POLARITY FOR A QUADRIC IN AN n - SPACE

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Abstract (*). H. F. BAKER [3] proved *analytically* that the $n+1$ $(n-2)$-spaces common to the pairs of corresponding primes of a pair of polar simplexes S and S' for a quadric Q in an n-space S_n, are *associated* in such a way that they are met by ∞^{n-2} lines, one line through each point of each $(n-2)$-space. Later J. A. TODD and H. S. M. COXETER [17] also proved *analytically* the *dually associated* character of the $n+1$ joins of the pairs of corresponding vertices of S and S' as a solution of an advanced problem proposed dy S. BAETTY [18]. It is suggested by COXETER (in the EDITOR's note there) that the same statement can be established *synthetically* by induction. This suggestion is followed up here to prove: «If ∞^{n-3} $(n-2)$-spaces, for n greater than 3, meeting n of the $n+1$ given lines $A_i B_i$ of general position in S_n, pass respectively through each of 2 points A_n, B_n of the $(n+1)$ th line, the $n+1$ lines $A_i B_i$ $(i=0, 1, \ldots, n)$ are associated in such a way that ∞^{n-3} $(n-2)$-spaces meeting them pass through every point of every one of these lines.»

Incidently we observe that $n(n+1)$ points, two on each edge of a simplex S in S_n, lie on a quadric, if, and only if, they lie, in $2^{n(n+1)/2}$ ways, in n-ads in the $n+1$ primes of another, polar to S for a quadric. As a result, we derive PASCAL's theorem for a quadric in S_n according to CHASLES [19] and its dual, BRIANCHON's theorem, in analogy with those for a conic, leading to a system of $(n+1)\,2^n$ lines, 2^n through each vertex of S, such that each line belongs to $2^{n(n-1)/2}$ of $2^{n(n+1)/2}$ associated sets of $n+1$ lines each. However interesting the relations of the lines of a system, they are not treated here.

A number of special cases of some interest are noted explaining the novelties in the paper of BAKER referred to above. Selfconjugate r-ads for Q arising from degenerate cases are also discussed. The paper is divided into 2 sections, one devoted to 4-space only, the other deals with developments in higher spaces.

(*) Published in the Proceedings of the 47th Session of the Indian Science Cong. Association held at Bombay in January 1960.

I. SPACE OF FOUR DIMENSIONS

1. 5 Associated Lines.

If 3 planes, meeting 4 given lines a, b, c, d of general position in a 4-space, pass respectively through each of two points E, E' of general position, ∞^1 such planes are possible and a plane through E or E' meeting three of a, h, c, d necessarily meets the fourth, EE' then is the fifth line e associated with a, b, c, d [9], [10].

The major work below is based on this proposition which is a necessary consequence of the observations made by BAKER ([3], p. 123) in regard to the character of a set of 5 associated lines.

2. Polar (Reciprocal) Simplexes.

2.1. Let i' be respectively the poles of the 5 solids $j\,k\,l\,m$

$$(i, j, k, l, m = A, B, C, D, E)$$

of a general simplex $S = A\,B\,C\,D\,E$ for a quadric Q in a 4-space. $S' = A'B'C'D'E'$ and S then form a pair of *polar simplexes* for Q.

The projection $j''k''l''m''$, of $j'k'l'm'$ from i' in its polar solid $j\,k\,l\,m$ for Q, forms a tetrahedron polar to $j\,k\,l\,m$ for the quadric section of Q by this solid. $j\,j''$, $k\,k''$, $l\,l''$, $m\,m''$ then generate a quadric w ([2], Ex. 7, p. 41) and have ∞^1 transversals which joined to i' give us ∞^1 planes meeting $j\,j'$, $k\,k'$, $l\,l'$, $m\,m'$. Similarly through i pass ∞^1 planes meeting them. Therefore, $i\,i'$ form a set of 5 associated lines (Art. 1).

2.2. Conversely, if the 5 pairs of corresponding vertices i, i' of 2 simplexes S and S' in a 4-space lie on 5 associated lines $i\,i'$, there exists a quadric Q, for which S and S' are polar, as follows.

Project the solid $j'k'l'm'$ of S' from its opposite vertex i', into $j''k''l''m''$ in the solid $j\,k\,l\,m$ of S, and the triangle $k''l''m''$ from j'' or the plane $k'l'm'$ of S' from its opposite edge $i'\,j'$ into $k'''\,l'''\,m'''$ in the plane $k\,l\,m$.

Now through i' pass an infinity of planes meeting the 5 given associated lines. They meet the solid $j\,k\,l\,m$ in an infinity of lines meeting the 4 lines $j\,j''$, $k\,k''$, $l\,l''$, $m\,m''$ which then generate a quadric w. Therefore the tetrahedra $j\,k\,l\,m$ and $j''\,k''\,l''\,m''$ are polar for a quadric Q_i ([2], Ex. 14, p. 53). Thence the triangles $k\,l\,m$ and $k'''\,l'''\,m'''$ are polar for the conic section Q_{ij} of Q_i by their plane. Therefore by CHASLES's theorem ([6], p. 62), they are in perspective from a point, say $(i\,j)$. It is easily verified that the plane $(i\,j)\;i'\,j'$ meets the 5 associated lines, and the line $(i\,j)\,j''$, lying in it, is a generator of w.

Similarly we construct the quadric Q_j for which the tetrahedron $i\,k\,l\,m$ is polar to the projection of $i'\,k'\,l'\,m'$ from j' in its solid, and show that the triangles $k\,l\,m$ and $k'''\,l'''\,m'''$ are polar for the conic section Q_{ji} of Q_j by their plane. The Desarguesian character of these two triangles fixes the conic for which they are polar ([6], p. 65). Hence $Q_{ij} = Q_{ji}$, that is, the quadrics Q_i and Q_j meet in a conic.

Thus the 5 quadrics Q_i, one in each solid of S, are such that every two of them meet in a conic. Therefore, they all lie on a 3-quadric Q determined by any three of them. Q is then seen to be the required quadric. Hence, *the 5 joins of the vertices of a simplex S, in a 4-space, to those of another, say S', one to one, form, in general, an associated set of 5 lines, if, and only if, S and S' are polar for a quadric, and consequently the 5 planes common to their corresponding solids too form an associated set sach that any line meeting four of them meeets the fifth and ∞^1 such lines lie in every solid through every plane of the set.*

3. Observations.

3.1. The 6 intersections, of the non-corresponding sides of a pair of triangles polar for a conic and therefore in perspective (Art. 2.2), lie on a conic ([1], p. 219; [6], Ex. 5, p. 80). Conversely, the 6 intersections, of a conic with the sides of a triangle, lie, in 8 ways ([5], p. 419), in pairs on the sides of another DESARGUES with it and therefore polar to it for a conic. Thus, *the 6 points, two on each side of a triangle, lie on a conic, if, and only if, they lie in 8 ways, in pairs on the sides of another polar to it for a conic and therefore* DESARGUES *with it.* The later part of this proposition speaks of the PASCAL's theorem, for a conic, in a form suitable for its extension into higher spaces ([5], p. 417; [15], pp. 141—42).

3.2. Let an edge ij of a simplex S meet a quadric W in g, h. An involution is set up by the 2 pairs of points $i, g;\ j, h$ and another by $i, h;\ j, g$. There are 2 pairs of foci of the 2 involutions, one pair for each, on ij, and thus 2 pairs of such foci on each edge of S. Now it follows from the preceding proposition that 3 pairs of them, one pair on each edge in a plane of S, lie on a conic, and there are 8 such conics in this plane. Thence the quadric Q, through 4 pairs of them, one pair on each edge through a vertex i of S, and 6 other foci, one on each other edge, passes through one of the said 8 conics in each plane of S through i, and therefore through one such conic in every plane of S. Q is one of the $2^{10} = 1024$ quadrics for which each intersection of W with each edge of S is conjugate to a vertex of S thereat. For there are 2 choices for a pair of foci on each edge of S independent of each other, there being 10 edges in all.

3.3. Conversely, the 20 points, two on each edge of a simplex S conjugate respectively to its two vertices thereat for a quadric Q, lie on a quadric W. For the 6 points in each plane of S lie on a conic as observed above (Art. 3.1). Now they distribute into 5 tetrads, each tetrad conjugate to a vertex of S for Q, in the 5 solids which determine the simplex S' polar to S for Q. Hence, *the 20 points, two on each edge of a simplex S in a 4-space, lie on a qaadric, if, and only if, they lie, in 1024 ways, in tetrads in the 5 solids of another polar to S for a quadric.* Dually, *the 20 solids, two through each plane of S, touch a quadric, if, and only if, they pass, in 1024 ways, in tetrads through the 5 vertices of another polar to S for a qaadric.* We may refer to it as an *S-theorem in a 4-space.*

3.4. Again consider the 20 points, two on each edge of S' (Art. 2.1) conjugate respectively to its two vertices thereat for Q. They form 5 tetrahedra of the type $j'' k'' l'' m''$ polar to the tetrahedron $j\,k\,l\,m$ of S for the quadric section of Q by its solid. Hence, *the 5 tetrahedra, each polar to a tetrahedron of a simplex in a 4-space for the section of a quadric by its solid, are inscribed in a qaadric.* It may he referred to as an *s-theorem in a 4-space.*

4. PASCAL's and BRIANCHON's theorems.

As an immediate consequence of the observations made above, we have the PASCAL's theorem in a 4-space, analogous to that for a conic, following COURT ([5], p. 418) and SALMON ([15], p. 142), and its dual, BRIANCHON's, as follows:

The 20 points, two on each edge of a simplex S in a 4-space, lie on a quadric, if, and only if, they distribute, in 1024 ways, into 5 tetrads, each tetrad consisting of 4 points on the 4 edges through a vertex of S, determining 5 solids which meet the 5 solids of S opposite its respective vertices in 5 associated planes.

For there are 2 choices for each point on each edge of S to belong to a tetrad independent of each other, and the 5 solids through the 20 points determine a simplex S' polar to S for a quadric.

Dually, *the 20 solids, two through each plane of S, touch a quadric, if, and only if, they distribute, in 1024 ways, into 5 tetrads, each tetrad consisting of 4 solids through the 4 planes in a solid of S, determining 5 points which join the vertices of S opposite its respective solids in 5 associated lines.*

The analogues theorems in a solid proved by BAKER ([2], Ex. 15, pp. 53—54) can also be established in this style.

5. Related Polar and Self-polar Simplexes.

5.1. Following BAKER ([4], pp. 516—518), we can derive 120 pairs of self-polar simplexes from a pair of polar simplexes, say S and S' (Art. 2.1), for the quadric Q as follows.

Let the vertices of S and S' be arranged in a cyclical order $j\,k\,l\,m\,i'\;j'\,k'\,l'\,m'\,i$. Every four consecutive points determine a solid. The first five consecutive solids in this order, *viz.*, $j\,k\,l\,m$, $k\,l\,m\,i'$, $l\,m\,i'\,j'$, $m\,i'\,j'\,k'$, $i'\,j'\,k'\,l'$, and the next or opposite five determine respectively two simplexes both self-polar for Q. The same two simplexes arise, if we arrange the solids opposite the respective vertices of S and S' in this order and take the points common to the tetrads of consecutive solids as their vertices, the first five for one and the next five for the other. Evidently then there are 120 pairs of such simplexes, one pair for each permutation of $i\,j\,k\,l\,m$ which settles that of $i'\,j'\,k'\,l'\,m'$ in the cycle.

But a simplex degenerates, if two of its solids coincide. That is the case, if the solid $j\,k\,l\,m$ contains its pole i' for Q, that is, when it touches Q and therefore i' lies on Q, or, if the plane $k\,l\,m$ meets its polar line $i'\,j'$ for Q, that is, when both touch Q. The former type of degeneration occurs for every permutation of $j\,k\,l\,m$, the later for every permutation of $k\,l\,m$ coupled with one of i', j'. Thus, *a pair of polar simplexes for a quadric Q, in a 4-space, give rise to 120 pairs of self-polar simplexes for Q. If a vertex of either lies on Q, 24 self-polar simplexes degenerate, and if an edge or a plane of either touches Q, 12 such simplexes degenerate.*

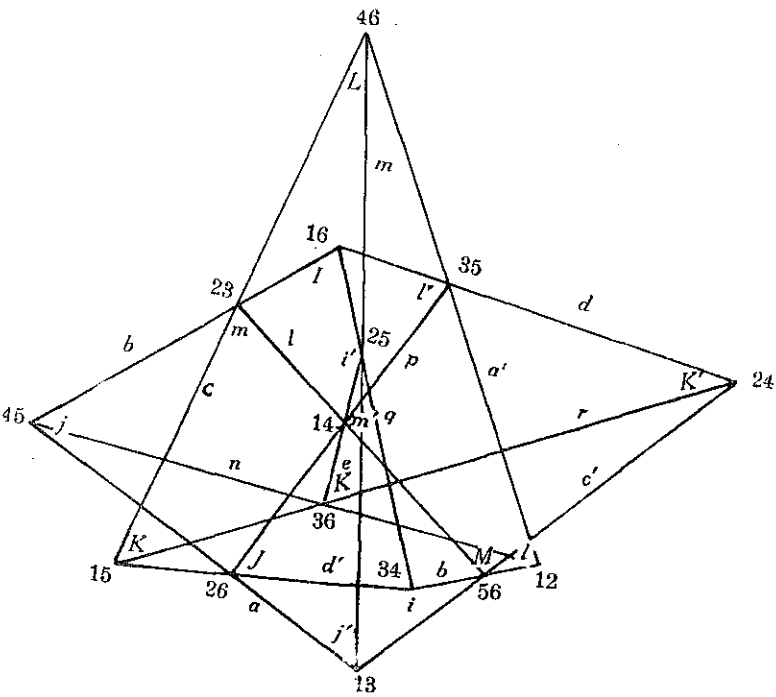
This explains the degeneration of the 2 self-polar simplexes derived by BAKER ([4], p. 417) from a pair of polar simplexes for a quadric.

5.2. Conversely, *we can derive 14400 pairs of polar simplexes from 2 self-polar simplexes for a quadric in a 4-space.* For, in this case, every permutation of the vertices of either of the two given simplexes can be coupled with every permutation of those of the other. Degeneration cases occur here also, if a vertex of one lies in a solid of the other, or, if an edge of one meets a plane of the other, that is, when a vertex, an edge, a plane or a solid of one is *conjugate* respectively to a vertex, an edge, a plane or a solid of the other for the quadric.

Definitions. *2 lines or 2 planes* in a 4-space are said to be *conjugate* for a quadric Q, if the polar of one for Q meets the other and consequently the polar of the second for Q meets the first; *a line l and a plane p* are said to be *conjugate* for Q, if the polar line of p, for Q, and l lie in a plane polar, therefore, to the line in which the polar plane of l, for Q, must meet p ([8], p. 171), in analogy with *conjugate* lines in a solid ([2], Ex. 5, p. 34) for a quadric there.

6. SEGRE's Figure 15_3.

6.1. 5 lines of an associated set and the 10 transversals of theirs, one to each triad of them, form a SEGRE's figure ([3], pp. 113—14), denoted $s\,f$ when arising from the 5 joins $i\,i'$ (Art. 2.1), of 15 lines and 15 CREMONA points ([1], p. 226), 3 lines through each point and 3 points on each line. The 15 points lie by fives in 45 T-planes ([1], p. 226), each containing either two transversals or a transversal and a line of the set meeting at a CREMONA point. Every two lines of the set determine a *singular* solid ([3], p. 115) which contains the transversal of the other three lines, 3 other transversals, one to each triad consisting of these two lines and one other, skew to each other but meeting the former one, 9 CREMONA points and 9 T-planes. There are 10 such solids. Thus, the transversals, of triads



having 2 lines common, are skew, and of those, having one line common, meet at a CREMONA point. Again 2 transversals and a line of the set concur at each CREMONA point and thus determine a solid which we may call a CREMONA solid. There are 15 such solids, each containing 7 CREMONA points and 3 T-planes.

6.2. The 4 edges through a vertex i of a simplex S (Art. 2.1) give us 3 pairs of opposite planes, viz., $i\,j\,k$, $l\,m\,i$; $i\,k\,l$, $j\,m\,i$; $i\,j\,l$, $k\,m\,i$. Let $i\,j\,k$, $l\,m\,i$ be conjugate for Q. Then $i\,j\,k$ meets the polar line $j'k'$ of $l\,m\,i$ which meets the polar line $l'm'$ of $i\,j\,k$ for Q. Thus i lies in the solids $j\,j'\,k\,k'$, $l\,l'\,m\,m'$, and, therefore, the common transversal of the triad of lines $i\,i'$, $j\,j'$, $k\,k'$ and that of $i\,i'$, $l\,l'$, $m\,m'$ both pass through i which is then a CREMONA point of $s\,f$.

Again if the edges $i\,k$, $m\,j$ of S be conjugate for Q, their respective polar planes $l'm'j'$, $i'k'l'$ are conjugate for Q, and, therefore, l', which is the pole of the solid $i\,j\,k\,m$ for Q, is a CREMONA point of $s\,f$. Thus, *if a pair of opposite planes through a vertex i of a simplex S, in a 4-space, and a pair of opposite edges in a solid of S be conjugate for a quadric Q, the pole of the solid for Q and i are both* CREMONA *points of $s\,f$* (cf. [13], Art. 5c).

6.3. Now if a plane ijk of S be conjugate to its two opposite planes kml, lmi for Q, it meets both the lines $i'j'$, $j'k'$ polar to them for Q, that is, j' lies in it and, therefore, its polar line $l'm'$ lies in the solid $klmi$ polar to j' for Q. ik then meets jj', say in J, and is, therefore, the common transversal of the triad of lines ii', jj', kk' with i, J, k as the 3 collinear CREMONA points of sf on it.

Again, if the edge ik of S be conjugate to its two opposite edges mj, jl for Q, they both meet its polar plane $j'l'm'$ for Q, and therefore j lies in this plane and ik in the polar solid $i'k'l'm'$ of j for Q, that is, $l'm'$ also meets jj' in the point no other than J and is the common transversal of the triad of the lines jj', ll', mm' with l', m', J as the 3 collinear CREMONA points of sf on it.

Consequently, ik, jj', $l'm'$, concurrent at J, are 3 mutually polar lines for Q, and, therefore, tangent to Q at J, lie in a CREMONA solid, of sf, tangent to Q at J.

6.4. Let us consider the conjugacy, for Q, of the alternate or opposite planes of S in a cycle $(ijklm)$ of its vertices along with that of the alternate or opposite edges of S in the cycle $(ikmjl)$, square of the former. To be specific for reference, we put down these conjugacies in the tabular form as follows:

ijk	is conjugate to	klm, lmi	....	(j')
jkl	—	lmi, mij	....	(k')
klm	—	mij, ijk	....	(l')
lmi	—	ijk, jkl	....	(m')
mij	—	jkl, klm	....	(i')
ik	—	mj, jl	....	(j)
km	—	jl, li	....	(l)
mj	—	li, ik	....	(i)
jl	—	ik, km	....	(k)
li	—	km, mj	....	(m)

We have discussed (j), (j') just above. Similarly behave the rest in like pairs from which we infer that jl, kk', $m'i'$; km, ll', $i'j'$; li, mm', $j'k'$; mj, ii', $k'l'$ concur respectively as triads of mutually polar lines for Q and, therefore, tangent to Q at the CREMONA points K, L, M, I of sf. Thus, *a pair of polar simplexes S and S' for a quadric Q, in a 4-space, can be so related that the 15 lines of sf all touch Q in triads of mutually polar lines for Q and, therefore, lie in 5* CREMONA *solids tangent to Q at the respective* CREMONA *points of sf.*

The simplex $s = IJKLM$ is inscribed to both of the skew pentagons $ikmjl$, $i'j'k'l'm'$ whose sides constitute the 10 transversals and vertices the 10 CREMONA points of sf other than I, J, K, L, M. The relation of the pair of polar simplexes S and S' (Art. 2.1) now is such that each vertex of either lies in a plane of the other and, therefore, each solid of either contains an edge of the other. We thus have some generalization ([4], p. 513) of MOEBIUS tetrads ([11], p. 471). The solids of S and S' constitute the 10 singular solids of sf.

6.5. Following BAKER ([1], p. 409), if we rename symbolically the vertices of S and S' as

$$i = 34, \quad j = 45, \quad k = 51, \quad l = 12, \quad m = 23,$$

$$i' = 52, \quad j' = 13, \quad k' = 24, \quad l' = 35, \quad m' = 41,$$

we find the novelty, in their relationship giving rise to 5 new pairs of polar simplexes for Q besides their being the 10 nodes of a SEGRE cubic primal, as observed by him, answered here in the annexed diagram. The 15 planes common to the pairs of the corresponding solids of these 6 pairs of simplexes ([4], p. 512) lie by threes in the 10 singular solids of sf, each plane occurring twice, and in the 5 CREMONA solids, tangent to Q at the vertices of s (Art. 6.4), of sf as its T-planes polar to its 15 lines for Q.

6.6. Now how to construct such a pair of polar simplexes, for a given quadric Q, leading to a SEGRE's figure, as illustrated above, is a problem before us answered below.

Take a point I on Q. Draw a triad of mutually polar lines through I, for Q, and, therefore, lying in the solid tangent to Q at I, to meet the solid, tangent to Q at another point J on it, in i, j, l'. Ii, Ij, Il'; Ji, Jj, Jl' thus form 2 triads of mutually polar lines for Q and, therefore, tangent to Q at I, J respectively.

Now take a point k on Ji and let kK be one of the two tangents, from k to Q, at K, in the plane Ikl' meeting Il' at k'. Evidently, Kj touches Q at K, for K lies in the polar solid $IiJl'$ of j for Q. The polar line of the plane jkk', tangent to Q at K, for Q, then touches Q at K and meets the lines Jl', Ii, polar respectively to the planes Jjk, Ijk' for Q, say in m', i'. Thus Ki', Kj, Kk form a third triad of mutually polar lines for Q in the solid tangent to Q at K.

Again let L be one of the two intersections of Q with the polar line of the plane $kl'i'$ for Q. Lk, Ll', Li' then form a fourth triad of mutually polar lines for Q and, therefore, lying in the solid tangent to Q at L. Let the lines Li', Jj polar respectively to the planes Lkl', Jkl' for Q, meet in j'; kL, Ij, polar to $Ll'i'$, $Ii'l'$, meet in m; Ll', Kj, polar to kLi', Kki', in l. il, mm', $j'k'$ are then seen to form the fifth triad of mutually polar lines for Q and, therefore, concurrent, say at M, lying in the solid tangent to Q at M.

That completes the construction of the needed pair of polar simplexes $ijklm$, $i'j'k'l'm'$ for Q (Art. 6.4).

6.7. Conversely, *the 15* CREMONA *points of a* SEGRE's *figure distribute, in 6 ways, into 2 parts, one consisting of 10 points as the nodes of a* SEGRE *cubic primal and, therefore, as the poles of the 10 singular solids of the figure for a 3-qaadric Q, or, as the vertices of a pair of polar simplexes, in 6 ways, for the same quadric, and the other consisting of 5 points as the points of contact on Q of the 15 lines of the figure which touch Q as 5 triads of mutually polar lines for it, one triad through each point, and each of the 15 sach triads occuring twice for two of the 6 quadrics, obtained similarly, for which the figure is thus self-reciprocal* ([3], Ex. 21, p. 148).

For the convenience of the argument, we rename symbolically the vertices of the simplex s as

$$I=16, \quad J=26, \quad K=36, \quad L=46, \quad M=56$$

in the manner we have done above (Art. 6.5) for S and S', and thus our figure now follows the notation of BAKER ([1], p. 225).

A quadric Q in a 4-space is determined by 14 conditions which are just necessary to let Q to circumscribe a simplex like s with vertices at the 5 CREMONA points, whose symbols all have one number in common, of the figure, with the 5 triads of lines through them, one triad through each point, ail mutually polar for Q and, therefore, lying in the 5 CREMONA solids respectively tangent to Q at these points, such that the polar, for Q, of every point of the figure is either a singular or a CREMONA solid and that of every line of the figure is a T-plane (Art. 6.5).

Evidently, there are 6 such quadrics *associated* with the figure, each determined by a simplex like s (cf. [1], Ex. 3, p. 232), every two like simplexes have a common vertex, and, thus, every two quadrics have a triad of mutually polar lines common through the common vertex of the corresponding simplexes.

The 6 fundamental points ([1], p. 224) of the figure, which have led to the symbolic representation of its 15 points making its study simple, symmetrical and fascinating, are discovered later (Art. 8.5) to disclose their fundamental existence in the make-up of the figure and thus complete its self-reciprocal character in regard to the 6 quadrics introduced here.

7. Self-conjugate Heptad

7.1. *If 2 pairs of opposite planes through a vertex i of a simplex S, in a 4-space, he conjugate for a quadric Q, the third pair of them is also conjugate for Q, and the 5 joins of the vertices of S to the respectively corresponding ones of its polar, say S', for Q are met by a line through i* (cf. [13], Art. 5d), *and are thus no longer associated* (cf. Art. 2.1.).

It follows immediately from its equivalent as well as corresponding proposition of BAKER ([2], Ex. 5, pp. 34—35) in the polar solid of i for Q. That is, if 2 pairs of opposite edges of a tetrahedron T are conjugate for Q, the third pair of them is also conjugate for Q and the 4 joins of the vertices of the corresponding ones of its polar for the quadric section of Q by the solid concur. The argument of Art. 1 does not hold good here.

With the aid of Art. 6.2 we can now prove the following theorem: *If 2 pairs of opposite planes through i as well as one pair through another vertex j of S be conjugate for Q, the edge ij of S meets the 5 joins of the corresponding vertices of S and S'. Further, if a pair of opposite planes through a vertex k' in the plane $k'l'm'$, of S', polar to ij for Q, be also conjugate for it, k' lies on ij, and, therefore, ij, $k'l'$, $k'm'$ form a triad of polar lines, for Q, lying in the solid $ijlm$, of S, tangent to Q at k'.*

7.2. *If 2 tetrahedra, lying in different solids in a 4-space, be projective*, they are polar* for a quadric Q, and the 5 joins of the corresponding vertices of the associated polar simplexes have a common transversal and are thus no longer associated.*

Let $T = BCDE$, $T' = B'C'D'E'$ be the 2 projective tetrahedra, lying respectively in the solids a, a', such that the 4 lines jj' are met by a line t, and the 4 points $klm \cdot k'l'm'$ lie on a line x in the plane $a \cdot a'$ $(j, k, l, m = B, C, D, E)$. We can now construct a quadric Q uniquely for which every j' is conjugate to the triad of points k, l, m, and t is polar to x.

Let A be the pole of a', and A' of a, for Q. $S = ABCDE$, $S' = A'B'C'D'E'$ are then the associated polar simplexes, for Q, to which belong T, T' respectively as required, and t meets AA' too. For, the polar plane $a \cdot a'$ of AA' and that of t, for Q, meet in the line x. The argument of Art. 1 fails here to prove the associated character of the 5 lines AA', BB', CC', DD', EE' (cf. Art. 2). For, the starting 4 lines are no longer general, as required there, when they have a common transversal, as is the case here.

7.3. Let F be a point on the common transversal t of the 5 joins ii', and its polar solid f, for the quadric Q, meet an edge $i'j'$ of the simplex S' of the preceding paragraph, in the pole $(Fklm)$ of the solid

$$Fklm \ (i, j, k, l, m = A, B, C, D, E).$$

Then, f contains the 10 points $ijk \cdot i'j'k'$ lying in the polar plane of t for Q; and the polar line (Flm) of the plane Flm, for Q, lies in the plane $i'j'k'$ and contains the 4 points $(Fklm)$, $(Flmi)$, $(Fjlm)$, $ijk \cdot i'j'k$. Thus a plane ijk and the line (Flm) lie in a solid which meets t in G, say. Now the plane

(*) **Definitions.** 2 tetrahedra T, T', lying in different solids in a 4-space, may be said to be *projective*, if the 4 joins of the vertices of T to those of T', one to one, are met by a line; and *polar* for a 3-quadric Q, if they belong to a pair of polar simplexes for Q, one to one, and correspond to each other.

It may be remarked here that T, T' *are projective, if, and only if, the 4 intersections of the corresponding planes of theirs are collinear* [12].

$i\,j\,(F\,k\,l\,m)$ lies in this solid as well as in the solid $i\,j\,i'\,j'$ which contains t, and, therefore, meets t in the point no other than G. Again, the 10 solids $i\,j\,k\,(F\,l\,m)$, one through each plane of the simplex S, and the 10 planes $i\,j\,(F\,k\,l\,m)$ all meeting t, one through each edge of S and meeting the corresponding edge of S', are such that each solid contains 3 planes and each plane iies in 3 solids. Thus, they all concur, and the 7 points A, B, C, D, E, F, G constitute a *self-conjugate heptad* for Q, in analogy with a self-conjugete hexad of BAKER ([2], Ex. 10, p. 47) for a quadric in a solid, such that the plane containing any three of them is conjugate for Q to the solid containing the other four. Its construction betrays that *the 5 joins of any five, of the 7 points of a self-conjugate heptad for a quadric Q in a 4-space, forming a simplex, to the respectively corresponding vertices of its polar for Q, are met by the join of the other two.*

Our self-conjugate heptad here apparently bears a resemblance to the self-polar heptagon of SCHUSTER ([16], p. 143), but it may be remarked that the two are never identical.

8. Self-conjugate Hexad.

8.1. *If 2 edges in a plane of a simplex, in a 4-space, be respectively conjugate* (by definition of Art. 5.2) *for a quadric Q to their opposite planes, the plane and the third edge in it are conjugate for Q to their respectively opposite edge and plane.*

Let an edge $I\,J$ of a simplex s (Art. 6.4) be conjugate for Q to the plane $K\,L\,M$ whose polar line $I'\,J'$, for Q, then meets $I\,J$, and its polar plane $K'\,L'\,M'$ for Q meets $K\,L\,M$ in a line. If, further, $J\,K$ be also conjugate for Q to $I\,L\,M$ which then meets its polar plane $I'\,L'\,M'$ in a line, $J'\,K'$ meets $J\,K$. Thus $I\,J\,K$ meets $I'\,J'\,K'$ in a line, $L\,M$ meets $L'\,M'$, and, therefore, $I\,J\,K$, and every other line therein, $I\,K$ in particular, are all conjugate, for Q, to $L\,M$ proving the first part of the proposition. For the second part, we refer to the dual proposition of Art. 7.1 in the solids $I\,J\,K\,L$, $I\,J\,K\,M$ where IJ, JK are conjugate respectively to KL, IL in one and to KM, IM in the other for Q, and, therefore, $I\,K$ is conjugate to both $J\,L$, $J\,M$ besides $L\,M$ and consequently to the plane $J\,L\,M$ for Q as required.

This is equivalent to saying that if $I\,I'$, $K\,K'$ both meet $J\,J'$, then $I\,I'$, $J\,J'$, $K\,K'$ concur and $L\,L'$, $M\,M'$ meet. Thus, *if two, of the five joins of the pairs of corresponding vertices of a pair of polar simplexes for a quadric in a 4-space, meet a third, the three concur and the other two meet.*

8.2. *If 3 consecutive edges, of a skew pentagon formed of the 5 vertices of a simplex S, in a 4-space, be conjugate to their respective opposite planes for a quadric Q, every edge of S is conjugate to its opposite plane for Q.*

Let the 3 edges $I\,J$, $J\,K$, $K\,L$ of s be conjugate for Q to their respective opposite planes KLM, LMI, MIJ. Then, by the preceding proposition IK, JL, LM,

MI and consequently IL, JM, KM also are conjugate for Q to their respective opposite planes.

This is equivalent to saying that *if four, of the 5 joins of the pairs of corresponding vertices of a pair of polar simplexes s and s′ for a quadric, in a 4-space concur, the fifth also concurs with them and thus s and s′ are in perspective.*

8.3. *Conversely, if two simplexes in a 4-space, be in perspective there is a unique quadric Q for which they are polar.*

Q can be constructed here also by the method adopted above (Art. 2.2), but no construction of Q can be simpler or more elegant than that of BAKER ([3], Ex. 22, p. 149).

8.4. Let $s = IJKLM$, $s' = I'J'K'L'M'$ be two simplexes in perspective from a centre O and polar for a quadric Q in a 4-space. Then the 6 points O, I', J', K', L', M' form a *self-conjugate hexad h* for Q, in analogy with a self-conjugate pentad of BAKER ([2], p. 37) for a quadric in a solid, such that the line joining any two of them is conjugate for Q to the solid containing the other four and consequently the polar line, for Q, of the plane containing any three of them lies in the plane containing the other three. Its construction betrays that *every one of the 6 points of a self-conjugate hexad for a quadric Q, in a 4-space, is the centre of perspective of the simplex formed by the other five and its polar for Q.*

8.5. It may happen that a simplex s is inscribed in a quadric Q and all its planes are conjugate to their respectively opposite edges for Q. It is seen that such is the case in Art. 6.7 where s is then in perspective with its polar simplex s', for Q, constituted by the 5 tangent solids of Q at its vertices, from a centre O. Therefore, the 5 vertices of s' together with O form a self-conjugate hexad h for Q. In fact, these are the wanted six fundamental points of the SEGRE's figure discussed above (Art. 6.7), as can be easily verified. Hence, h is the common self-conjugate hexad, as a common link, of the 6 quadrics, mentioned there, associated with the figure which is noticed to be self-polar for all of them. Thus, *the 6 fundamental points of a SEGRE's figure 15_3 constitute a common self-conjugate hexad for the 6 quadrics associated with it.*

9. Analytical Expressions for Q.

It is proved by BAKER ([2], pp. 34, 39, 49) that a quadric Q in a solid can be expressed tangentially as the sum of squares of 4, 5 or 6 points according as they form a self-conjugate tetrad, pentad or hexad for Q. Similarly it can be shown here too that a quadric Q in a 4-space can also be expressed as the sum of squares of 5, 6 or 7 points according as they form a self conjugate pentad, hexad or heptad for Q. In fact, HODGE and PEDOE ([7], pp. 219—225) have done the needful for all spaces and thus established their existence analytically as polar r-ads for Q in each space.

II. SPACE OF n DIMENSIONS

10. $n+1$ Associated Lines.

Let us assume that, in an $(n-1)$-space S_{n-1}, if ∞^{n-4} $(n-3)$-spaces $(n>4)$ *) meeting $n-1$ of n given lines $A_i' B_i'$ $(i=1, \ldots, n)$ of general position pass respectively through each of 2 points A_n', B_n' of the nth line, the n lines are *associated* such that ∞^{n-4} $(n-3)$-spaces meeting them pass through every point of every one of these lines and therefore ∞^{n-3} $(n-3)$-spaces, in all, meet all of them.

If, in an n-space $(n>3)$, ∞^{n-3} $(n-2)$*-spaces meeting n of* $n+1$ *given lines* $A_j B_j$ $(j=0, 1, \ldots, n)$ *of general position pass respectively through each of two points* A_n, B_n *of the* $(n+1)$ *th line, the* $n+1$ *lines are associated such that* ∞^{n-3} $(n-2)$*-spaces meeting them pass through every point of every one of these lines and thus any* $(n-2)$*-space meeting n of them meets the* $(n+1)$*th too.* $A_j B_j$ will be referred to form *an associated set of* $n+1$ *lines.*

To prove the proposition, project the n lines $A_i B_i$ from a point P on the $(n+1)$th line $A_0 B_0$ into then n lines $A_i' B_i'$ in S_{n-1} such that A_i projects into A_i' and B_i into B_i'. Now from hypothesis it follows that ∞^{n-4} $(n-2)$-spaces meeting the $n+1$ lines pass respectively throagh PA_n, PB_n and therefore ∞^{n-4} $(n-3)$-spaces meeting the n lines $A_i' B_i'$ pass respectively through A_n', B_n'. Then, by the assumption, these n lines are associated in such a way that they are all met by ∞^{n-3} $(n-3)$-spaces which joined to P give us the same number of $(n-2)$-spaces through P meeting all the lines $A_j B_j$. Therefore, if Q be a point on $A_n B_n$, there pass ∞^{n-4} $(n-2)$-spaces meeting these $n+1$ lines through PQ, P being an arbitrary point on $A_0 B_0$ whieh itself is an arbitrarily chosen one of the given lines. Thus ∞^{n-3} $(n-2)$-spaces pass through an arbitrary point Q on $A_n B_n$ to meet the $n+1$ lines.

Hence the proposition under consideration holds good, if, and only if, our assumption be true. But the same is so in a 4-space as seen above (Art. 1). Thus it holds when $n=5$, and therefore for $n=6$, and so on.

(*) **Remark:** In a plane any three concurrent lines and in a solid any four generators of one system of a quadric may also be said to be *associated* as a limiting case, if we agree to take $\infty^0=1$ and modify a bit the proposition to suit the circumstances there, for any line in a plane meets any other therein and every four lines, in a solid, of general position always have two transversals ([2] p. 184).

11. Polar Simplexes.

The joins of the corresponding vertices of two simplexes, in an n-space, form, in general, an associated set of $n+1$ lines, if, and only if, the simplexes are polar for a qaadric therein, and consequently the $n+1$ $(n-2)$-spaces common to their corresponding primes too form an associated set such that any line meeting n of them meets the $(n+1)$th and ∞^{n-3} such lines lie in every hyperplane through every $(n-2)$-space of the sei.

For $n=4$, it has been established (Art. 2) by making use of Art. 1, which is the basis of the last article, and the corresponding propositions in a solid and in a plane. Following similar arguments, it can be now proved, by the method of induction, for higher spaces too, with the help of the preceding proposition.

12. S- and s-theorems.

Following the arguments of Art. 3, we may simply state these theorems as follows:

S-theorem: *The $n(n+1)$ points, 2 on each edge of a simplex S in an n-space, lie on quadric, if, and only if, they lie, in $2^{n(n+1)/2}$ ways, in n-ads in the $n+1$ primes of another, polar to S for a quadric.* Dually: *The $n(n+1)$ hyperplanes, 2 through each $(n-2)$-space of S, touch a quadric, if, and only if, they pass, in $2^{n(n+1)/2}$ ways, in n-ads through the $n+1$ vertices of another, polar to S for a quadric.*

For $n(n+3)/2$ general points in an n-space determine uniquely an $(n-1)$-quadric, referred to simply as a quadric here unless otherwise stated, therein.

s-theorem: *The $n+1$ $(n-1)$-simplexes, each polar to the $(n-1)$-simplex, formed of n vertices of a simplex in an n-space, for the $(n-2)$-quadric section of an $(n-1)$-qaadric therein by its hyperplane, are all inscribed in an $(n-1)$-quadric.*

13. Pascal's and Brianchon's Theorems.

As a result of the S-theorem in an n-space, we are now in a position to state these analogues of these theorems (cf. Art. 4) as follows:

Pascal's Theorem: *The $n(n+1)$ points, 2 on each edge of a simplex S in an n-space, lie on a quadric, if, and only if, they distribute, in $2^{n(n+1)/2}$ ways, into $n+1$ n-ads, each n-ad consisting of n points on the n edges through a vertex of S, determining $n+1$ hyperplanes which meet the $n+1$ primes of S opposite its respective vertices in $n+1$ associated $(n-2)$-spaces* (Art. 11). Dually: *The $n(n+1)$ hyperplanes, 2 through each $(n-2)$-space of S, touch a quadric, if, and only if, they distribute, in $2^{n(n+1)/2}$ ways, into $n+1$ n-ads, each n-ad consisting of n hyperplanes* *through the n $(n-2)$-spaces in a prime of S, determining $n+1$ points which join the vertices of S opposite its respective primes in $n+1$ associated lines* (BRIANCHON's theorem).

14. Related Polar and Self-polar Simplexes.

Following the argument of Art. 5, we may state that:

A pair of polar simplexes for a quadric Q, in an n-space, give rise to $(n+1)!$ pairs of self-polar simplexes for the same quadric. If a vertex of either lie on Q, $n!$ self-polar simplexes degenerate, and if an r-space or $(n-r-1)$-space of either touch Q, $(r+1)! \cdot (n-r)!$ such simplexes degenerate, 1-space being an edge, 2-space a plane and 3-space a solid.

Conversely: *We can derive $[(n+1)!]^2$ pairs of polar simplexes from 2 self-polar simplexes for the same quadric in an n-space* (cf. 4, pp. 516—518). Degeneration cases occur here also, if a vertex of one lies in a prime of the other, or, if an r-space of one meets an $(n-r-1)$-space of the other, that is when a vertex or an r-space of one is *conjugate* respectively to a vertex or an r-space of the other for the quadric.

Definitions: A q-space $\underline{q}$ and an r-space $\underline{r}$ $(q \geq r)$ in an n-space may be said to be *conjugate* for a quadric Q, if the polar q' of $\underline{q}$ for Q meets $\underline{r}$ in a point which, therefore, is the pole of the hyperplane where, then, must lie $\underline{q}$ and the polar r' of $\underline{r}$ for Q; and *p-conjugate*, if q' meets $\underline{r}$ in a p-space and consequently $\underline{q}$ meets r' in a $(p+q-r)$-space polar, therefore, to the $(n+r-p-q-1)$-space $q'\underline{r}$ for Q (cf. Definitions of Art. 5.2).

15. Special cases.

Evidently the cases of conjugacies, for a quadric Q, of various elements of a simplex in an n-space, increase with n and it is impossible to exhaust all of them unless n is specified. Hence we shall take up below only those cases which are of general interest.

Let $S = A_0, \ldots, A_n$, $S' = B_0, \ldots, B_n$ be a pair of polar simplexes for Q and the $n+1$ joins $A_i B_i$ of their corresponding vertices A_i, B_i be referred to just as joins for brevity.

15.1. Let $n = 2r-1$, and an r-space, say $A_0 \ldots A_r$, of S be conjugate, for Q, to an opposite $(r-1)$-space, say $A_r \ldots A_{2r-1}$, of S. Then the polar $(r-2)$-space $B_{r+1} \ldots B_{2r-1}$, of the r-space for Q, meets the $(r-1)$-space in a point which is, therefore, the pole, for Q, of the hyperplane where, then, the r-space and the polar $(r-1)$-space $B_0 \ldots B_{r-1}$ of the $(r-1)$-space for Q must lie. Thus the r joins $A_0 B_0, \ldots, A_{r-1} B_{r-1}$ lie in a hyperplane and so do the other r joins. Such is evidently also the case when the $(r-1)$-space is conjugate, for Q, to its opposite $(r-1)$-space, the difference being that in the former case the unique $(r-2)$-space meeting the r joins in the hyperplane [14] $A_r \dots A_{2r-1}\, B_r \dots B_{2r-1}$ passes through A_r. For in the $(2r-3)$-space $A_r \dots A_{2r-1}\, B_{r+1} \dots B_{2r-1}$, it is the definite $(r-2)$-space through A_r meeting the $r-1$ joins $A_{r+1}\, B_{r+1}, \dots, A_{2r-1}\, B_{2r-1}$.

If the said r-space and $(r-1)$-space be p-conjugate for Q, the first r joins lie in a $(2r-p-2)$-spaee and so do the others. Such is also the case when the $(r-1)$-space and its opposite $(r-1)$-space are p-conjugate for Q, the difference being that in the former case the 2 $(2r-p-2)$-spaces both pass through A_r.

Thus: *If an $(r-1)$-space of a simplex S in a $(2r-1)$-space be p-conjugate, for a quadric Q therein, to its opposite $(r-1)$-space or an opposite r-space, the r joins of the vertices of S in the $(r-1)$-space to the corresponding ones of the polar of S for Q lie in a $(2r-p-2)$-space and so do the other r joins, and the common vertex, say A, of the $(r-1)$-space and the r-space lies in both the 2 $(2r-p-2)$-spaces sach that, if $p=0$ or when they are simply conjugate, the unique $(r-2)$-space meeting the first r joins passes through A.*

15.2. Similarly: *If an r-space, of a simplex S in a $(2r)$-space, be p-conjugate, for a quadric Q therein, to its opposite $(r-1)$-space, the $r+1$ joins of the vertices of S in the r-space to the corresponding ones of the polar, say S', of S for Q lie in a $(2r-p-1)$-space and the other r joins in a $(2r-p-2)$-space; if an r-space of S be p-conjugate, for Q, to an opposite r-svace, the $r+1$ joins of the vertices of S in either r-space to the corresponding ones of S' lie in a $(2r-p)$-space such that the join through their common vertex, say A, lies in both the $(2r-p)$-spaces, and when they are just conjugate, the unique $(r-1)$-space meeting either $r+1$ joins passes through A.*

15.3. Evidently: *If an edge of a simplex S in an n-space be conjugate to its opposite $(n-2)$-space for a quadric Q therein, the joins of the two vertices comprising the edge to the corresponding two of the polar of S for Q meet in a point whose polar hyperplane for Q contains the other $n-1$ joins.*

15.4. Let 2 edges $A_0 A_1$, $A_1 A_2$ in a plane of S, be conjugate for Q to their respective opposite $(n-2)$-spaces $(n>4)$ whose polar lines $B_0 B_1$. $B_1 B_2$, for Q, then meet them respectively each in a point. Thus the plane $A_0 A_1 A_2$ meets $B_0 B_1 B_2$ in a line and is therefore 1-conjugate, and consequently every other line therein, $A_0 A_2$ in particular, simply coujugate, for Q,' to its opposite $(n-3)$-space. Hence the 2 joins $A_0 B_0$, $A_2 B_2$ both meet the third $A_1 B_1$, and the other $n-2$ joins lie in the polar $(n-2)$-space of the line of intersection $A_0 A_1 A_2 \cdot B_0 B_1 B_2$ for Q.

Again $A_0 A_1$ is evidently conjugate for Q to every edge of S opposite to it, in particular to $A_2 A_3$. $A_1 A_2$ is Similarly related to $A_0 A_3$. Hence, 2 pairs of opposite edges of the tetrahedron $A_0 A_1 A_2 A_3$ are conjugate for a 2-quadric section Q_2 of

Q by its solid, and therefore the third pair, *viz.* $A_0 A_2$, $A_1 A_3$ are so (Art. 7.1). That is, the polar line, of $A_1 A_3$ for Q_2, which is the meet of the solid with the polar $(n-2)$-space of $A_1 A_3$ for Q, meets $A_0 A_2$. In other words, $A_0 A_2$ is conjugate to $A_1 A_3$ for Q. Similarly it is conjugate for Q to the other $n-3$ edges $A_1 A_4, \ldots, A_1 A_n$ besides the $(n-3)$-space $A_3 \ldots A_n$. Hence $A_0 A_2$ is conjugate for Q to the $(n-2)$-space $A_1 A_3 \ldots A_n$ whose polar line $B_0 B_2$, for Q, therefore meets it. Consequently $A_0 B_0$, $A_2 B_2$ meet and therefore $A_1 B_1$ concurs with them. Thus:

If two edges in a plane of a simplex S in an n-space $(n>4)$ be respectively conjugate for a quadric Q therein to their opposite $(n-2)$-spaces, the plane is 1-conjugate to its opposite $(n-3)$-space and the third edge in it simply conjugate to its opposite $(n-2)$-space for Q. This is equivalent to saying that:

If two of the $n+1$ joins of the vertices of S to the corresponding ones of its polar for Q meet a third, the 3 joins concur and the other $n-2$ joins lie in an $(n-2)$-space (cf. Art. 8.1).

15.5. As an immediate consequence of what proceeds we have the following results:

If r $(2<r<n-1)$ consecutive edges, of a skew $(n+1)$-gon formed of the vertices of a simplex S in an n-space $(n>4)$, be conjugate to their respective opposite $(n-2)$-spaces for a quadric Q, their r-space is conjugate, p-conjagate $(p<r-1)$ or $(r-1)$-conjugate for Q to its opposite $(n-r-1)$-space according as $n=r+2$, $r+p+2$ or $>2r$; every q-space of S in this r-space is p-conjugate $(p<q-1)$ or $(q-1)$-conjugate, for Q, to its opposite $(n-q-1)$-space according as $n=q+p+2$ or $>2q$; every plane of S therein is 1-conjugate for Q to its opposite $(n-3)$-space; every edge of S therein is conjugate for Q to its opposite $(n-2)$-space; the joins of the $r+1$ vertices of S therein to the corresponding ones of the polar, say S', of S for Q concur, and the other $n-r$ joins lie in an $(n-r)$-space. If $n-1$ consecutive edges of the $(n+1)$-gon be respectively conjugate for Q to their opposite $(n-2)$-spaces, every r-space of S is p-conjagate $(p<r-1)$ or $(r-1)$-conjugate, for Q, to its opposite $(n-r-1)$-space according as $n=r+p+2$ or $>2r$; every plane of S is 1-conjugate for Q to its opposite $(n-3)$-space; every edge of S is conjugate for Q to its opposite $(n-2)$-space, and therefore all the $n+1$ joins of the corresponding vertices of S and S' concur or S and S' are in perspective.

15.6. The vertices of the polar of the r-simplex $(r>2)$ formed of $r+1$ vertices, say $A_0, \ldots, A_r$, of S for the $(r-1)$-quadric section Q_{r-1} of Q by their r-space lie at its intersections with the polar $(n-r)$-spaces, for Q of the $r+1$ $(r-1)$-spaces of the r-simplex, which all pass through its polar $(n-r-1)$-space $B_{r+1} \ldots B_n$ for Q. Now if $r-1$ consecutive edges of the $(r+1)$-gon formed of these vertices be conjugate for Q to their respective opposite $(r-2)$-spaces, their $r+1$ joins to the corresponding vertices of the polar of the r-simplex for Q_{r-1} concur, say at O, and therefore their $r+1$ joins to the corresponding ones of S' are all met by the $(n-r)$-space $OB_{r+1} \ldots B_n$.

Further if $n = 2r - 2$, the 2 r-simplexes $A_0 \dots A_r$, $B_0 \dots B_r$ are *projective* [14] from the $(r-2)$-space, which meets the $r+1$ joins of their corresponding vertices, such that the $r+1$ points of intersection of their corresponding $(r-1)$-spaces are collinear. Thus:

If r consecutive edges of the $(r+2)$-gon formed of $r+2$ vertices of a simplex S in an n-space be conjugate for a quadric Q therein to their respective opposite $(r-1)$-spaces, the $r+2$ joins of these vertices to the corresponding ones of the polar, say S', of S for Q are met by an $(n-r-1)$-space through the polar $(n-r-2)$-space for Q of their $(r+1)$-space, and therefore, if $n = 2r$, their $(r+1)$-simplex is projective to the corresponding one of S' from the $(r-1)$-space meeting the said $r+2$ joins.

16. Self-conjugate $(n+r)$-ads $(1 < r \leq n)$.

16.1. *If 2 simplexes in an n-space be in perspective, there is a unique quadric Q therein for which they are polar* (cf. Art. 8.3). The method adopted here (Art. 2.2) as well as that of BAKER ([2], Ex. 22, p. 149) to construct Q in a 4-space can be extended to n-space too.

16.2. If S and S' (Art. 15) be 2 simplexes in perspective, say from O, and therefore polar for Q, the $n+2$ points $O, A_0, \dots, A_n$ form a *self-conjugate $(n+2)$-ad* (cf. Art. 8.4) for Q such that the line joining any two of them is conjugate for Q to the hyperplane containing the other n points and consequently the polar for Q of the p-space containing any $p+1$ of them lies in the $(n-p)$-space containing the other $n-p+1$ points. Its construction betrays that: *Every one of the $n+2$ points of a self-conjugate $(n+2)$-ad for a quadric Q in an n-space is the centre of perspective of the simplex formed of the other $n+1$ points and its polar for Q.*

16.3. The possible $(r-2)$-spaces meeting the $n+1$ joins of the corresponding vertices of S and S' (Art. 15.6, 11) indicate the possibility of the formation of the other self-conjugate $(n+r)$-ads for Q (cf. Art. 7.3), $r = 2$ having been just considered.

Thanks are due to Prof. B. R. SETH for his generous, kind and constant encouragement in our pure pursuits.

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ÖZET

n-boyutlu S_n uzayında bir Q kuadriğine nazaran poler olan bir S, S' semplcks çiftinin mütekabil hiperdüzlem çiftlerine müşterek $n+1$ tane $(n-2)$-boyutlu uzayın, doğruların her biri her $(n-2)$-boyutlu uzayın bir tek noktasından geçmek suretiyle ∞^{n-2} doğru tarafından kesilecek tarzda *asosye* oldukları H. F. BAKER [2] tarafından *analitik* yoldan ispat edilmiştir. Daha sonra, S ve S' 'nün mütekabil tepelerinin $n+1$ tane birleşiminin *düal asosye* oluşları yine *analitik* olarak, S. BEATTY [18] tarafından ortaya atılan güç bir probleme cevaben, J. A. TODD ve H. S. M. COXETER [17] tarafından gösterilmiştir. Sözü geçen makaledeki bir notta, aynı iddianın endüksiyon metodu ile *sentetik* yoldan elde edilebileceği COXETER tarafından kaydedilmektedir. Bu görüş burada aşağıdaki teoremi ispat etmek üzere kullanılmıştır: «$n>3$ olmak şartıyle, n-boyutlu S_n uzayında umumî bir durumda bulunan $n+1$ tane $A_i\,B_i$ doğrularından n tanesi ile kesişen ∞^{n-3} tane $(n-2)$-boyutlu uzay aynı zamanda $(n+1)$-inci doğrunun A_n ve B_n noktalarından da geçerse; bu $n+1$ tane $A_i\,B_i$ $(i=0, 2, \ldots, n)$ doğrusu o şekilde asosyedir ki bunlarla kesişen ∞^{n-2} tane $(n-2)$-boyutlu uzay bu doğruların her birinin her noktasından geçer.»

Bu vesile ile, S_n uzayı içindeki bir S sempleksinin her bir kenarı üzerinde içinden iki tanesi bulunan $n(n+1)$ noktanın bir kuadrik üzerinde bulunmaları için gerek ve yeter şartın bunların n noktalık takımları olarak bir kuadriğe göre S e poler diğer bir sempleksin $n+1$ tane hiperdüzlemi üzerinde $2^{n(n+1)/2}$ türlü tevzi edilebilmeleri olduğuna işaret edelim. Bunun neticesi olarak, S_n uzayında bir kuadrik için PASCAL teoreminin CHASLES'a göre ispatıyle düalı olan BRIANCHON teoremini koniklerde elde edildikleri gibi bulunmuştur: bu surette S' nin her tepesinden 2^n tanesi geçmek üzere ve her biri $2^{n(n-1)/2}$ tane $n+1$ adet asosye doğrudan ibaret takımdan $2^{n(n-1)/2}$ 'sine ait olmak üzere $(n+1)2^n$ doğrudan ibaret bir sisteme varılır. Bu sistemin doğruları arasındaki münasebetler çok alâka çekici olmakla beraber, bunlar burada incelenmemiştir.

Yukarda zikredilen BAKER'in makalesindeki bazı yenilikleri aydınlatıcı birkaç hususî hal bu meyanda kaydedilmiş bulunmaktadır. Dejenere haltere tekabül eden Q kuadriğine nazaran kendisine eşlenik r-li nokta takımları da tetkik edilmiştir. Makale iki kısıma ayrılmış bulunmaktadır: birincisi sade 4 boyutlu uzayın diğeri ise daha yüksek uzayların tetkikine hasredilmiştir.