; X^k{}_b\; X^l{}_c\; X^h{}_e$$
$$= C_{(r)\,de}\; R^d{}_{abc} + 2 \sum_{(r)} B_{(r)\,a\,[b}\; A^d{}_{|\,(r)\,|\,(c)]}\; C_{(r)\,de}$$
$$+ 2\, g_{ih}\,(x, n_{(r)})\; X^h{}_e\; W^i{}_{a[b\,;\,c]},$$

(3.19) $$g_{ih}\,(x, n_{(r)})\, R^i{}_{jkl}\, X^j{}_a\, X^k{}_b\, X^l{}_c\, n^h{}_{(c)}$$
$$= 2 \sum_{(r)} \{B_{(r)\,a\,[b\,;\,c]} + \sum_{(x)} B_{(x)\,a\,[b}\, R^{(r)}{}_{|\,(x)\,|\,c]}\}$$
$$+ 2\, g_{ih}\,(x, n_{(c)})\, n^h{}_{(c)}\; W^i{}_{a\,[b\,;\,c]},$$

(3.20) $$R^i{}_{jkl}\, X^j{}_a\, X^k{}_b\, X^l{}_c$$
$$= R^d{}_{abc}\, X^i{}_d + 2\, \Omega_{a\,[b}\, A^d{}_{c]}\, X^i{}_d + 2\, n^i\, \{\, \Omega_{a\,[b\,;\,c]}$$
$$+ \Omega_{a\,[b}\, R_{c]}\,\} + 2\, \omega^i{}_{a\,[b\,;\,c]},$$

(3.21) $$g_{ih}\,(x, n)\, R^i{}_{jkl}\, X^j{}_a\, X^k{}_b\, X^l{}_c\, X^h{}_e$$
$$= C_{de}\, R^d{}_{abc} + 2\, \Omega_{a\,[b}\, A^d{}_{c]}\, C_{de}$$
$$+ 2\, g_{ih}\,(x, n)\, X^h{}_e\, \omega^i{}_{a\,[b\,;\,c]},$$

and

(3.22) $$g_{ih}\,(x, n)\, R^i{}_{jkl}\, X^j{}_a\, X^k{}_b\, X^l{}_c\, n^h$$
$$= 2\, \{\, \Omega_{a\,[b\,;\,c]} + \Omega_{a\,[b}\, R_{c]}\,\} + 2\, g_{ih}\, n^h\, \omega^i{}_{a\,[b\,;\,c]}.$$

The equations (3.17) to (3.19) are the generalised forms of Gauss and Codazzi's equations for the subspace of a Finsler space [1] and the equations (3.20) to (3.22) are the equations for a hypersurface [3].

4. Equation for $P^*{}_{(qr)\,\alpha}$. In this article we decompose the unit vector in the direction of the normal $n^{*i}{}_{(q)}\,(x, x')$ instead of $n^i{}_{(q)}$ as in article 2. Thus

(4.1) $$p^{*i}{}_{(q)} = t^{*a}{}_{(q)}\, X^i{}_a + \sum_{(r)} c^*{}_{(qr)}\, n^{*i}{}_{(r)}$$

where $t^{*a}{}_{(q)}$ and $c^*{}_{(qr)}$ are given by

(4.2) $$c^*{}_{(qr)}\, \psi_{(r)} = g_{ij}\,(x, x')\, n^{*j}{}_{(r)}\, p^{*i}{}_{(q)}$$

and

(4.3) $$g_{ab}\,(u, u')\, t^{*a}{}_{(q)} = g_{ij}\,(x, x')\, p^{*i}{}_{(q)}\, X^j{}_b \equiv t^*{}_{(q)\,b}\,.$$

Taking the covariant derivative of (4.1) with respect to u^b, we have

(4.4) $$p^{*i}{}_{(q)\,;\,b} = X^i{}_{ab}\, t^{*a}{}_{(q)} + X^i{}_a\, t^{*a}{}_{(q)\,;\,b} + \sum_{(r)} c^*{}_{(qr)\,;\,b}\, n^i{}_{(r)}$$
$$+ \sum_{(r)} c^*{}_{(qr)}\, n^{*i}{}_{(r)\,;\,b}\,.$$

Substituting the values of $n^{*i}{}_{(q)\,;\,b}$ and $X^i{}_{ab}$ from the equations (1.28) and (1.17), the equation (4.4) becomes

(4.5) $$p^{*i}{}_{(q)\,;\,b} = v^{*a}{}_{(q)\,b}\; X^i{}_a + \sum_{(r)} p^*{}_{(qr)\,b}\; n^{*i}{}_{(r)}$$

in which we have put

(4.6) $$q^{*a}{}_{(q)\,b} \;\underline{\underline{\text{def}}}\; t^{*a}{}_{(q)\,:\,b} + \sum_{(r)} c^*{}_{(qr)}\; B^a{}_{(r)\,b}$$

and

(4.7) $$P^*{}_{(qr)\,b} \;\underline{\underline{\text{def}}}\; t^{*a}{}_{(q)}\; \Omega^*{}_{(r)\,ab}\;(u, u') + \sum_{(x)} c^*{}_{(qx)}\; N^{(r)}{}_{(x)\,b} + c^*{}_{(qr)\,;\,b}\,.$$

Taking the covariant derivative of (4.5) with respect to u^c, we get

$$P^{*i}{}_{(q)\,;\,bc} = q^{*a}{}_{(q)b\,;\,c}\; X^i{}_a + q^{*a}{}_{(q)\,b}\; X^i{}_{ac} + \sum_{(r)} P^*{}_{(qr)b\,;\,c}\; n^{*i}{}_{(r)}$$
$$+ \sum_{(r)} P^*{}_{(qr)\,b}\; n^{*i}{}_{(r)\,;\,c}\,.$$

Again substituting the values of $X^i{}_{ac}$ and $n^{*i}{}_{;\,c}$, we get

(4.8) $$p^{*i}{}_{(q)\,;\,br} = (q^{*a}{}_{(q)\,;\,br} + \sum_{(r)} P^*{}_{(qr)\,b}\; B^a{}_{(r)\,c})\; X^i{}_a$$
$$+ \sum_{(r)} (q^*{}_{(q)\,b}\; \Omega^*{}_{(r)\,ac} + P^*{}_{(qr)\,b\,;\,r}$$
$$+ \sum_{(x)} P^*{}_{(qx)\,b}\; N^{(r)}{}_{(x)\,c})\;\; n^{*i}{}_{(r)}\,.$$

With the help of

$$2\, p^{*i}{}_{(q)\,;\,[bc]} = R^i{}_{jkl}\; p^{*j}{}_{(q)}\; X^k{}_b\; X^l{}_c$$

and the above equation, we have

(4.9) $$R^i{}_{jkl}\; p^{*j}{}_{(q)}\; X^k{}_b\; X^l{}_c$$
$$= 2\, X^i{}_a\; (q^{*a}{}_{(q)\,[b\,\cdot\,c]} + \sum_{(r)} P^*{}_{(qr)\,[b}\; B^a{}_{|\,(r)\,|\,c]})$$
$$+ 2 \sum_{(r)} (q^{*a}{}_{(q)\,[b}\; \Omega^*{}_{|\,(r)\;a\,|\,c]} + P^*{}_{(qr)\,[b\,;\,r]}$$
$$+ \sum_{(x)} P^*{}_{(qx)\,[b}\; N^{(r)}{}_{|\,(x)\,|\,c]})\;\; n^{*i}{}_{(r)}.$$

Multiplying (4.9) by $g_{ih}\,(x, x')\; n^{*h}{}_{(\sigma)}$, we get.

(4.10)
$$R_{jhkl}\, p^{*j}{}_{(q)}\, n^{*h}{}_{(r)}\, X^k{}_b\, X^l{}_c = 2\,\psi_{(c)} \{ q^{*a}{}_{(q)} [_b\, \Omega^*{}_{|(r)a|c]} + P^*{}_{(qr)}[_{b:c]} + \sum_{(x)} P^*{}_{(qx)}[_b\, N^{(r)}{}_{|(x)|c]} \}.$$

Similarly, multiplying (4.9) by $G\, g_{ih}\,(x, x')\, X^h{}_d\, x$, we find that

(4.11)
$$R_{jhkl}\, p^{*j}{}_{(q)}\, X^h{}_d\, X^k{}_b\, X^l{}_c = 2\, g_{ad}\,(u, u')\, \{ q^{*a}{}_{(q)}[_{b:c]} + \sum_{(r)} P^*{}_{(qr)}[_b\, B^a{}_{|(r)|c]} \}.$$

The equations (4.10) and (4.11) are the fundamental equations which determine the subspace F_m of a Finsler space F_n. From (4.5) we get

(4.12)
$$P^*{}_{(qc)h} = \frac{1}{\psi_{(r)}}\, g_{ih}\,(x, x')\, p^{*i}{}_{(q):b}\, n^{*h}{}_{(r)}$$

and

(4.13)
$$q^*{}_{(q)bd} = q^{*a}{}_{(q)b}\, g_{ad}\,(u, u') = g_{ih}\,(x, x')\, X^h{}_d\, p^{*i}{}_{(q);b}.$$

In the case of a hypersurface, p^{*i} is given by.

(4.14)
$$p^{*i} = t^{*a}\, X^i{}_a + \Gamma^*\, n^{*i}$$

where t^{*a} and Γ^* are given by

(4.15)
$$t^{*a}\, g_{ac}\,(u, u') = g_{ij}\,(x, x')\, X^j{}_c\, p^{*i} \equiv t^*{}_c$$

and

(4.16)
$$\Gamma^* = \frac{1}{\psi}\, g_{ij}\,(x, x')\, p^{*i}\, n^{*j}.$$

Equations (4.6) and (4.7) in this case become

(4.17)
$$q^{*a} = t^{*a}{}_{;b} + \Gamma^*\, B^a{}_b$$

and

(4.18)
$$P^*{}_b = \Omega^*{}_{ab}\, t^{*a} + \Gamma^*{}_{;b} + \Gamma^*\, N_b.$$

For the case of a hypersurface the equations (4.9) - (4.13) take the following forms respectively:

(4.19)
$$R^i{}_{jkl}\, p^{*j}\, X^k{}_b\, X^l{}_c = 2\, X^i{}_a \{ q^{*a}[_{b;c]} + P^*[_b\, B_{c]} \} + 2 \{ \Omega^*{}_a[_c\, q^{*a}{}_{b]} + P^*[_{b;c]} + P^*[_b\, N_{c]} \}\, n^{*i},$$

(4.20)
$$R_{jhkl}\, p^{*j}\, n^{*h}\, X^k_b\, X^l_c = 2\,\psi\,\{\Omega^*_{a}{}_{[c}\, q^{*a}{}_{b]} + P^*{}_{[b\,;\,c]} + P^*{}_{[b}\, N_{c]}\}\,,$$

(4.21)
$$R_{jhkl}\, p^{*j}\, X^h_d\, X^k_b\, X^l_c = 2\, g_{ad}\,(n, u')\,(q^{*a}{}_{[b\,;\,c]} + P^*{}_{[b}\, B^a{}_{c]})\,,$$

(4.22)
$$P^*{}_b = \frac{1}{\psi}\, g_{ih}\,(x, x')\, p^{*i}{}_{;\,b}\, n^{*h},$$

and

(4.23)
$$q^*{}_{bd} \equiv q^{*a}{}_b\, g_{ad}\,(a, u') = g_{ih}\,(x, x')\, X^h_d\, p^{*i}{}_{;\,b}\,.$$

5. Particular Cases.

(*a*) Let us consider a congruence of curves, which is such that the vector $p^{*i}{}_{(q)}$ in the direction of the curve of the congruence, is normal to F_m. Thus

(5.1)
$$p^{*i}{}_{(q)} = \sum_{(r)} c^*{}_{(qr)}\, n^{*i}{}_{(r)}\,.$$

The equation (4.4) becomes

$$p^{*i}{}_{(q)\,;\,b} = q^{*a}{}_{(q)\,b}\, X^i_a + P^*{}_{(qr)\,b}\, n^{*i}{}_{(r)}$$

where

(5.2)
$$q^{*a}{}_{(q)\,b} \underset{=\!=}{\text{def}} \sum_{(r)} c^*{}_{(qr)}\, B^a{}_{(r)\,b}$$

and

(5.3)
$$P^*{}_{(qr)\,b} \underset{=\!=}{\text{def}} \sum_{(x)} c^*{}_{(qr)}\, N^{(r)}{}_{(x)\,b} + c^*{}_{(qr)\,;\,b}\,.$$

The equations (4.9) to (4.11), become

(5.4)
$$\begin{aligned} R^i{}_{jkl}\, p^{*j}{}_{(q)}\, X^k_b\, X^l_c &= 2\, X^i_a\, \{q^{*a}{}_{(q)}\,{}_{[b\,;\,c]} + \sum_{(r)} P^*{}_{(qr)}\,{}_{[b}\, B^a{}_{|(r)|\,c]}\} \\ &+ 2\sum_{(r)} \{q^*{}_{(q)}\,{}_{[b}\, \Omega^*{}_{|(r)\,a|\,c]} + P^*{}_{(qr)}\,{}_{[b\,;\,c]} \\ &+ \sum_{(x)} P^*{}_{(qx)}\,{}_{[b}\, N^{(r)}{}_{|(x)|\,c]}\}\, n^{*i}{}_{(r)}\,, \end{aligned}$$

(5.5)
$$R_{jhkl}\, p^{*j}{}_{(q)}\, n^{*h}{}_{(r)}\, X^k{}_b\, X^l{}_c = 2\,\psi_{(r)}\,\{ q^{*a}{}_{(q)}\,[_b\, \Omega^*{}_{|(r)a|\, c]} + P^*{}_{(qr)}\,[_{b\,;\,c]} + \sum_{(x)} P^*{}_{(qx)}\,[_b\, N^{(r)}{}_{|(x)|\,c]}\},$$

and

(5.6)
$$R_{jhkl}\, p^{*j}{}_{(q)}\, X^h{}_d\, X^k{}_b\, X^l{}_c = 2\, g_{ad}\,(u, u')\,\{ q^{*a}{}_{(q)}\,[_{b\,;\,c]} + \sum_{(r)} P^*{}_{(qr)}\,[_b\, B^a{}_{|(r)|\,c]}\}.$$

For a hypersurface, $p^{*i} = n^{*i}$, the equations (5.4) to (5.6) become

(5.7)
$$R^i{}_{jkl}\, n^{*j}\, X^k{}_b\, X^l{}_c = 2\,\{ B^a[_{b\,;\,c]} + N[_b\, B^a{}_{c]}\}\, X^i{}_a + 2\, n^{*i}\,\{\Omega^*{}_a\,[_c\, B^a{}_{b]} + N[_{b\,;\,c]}\},$$

(5.8)
$$R_{jhkl}\, n^{*j}\, n^{*h}\, X^k{}_b\, X^l{}_c = 2\,\psi\,\{\Omega^*{}_a[_c\, B^a{}_{b]} + N\,[_{b\,;\,c]}\},$$

and

(5.9)
$$R_{jhkl}\, n^{*j}\, X^h{}_d\, X^k{}_b\, X^l{}_c = 2\, g_{ad}\,(u, u')\,\{ B^a\,[_{b\,;\,c]} + N\,[_b\, B^a{}_{c]}\}.$$

(*b*) If $p^{*i}{}_{(q)} = X^i{}_a\, t^{*a}{}_{(q)}$, then

$$p^{*i}{}_{(q)\,;\,b} = \sum_{(r)} \Omega^*{}_{(r)ab}\, n^{*i}{}_{(r)}\, t^{*a}{}_{(q)} + X^i{}_a\, t^{*a}{}_{(q)\,;\,b}.$$

Consequently, the equations (4.9) to (4.11) change into the following equations respectively:

(5.10)
$$R^i{}_{jkl}\, p^{*j}{}_{(q)}\, X^k{}_b\, X^l{}_c = 2\, t^{*a}{}_{(q)\,;\,[bc]}\, X^i{}_a + 2\sum_{(r)} t^{*a}{}_{(q)}\, \Omega^*{}_{(r)a}[_b\, B^d{}_{|(r)|\,c]}\, X^i{}_d + 2\sum_{(r)} t^{*a}{}_{(q)}\, n^{*i}{}_{(r)}\, \Omega^*{}_{(r)a}[_{b\,;\,c]} + 2\sum_{(r)}\sum_{(x)} t^{*a}{}_{(q)}\, n^{*i}{}_{(x)}\, \Omega^*{}_{(r)a}[_b\, N^{(x)}{}_{|(r)|\,c]},$$

(5.11)
$$R_{jhkl}\, p^{*j}{}_{(q)}\, X^k{}_b\, X^l{}_c\, n^{*h}{}_{(\sigma)} = 2\, t^{*a}{}_{(q)}\, \Omega^*{}_{(\sigma)a}\,[_{b\,;\,c]}\,\psi_{(\sigma)} + 2\sum_{(r)} t^{*a}{}_{(q)}\, \Omega^*{}_{(r)a}\,[_b\, N^{(\sigma)}{}_{|(r)|\,c]}\,\psi_{(\sigma)},$$

and

(5.12)
$$R_{jhkl}\ p^{*j}{}_{(q)}\ X^h{}_d\ X^k{}_b\ X^l{}_c = 2\ g_{ad}\ (u, u')\ t^{*a}{}_{(q)\,;[bc]} + 2 \sum_{(r)} t^{*a}{}_{(q)}\ \Omega^*{}_{(r)a[b}\ B^e{}_{|(r)|c]}\ g_{de}\ (u, u').$$

For a hypersurface, equations (5.10) to (5.12) become

(5.13)
$$R^i{}_{jkl}\ p^{*j}\ X^k{}_b\ X^l{}_c = 2\ t^{*a}{}_{;[bc]}\ X^i{}_a\ 2\ t^{*a}\ \Omega^*{}_{a[b}\ B^d{}_{c]}\ X^i{}_d + 2t^{*a}\ n^{*i}\ \{\Omega^*{}_{a[b;c]} + \Omega^*{}_{a[b}\ N_{c]}\},$$

(5.14)
$$R_{jhkl}\ p^{*j}\ X^k{}_b\ X^l{}_c\ n^{*h} = 2\ \psi\ t^{*a}\ \{\Omega^*{}_{a[b;c]} + \Omega^*{}_{a[b}\ N_{c]}\},$$

and

(5.15)
$$R_{jhkl}\ p^{*j}\ X^h{}_d\ X^k{}_b\ X^l{}_c = 2\ g_{ad}\ t^{*a}{}_{;[bc]} + 2\ t^{*a}\ g_{de}\ (u, u')\ \Omega^*{}_{a[b}\ B^e{}_{c]}.$$

(c) if we further assume that $t^{*a}{}_{(q)}$ and $p^{*i}{}_{(q)}$ are tangential to the curve of the congruence, then they are connected by the relation $x'^i = u'^a\ X^i{}_a$. Due to this change, the equations (5.10) to (5.12) take the following forms respectively :

(5.16)
$$R^i{}_{jkl}\ X^j{}_a\ X^k{}_b\ X^l{}_c = R^e{}_{abc}\ X^i{}_e + 2 \sum_{(r)} X^i{}_d\ \Omega^*{}_{(r)a[b}\ B^d{}_{|(r)|c]} + 2 \sum_{(r)} \Omega^*{}_{(r)a[b;c]}\ n^{*i}{}_{(r)} + 2 \sum_{(r)} \sum_{(x)} n^{*i}{}_{(x)}\ \Omega^*{}_{(r)a[b}\ N^{(x)}{}_{|(r)|c]},$$

(5.17)
$$R_{jhkl}\ X^j{}_a\ X^k{}_b\ X^l{}_c\ n^{*h}{}_{(\sigma)} = 2\ \Omega^*{}_{(\sigma)a[b;c]}\ \psi_{(\sigma)} + 2\ \psi_{(\sigma)} \sum_{(r)} \Omega^*{}_{(r)a[b}\ N^{(\sigma)}{}_{|(r)|c]},$$

and

(5.18)
$$R_{jhkl}\ X^j{}_a\ X^k{}_h\ X^l{}_c\ X^h{}_e = R_{aehc} + 2 \sum_{(r)} g_{de}\ (u, u')\ \Omega^*{}_{(r)a[b}\ B^d{}_{|(r)|c]}.$$

In the case of a hypersurface, equations (5.16) to (5.18) become

$$R^i{}_{jkl}\ X^j{}_a\ X^k{}_b\ X^l{}_c = R^e{}_{ahc}\ X^i{}_e + 2\ X^i{}_d\ \Omega^*{}_{a\,[b}\ B^d{}_{c]} + 2\ \Omega^*{}_{a[b\,;\,c]}\ n^{*i} + 2\ \Omega^*{}_{a\,[b}\ N_{c]}\ n^{*i}\,, \tag{5.19}$$

$$R_{jhkl}\ X^j{}_a\ X^k{}_b\ X^l{}_c\ n^{*h} = 2\ \psi\,\{\ \Omega^*{}_{a\,[b\,;\,c]} + \Omega^*{}_{a\,[b}\ N_{c]}\}\,, \tag{5.20}$$

and

$$R_{jhkl}\ X^j{}_a\ X^k{}_b\ X^l{}_c\ X^h{}_e = R_{aebc} + 2\ g_{de}\ \Omega^*{}_{a\,[b}\ B^d{}_{c]}\,. \tag{5.21}$$

The equations (5.16) to (5.18) are the secondary generalised forms of GAUSS and GODAZZI equations for the subspace F_m, [1], and the equations (5.19) to (5.21) are for the hypersurface F_{n-1} [3].

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ÖZET

V_n hiperyüzeyinin her noktasından bir eğrisi geçen eğri kongrüansı göz önüne alınırsa, hiperyüzeyin genel noktasındaki normal birim vektörünün kontravaryant bileşenleri N^i ve aynı noktadan geçen eğrinin teğet birim vektörünün kontravaryant bileşenleri λ^j ile gösterilmek üzere v_k (yani $a_{ij}\ N^i \lambda^j{}_{;\,k}$) için bir ifadenin R. S. MISHRA tarafından elde edildiği malûmdur [2]. Bu araştırmanın gâyesi bu formülün bir FINSLER uzayında bulunan bir hiperyüzey veya bir altuzayından hareket edilmesi halindeki genelleştirilmiş şeklini bulmaktır. Bazı özel haller de ayrıca incelenmiştir.