HAMILTONIAN ALGEBRAS

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Algebras of rank 2, with an involution which is an anti-isomorphism with respect to multiplication, were considered by HAMILTON (refs. 1, 2, 3). In this paper, such algebras are shown to have three different types of difference algebras with respect to the radical. An algebra is also obtained for any value of n.

1. A linear associative algebra H_n of dimension $(n+1)$ over a field F (not of characteristic 2), with basis $(i_0, i_1, i_2, \ldots, i_n)$, will be called «Hamiltonian» if:

(a) $i_0 = 1$, the multiplicative unit of F.

(b) Each element x of H_n,

where $$x = x_0 + x_1 i_1 + x_2 i_2 + \cdots + x_n i_n,$$

has a «conjugate»

$$\bar{x} = x_0 - x_1 i_1 - x_2 i_2 - \cdots - x_n i_n.$$

(c) $\overline{xy} = \bar{y}\,\bar{x}$, where x and y are any two elements of H_n.

2. Let $$i_r i_s = \sum_{t=0}^{n} a^t{}_{rs}\, i_t \qquad \text{where} \qquad a^t{}_{rs} \in F.$$

Since $i_0 = 1$, $i_0 i_s = i_s i_0 = i_s$ and $a^t{}_{0r} = \delta^t{}_r = a^t{}_{r0}$.

Since $\overline{i_r i_s} = i_s i_r$, $a^0{}_{sr} = a^0{}_{rs}$ and $a^j{}_{rs} = -a^j{}_{sr}$ if $j > 0$.

When $r = s$

$$\bar{i}^2{}_r = i^2{}_r = a^0{}_{rr} \quad \text{and} \quad a^j{}_{rr} = 0 \quad \text{if} \quad j > 0.$$

3. Let $\mathbf{x}' = (x_0, x_1, x_2, \ldots, x_n)$ and let $\mathbf{x}$ be the corresponding column vector. Let A_0 be the $(n \times n)$ symmetric matrix:

$$A_0 = \begin{pmatrix} a^0_{11} & a^0_{12} & \cdots\cdots & a^0_{1n} \\ a^0_{21} & a^0_{22} & \cdots\cdots & a^0_{2n} \\ \vdots & \vdots & & \vdots \\ a^0_{n1} & a^0_{n2} & \cdots\cdots & a^0_{nn} \end{pmatrix}.$$

Then

$$x + \bar{x} = 2x_0 \varepsilon F,$$

and

$$x\bar{x} = \bar{x}x = \mathbf{x}' A \mathbf{x} = N(x) \varepsilon F,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -A_0 \end{pmatrix}.$$

$N(x) = N(\bar{x})$ is called the «norm» of x.

Every element $x \varepsilon H_n$ satisfies a unique quadratic equation with coefficients in F:

$$x^2 - 2x_0 x + N(x) = 0.$$

4. Since $N(xy) = xy\,\overline{xy} = xy\,\bar{y}\,\bar{x} = x\,N(y)\,\bar{x} = N(x)\,N(y)$ it follows that

$$N(x^n) = [N(x)]^n$$

for any positive integer n.

If $N(x) \neq 0$, x possesses an inverse

$$x^{-1} = \frac{\bar{x}}{N(x)}.$$

5. Let x be a nil-potent element of H_n, $x^r = 0$.

Then

$$N(x^r) = [N(x)]^r = 0 \quad \text{and} \quad x^2 - 2x_0 x = 0.$$

Hence

$$x^r = (2x_0)\,x^{r-1} = (2x_0)^{r-1}\,x = 0$$

and

$$x^2 = N(x) = x + \bar{x} = 0.$$

Let $x^2 = c$ where $c \varepsilon F$.

Then

$$c - 2x_0 x + N(x) = 0$$

and

$$x = \frac{c + N(x)}{2x_0} \varepsilon F, \quad \text{if} \quad x_0 \neq 0.$$

If $x_0 = 0$, $x^2 = -N(x) \varepsilon F$ also.

Let N_n be the radical of H_n, and let x and y belong to N_n. Then

$$(xy)^2 = (yx)^2 = 0 \quad \text{and} \quad (x+y)^2 = 0.$$

Hence $$xy = -yx.$$

Also $$(xy+z)^2 = 0, \quad \text{hence} \quad xyz + zxy = 0.$$

But $$zxy = -xzy = xyz \therefore 2xyz = 0.$$

Hence $$N^3{}_n = 0.$$

6. If x is an idempotent element of H_n,

$$x^2 = x \neq 0.$$

Then $$N(x^2) = [N(x)]^2 = N(x).$$

The quadratic equation satisfied by x is unique and is $x^2 - x = 0$. Hence

$$N(x) = 0 \quad \text{and} \quad x + \bar{x} = 1 \quad \text{or} \quad x = 1.$$

Then
$$x = \frac{1}{2} + \sum_{r=1}^{n} x_r i_r$$

is idempotent if $N(x) = 0$.

7. Let H_n' be the difference algebra of H_n with respect to N_n.

$$H_n' = H_n / N_n$$

Let Z_n be the centre of H_n'. Then Z_n is the direct sum of fields, by DEDEKIND's theorem.

Let $e = e_1 + e_2 + \cdots + e_p$ be the multiplicative unit of H_n' where

$$e_i e_j = \delta_{ij} e_i.$$

Now $\bar{e} = e$ hence

$$\bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_p = e_1 + e_2 + \cdots + e_p.$$

Let $e_i + \bar{e}_i = k_i$ and $e_i \bar{e}_i = h_i$ where h_i and $k_i \varepsilon F$.

If $$e_i = \pm \bar{e}_i$$

$$e_i \bar{e}_i = \pm e_i^2 = \pm e_i = h_i \varepsilon F.$$

But e_i does not belong to F, hence $e_i \neq \pm \bar{e}_i$.

Then there must exist a one-to-one correspondence between the e_i's and $\bar{e}_i$'s.

Let $\bar{e}_i = e_j$ where $i \neq j$.

Since $e_i + \bar{e}_i = k_i \neq 0$, then

$$e_m(e_i + e_j) = k_i e_m, \quad \text{where} \quad m \neq i, \; m \neq j.$$

But $e_m e_i = 0 = e_m e_j$, hence $p \leqq 2$.

8. Let $p = 2$, then

$$e = e_1 + e_2, \; \bar{e}_1 = e_2, \; \bar{e}_2 = e_1$$

$$e_1 e_2 = 0$$

$$e_1{}^2 = e_1 \quad e_2{}^2 = e_2.$$

Then

$$\begin{aligned} Z_n &= Z_n e \\ &= Z_n e_1 + Z_n e_2 \\ &= F_1 + F_2 \end{aligned}$$

where F_1 and F_2 are fields and

$$H_n' = H_n' e = H_n' e_1 + H_n' e_2.$$

Let $y \varepsilon H_n' e_1$, then $\bar{y} \varepsilon H_n' e_2$.

$$y + \bar{y} = 2y_0 = 2y_0 e = 2y_0 e_1 + 2y_0 e_2$$

$$\therefore \; y = 2y_0 e_1, \quad \bar{y} = 2y_0 e_2$$

or

$$H_n' e_1 = F e_1, \quad H_n' e_2 = F e_2.$$

Hence H_n' is the direct sum of two fields, each equal to F.

9. Let $p = 1$ $e = e_1$.

H_n' is then a simple algebra with no two sided ideals, and by a theorem of WEDDERBURN, isomorphic to an algebra D_q of $(q \times q)$ matrices over a division ring D.

The centre of H_n', Z_n', is isomorphic to scalar matrices over F

$$Z_n' \cong k E_q \quad \text{where} \quad k \varepsilon F.$$

Let $y \varepsilon H_n'$, then

$$y = \sum_{i,j=1}^{q} y_{ij} e_{ij} \quad \text{where} \quad y_{ij} \varepsilon D$$

$$e_{ij} + \bar{e}_{ij} = k F_q, \; e_{ij} \bar{e}_{ij} = h E_q \quad \text{where} \quad h \text{ and } k \varepsilon F,$$

then

$$e_{ij} \bar{e}_{ij} = e_{ij}(k E_q - e_{ij}) = k e_{ij} = h E_q \quad \text{if} \quad i \neq j.$$

Hence $k=0$, and $e_{ij}+\bar{e}_{ij}=0$ when $i\neq j$.

Again $\bar{e}_{ii}=\overline{e_{ij}\,e_{ji}}=\bar{e}_{ji}\,\bar{e}_{ij}=e_{ji}\,e_{ij}=e_{jj}$.

Likewise $\bar{e}_{kk}=e_{jj}$. Hence $q\leqq 2$.

10. Let $e_{ii}+\bar{e}_{ii}=k\,E_q$, $\quad e_{ii}\,\bar{e}_{ii}=h\,E_q$.

Then $e_{ii}\,\bar{e}_{ii}=e_{ii}\,(k\,E_q-e_{ii})=(k-1)\,e_{ii}$, hence $k=1$ and

$$e_{ii}+\bar{e}_{ii}=E_q.$$

Let $q=1$. Then $H_n'\cong D$, a division ring containing F.

Let $q=2$. Then $H_n'\cong D_2$, an algebra of 2×2 matrices over a division ring D.

Let d be any element of D and y an element of H_n' such that

$$y=\begin{pmatrix} d & 0\\ 0 & 0\end{pmatrix} \quad\text{and}\quad \bar{y}=\begin{pmatrix} 0 & 0\\ 0 & d\end{pmatrix}.$$

Then $y+\bar{y}=d\,E_2$ and $y+\bar{y}$ is contained in F, hence $\underline{d}$ is contained in F, and so $D=F$.

Hence we have proved the following

Theorem. *The difference algebra* $H_n'=H_n/N_n$ *is isomorphic to one of the following*

a) *The direct sam of two fields, each equal to* F.

b) *A division ring containing* F *as a sub field.*

c) *The algebra of* (2×2) *matrices over* F.

11. Multiplication Tables for H_n.

$n=2$

$$i^2{}_1=a^0{}_{11} \qquad\qquad i^2{}_2=a^0{}_{22}$$

$$i_1\,i_2=a_{12}{}^0+a_{12}{}^1\,i_1+a_{12}{}^2\,i_2$$

$$a_{12}{}^0=+\,a_{21}{}^0 \qquad a_{12}{}^1=-\,a_{21}{}^1 \qquad a_{12}{}^2=-\,a_{21}{}^2.$$

The associativity conditions

$$(i_1\,i_2)\,i_2=i_1\,i_2{}^2 \qquad (i_2\,i_1)\,i_1=i_2\,i_1{}^2$$

give the following relations

$$a_{12}{}^0\,a_{12}{}^1+a_{12}{}^2\,a_{22}{}^2=0 \qquad\qquad a_{12}{}^0\,a_{12}{}^2+a_{12}{}^1\,a_{11}{}^0=0$$

$$(a_{12}{}^1)^2-a_{22}{}^0=0 \qquad\qquad a_{12}{}^1\,a_{12}{}^2+a_{12}{}^0=0$$

$$(a_{12}{}^2)^2-a_{11}{}^0=0$$

putting $a_{12}^1 = l$, $a_{12}^2 = m$, the multiplication table becomes

$$i_1^2 = m^2 \qquad i_2^2 = l^2$$

$$i_1 i_2 = -l\,m + l\,i_1 + m\,i_2\,.$$

Then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -m^2 & lm \\ 0 & lm & -l^2 \end{pmatrix}$$

which is singular.

Taking $l = -m = 1$, the multiplication table becomes

$$i_1^2 = i_2^2 = 1 \qquad i_1 i_2 = 1 + i_1 - i_2.$$

12. $n = 3$

The equations of associativity give the following sets of solutions

$$a_{31}^3 = a_{12}^2 \qquad a_{12}^1 = a_{23}^3 \qquad a_{31}^1 = a_{23}^2\,.$$

Putting these coefficients equal to u, v and w respectively and

$$a_{23}^1 = l, \qquad a_{31}^2 = m \qquad a_{12}^3 = n,$$

and using the other equations of associativity, the multiplication table becomes:

$$i_1^2 = u^2 - mn, \qquad i_2^2 = v^2 - nl, \qquad i_3^2 = w^2 - lm$$

$$i_2 i_3 = (lu - vw) + l\,i_1 + w\,i_2 + v\,i_3$$

$$i_3 i_1 = (mv - wu) + w\,i_1 + m\,i_2 + u\,i_3$$

$$i_1 i_2 = (nw - uv) + v\,i_1 + u\,i_2 + n\,i_3$$

If $l = m = n = 1$ and $u = v = w = 0$, the table becomes that of quaternions.

A second set of solutions is given by:

$$a_{23}^1 = a_{31}^2 = a_{12}^3 = 0$$

$$a_{31}^3 + a_{12}^2 = a_{23}^3 + a_{12}^1 = a_{31}^1 + a_{23}^2 = 0.$$

On putting

$$a_{21}^2 = r, \qquad a_{12}^1 = s, \qquad a_{23}^2 = t$$

the multiplication table becomes

$$i_1^2 = r^2 \qquad i_2^2 = s^2 \qquad i_3^2 = t^2$$

$$i_1 i_2 = r s + s i_1 - r i_2$$

$$i_2 i_3 = s t + s i_1 - t i_2$$

$$i_3 i_1 = r t - t i_1 + r i_3$$

Putting $r = s = t = 1$, the table becomes

$$i_1^2 = i_2^2 = i_3^2 = 1$$

$$i_1 i_2 = 1 + i_1 - i_2$$

$$i_2 i_3 = 1 + i_2 - i_3$$

$$i_3 i_1 = 1 + i_3 - i_1$$

13. $n = 4$

From the conditions of associativity, the following relatives are obtained:

(A)

$$\left.\begin{array}{l} a_{23}{}^1 = a_{34}{}^1 = a_{42}{}^1 = 0 \\ a_{34}{}^2 = a_{41}{}^2 = a_{13}{}^2 = 0 \\ a_{41}{}^3 = a_{12}{}^3 = a_{24}{}^3 = 0 \\ a_{23}{}^4 = a_{31}{}^4 = a_{12}{}^4 = 0 \end{array}\right\} \quad \text{(A1)}$$

or

$$\begin{vmatrix} a_{42}{}^2 & a_{42}{}^4 & a_{13}{}^1 \\ a_{41}{}^1 & a_{23}{}^3 & a_{23}{}^2 \\ a_{34}{}^3 & a_{12}{}^1 & a_{43}{}^4 \end{vmatrix} = \begin{vmatrix} a_{31}{}^1 & a_{24}{}^2 & a_{31}{}^3 \\ a_{34}{}^4 & a_{43}{}^3 & a_{12}{}^2 \\ a_{32}{}^2 & a_{11}{}^1 & a_{14}{}^4 \end{vmatrix} = 0,$$

(A2)

$$\begin{vmatrix} a_{41}{}^1 & a_{23}{}^3 & a_{14}{}^4 \\ a_{34}{}^3 & a_{13}{}^1 & a_{12}{}^2 \\ a_{42}{}^2 & a_{42}{}^4 & a_{31}{}^3 \end{vmatrix} = \begin{vmatrix} a_{43}{}^4 & a_{12}{}^2 & a_{12}{}^1 \\ a_{13}{}^1 & a_{31}{}^3 & a_{42}{}^4 \\ a_{23}{}^2 & a_{14}{}^4 & a_{23}{}^3 \end{vmatrix} = 0,$$

(B)

$$\left.\begin{array}{l} a_{12}{}^1 = a_{13}{}^1 = a_{14}{}^1 = 0 \\ a_{21}{}^2 = a_{23}{}^2 = a_{24}{}^2 = 0 \\ a_{31}{}^3 = a_{32}{}^3 = a_{34}{}^3 = 0 \\ a_{41}{}^4 = a_{42}{}^4 = a_{43}{}^4 = 0 \end{array}\right\} \quad \text{(B1)}$$

or

$$\begin{vmatrix} a_{23}{}^2 & a_{23}{}^3 & a_{23}{}^4 \\ a_{34}{}^2 & a_{34}{}^3 & a_{34}{}^4 \\ a_{42}{}^2 & a_{42}{}^3 & a_{42}{}^4 \end{vmatrix} = \begin{vmatrix} a_{34}{}^3 & a_{34}{}^4 & a_{34}{}^1 \\ a_{41}{}^3 & a_{41}{}^4 & a_{41}{}^1 \\ a_{13}{}^3 & a_{13}{}^4 & a_{13}{}^1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_{41}{}^4 & a_{41}{}^1 & a_{41}{}^2 \\ a_{12}{}^4 & a_{12}{}^1 & a_{12}{}^2 \\ a_{24}{}^4 & a_{24}{}^1 & a_{24}{}^2 \end{vmatrix} = \begin{vmatrix} a_{12}{}^1 & a_{12}{}^2 & a_{12}{}^3 \\ a_{23}{}^1 & a_{23}{}^2 & a_{23}{}^3 \\ a_{31}{}^1 & a_{31}{}^2 & a_{31}{}^3 \end{vmatrix} = 0. \qquad \text{(B2)}$$

Putting $a_{23}{}^1 = l$, $a_{31}{}^2 = m$ and $a_{12}{}^3 = n$, the table:

$$i_1{}^2 = -mn, \quad i_2{}^2 = -nl, \quad i_3{}^2 = -ml, \quad i_4{}^2 = 0$$

$$i_1 i_4 = i_2 i_4 = i_3 i_4 = 0$$

$$i_2 i_3 = l i_1, \quad i_3 i_1 = m i_2, \quad i_1 i_2 = n i_3$$

is obtained. If $l = m = n = 1$, the algebra is the quaternion algebra.

Putting $a_{23}{}^1 = l$, $a_{34}{}^1 = m$ and $a_{42}{}^1 = n$ the table:

$$i_1{}^2 = i_2{}^2 = i_3{}^2 = i_4{}^2 = 0$$

$$i_1 i_2 = i_1 i_3 = i_1 i_4 = 0$$

$$i_2 i_3 = l i_1, \quad i_3 i_4 = m i_1, \quad i_4 i_2 = n i_1$$

is obtained.

Putting $a_{34}{}^3 = r$, $a_{12}{}^1 = s$, $a_{23}{}^2 = t$, $a_{14}{}^1 = u$ the table:

$$i_1{}^2 = r^2, \quad i_2{}^2 = s^2, \quad i_3{}^2 = t^2, \quad i_4{}^2 = u^2,$$

$$i_1 i_2 = r s + s i_1 - r i_2$$

$$i_1 i_3 = r t + t i_1 - r i_3$$

$$i_1 i_4 = r u + u i_1 - r i_4$$

$$i_2 i_3 = s t + t i_2 - s i_3$$

$$i_2 i_4 = s u + u i_2 - s i_4$$

$$i_3 i_1 = t a + u i_3 - t i_4$$

is obtained.

If $r = a = s = t = 1$, the table becomes

$$i_p{}^2 = 1, \quad i_p i_q = 1 + i_p + i_q, \quad p, q = 1, 2, 3, 4.$$

14. *An Algebra of Arbitrary Dimension n.*

The algebra H_n, where:

$$i_p^2 = 1, \quad i_p i_q = 1 + i_p - i_q, \quad p, q = 1, 2, \ldots, n$$

is clearly associative, and

$$N(x) = x_0^2 - (x_1 + \cdots + x_n)^2$$

$$x^2 = 2x_0 x - N(x).$$

Hence if $x^2 = 0$, $x_0 = 0$ and $N(x) = 0$ and

$$x_0 = (x_1 + \cdots + x_n) = 0$$

Let R be the set of elements of H_n whose squares are zero, and let

$$x, y \varepsilon R.$$

Then

$$x = x_1 i_1 + \cdots + x_n i_n \quad \text{where} \quad x_1 + x_2 + \cdots + x_n = 0$$

$$y = y_1 i_1 + \cdots + y_n i_n \quad \text{where} \quad y_1 + y_2 + \cdots + y_n = 0.$$

Hence $\quad (x - y)^2 = 0, \quad$ for $\quad (x_1 - y_1) + \cdots + (x_n - y_n) = 0.$

Further $\quad x i_r = x = - i_r x, \quad r = 1, 2, \ldots, n,$

then

$$x z = k_1 x$$

$$z x = k_2 x$$

where

$$k_1 = z_0 + (z_1 + \cdots + z_n) \ \varepsilon F,$$

$$k_2 = z_0 - (z_1 + \cdots + z_n) \ \varepsilon F,$$

which proves that R is a two-sided ideal, and

$$R \subseteq N_n$$

Let $w \varepsilon N_n$, then $w^2 = 0$, hence $N_n \subseteq R$.

Hence $R = N_n$, i.e. the radical of H_n consists of all elements whose squares are zero.

Geometrical Representation.

Any element $x \varepsilon H_n$ con be represented by the point $\mathrm{r} = (x_0, x_1, x_2, \ldots, x_n)$ in $(n+1)$ dimensional euclidean space. The radical N_n is then represented by the set of points in the $(n-1)$ dimensional sub-space

$$x_0 = 0, \quad x_1 + x_2 + \cdots + x_n = 0.$$

The «perpendicular distance» to the space of N_n from a point

$$(0, x_1, x_2, \ldots, x_n) \quad \text{is} \quad \frac{1}{\sqrt{n}}(x_1 + x_2 + \cdots + x_n) = \frac{\sigma_n}{\sqrt{n}}$$

and direction cosines are

$$\frac{1}{\sqrt{n}}(0, 1, 1, \ldots, 1).$$

Resolving x into components in, and perpendicular to, the radical space

$$x = x_0 + \frac{\sigma_n}{n}(i_1 + i_2 + \cdots i_n)$$

$$+ \frac{1}{n}\left\{(n\,a_1 - \sigma_n)\,i_1 + (n\,a_2 - \sigma_n)\,i_2 + \cdots + (n\,a_n - \sigma_n)\,i_n\right\}$$

hence
$$x/N_n = x_0 + \frac{\sigma_n}{n}(i_1 + i_2 + \cdots + i_n)$$

Put
$$j_1 = \frac{1}{n}(i_1 + i_2 + \cdots + i_n) \quad \text{then} \quad j_1^2 = 1$$

and
$$x/N_n = x_0 + \sigma_n\,j_1 \cong \begin{pmatrix} x_0 & \sigma_n \\ \sigma_n & x_0 \end{pmatrix}.$$

By a suitable nonsingular transformation, the algebra

$$j_1^2 = 1 \qquad j_2^2 = j_3^2 = \cdots = j_n^2 = 0$$

$$j_1\,j_r = -\,j_r, \qquad j_r\,j_1 = j_r, \qquad (r = 2, 3, \ldots, n)$$

$$j_r\,j_s = 0 \qquad (s = 2, 3, \ldots, n)$$ is obtained.

For example in the case $n = 3$, we can take

$$j_1 = \frac{1}{3}(i_1 + i_2 + i_3)$$

$$j_2 = i_1 - i_2$$

$$j_3 = i_1 + i_2 - 2i_3.$$

The vector $(1, 1, 1)$ is normal to the plane $x_1 + x_2 + x_3 = 0$ and $(1, -1, 0)$ and $(1, 1, -2)$ are two orthogonal vectors in the plane. Similarly, in the general case, j_1 is taken along the n-vector $(1, 1, 1, \ldots, 1)$ which is normal to the $(n-1)$ dimensional space

$$x_1 + x_2 + \cdots + x_n = 0, \quad \text{and} \quad j_2, j_3, \cdots j_n$$

are chosen along $(n-1)$ mutually orthogonal vectors in the space.

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ÖZET

Çarpıma nazaran bir anti-izomorfism olan involüsyonları haiz ikinci dereceden cebirler HAMILTON tarafından tetkik edilmiştir. [1, 2, 3]. Bu makalede bu çeşit cebirlerin radikaln nazaran üç farklı tipten bölüm cebirleri olduğu gösterilmektedir. Üstelik n'nin herhangi bir değeri için de $n+1$ boyutlu bir cebir elde edilmektedir.