A CONFIGURATION OF 600 LINES IN [4]

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Abstract*. The 600 generating lines or g-lines of the configuration lie by twos in 900 g-planes and by nines in 200 h-planes; by sixes on 1300 2-quadries; by 18s in 100 h-solids determined by 100 pairs of h-planes meeting in 100 h-lines; by 72s in 25 f-solids; by 24s on 100 3-quadrics, the 24 lines on each quadric lying also on the quartic primals of a net determined by any two of them and therefore on the octic surfaces of its intersection with them too.

Every g-line meets 27 others, 3 in 3 g-, 6 in 6 h- and 6 in 6 h'-points and the other 12 concurring by threes in 4 G-points on it. The 900 g-, 1800 h- and 1800 h'-points lie by sixes on 6600 conics which lie by 156s on the 100 3-quadrics such that the 6 planes determined by the 6 pairs of g-lines through the 6 points determining one such conic concur in a point and form 2 triads of generating planes of the 2 opposite systems of a quadric point-cone. There are 3200 such hypercones.

The 24 g-lines of a 3-quadric form 18 skew quadrilaterals with vertices at the 36 g- and 36 h- or h'-points of their intersection which distribute thus uniquely into 18 tetrads, 3 g-points coupling with 3 h-or h'-points on each g-line as mates in an involution.

The 9 g-lines in an h-plane form 2 triads of mutually perspective triangles, each triad having the same centre of perspectivity at an S- or S'-point, and thus form a familiar figure $(20_3, 15_4)$ with the 6 axes of perspectivity [2]. The non-corresponding sides of each pair of perspective triangles meet in 6 G-points forming a PASCAL hexagram inscribed in a G-conic.

The 000 G-points lie by 24s on 225 2-quadrics, reducing to 25 only under specialised circumstances, when they become the 120 GAUSS points [4] referred to a pair of simplexes. The 2 triads of mutually perspective triangles in an h-plane then form the dual of mutually *a(joint veronesian systems* [3], and the S-, S'-points become a pair of STEINER *points* [2] for the hexagram of the 6 GAUSS points therein.

There arise a good number of interesting subconfigurations from the various elements of the configuration and their mutual relations.

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1. INTRODUCTION

1.1. Preliminaries. Let x, y, z, u, v be the coordinates of a point with reference to a simplex S in a 4-dimensional projective space [4] and x', y', z', u', v' referred to another, say S'.

The 600 lines common to the 600 triads of solids given by the equations $p=i, q=j, r=k$ and lying by sixes in the 100 solids given by

$$p+q+r=i+j+k,$$

denoted as the *silm-solids* $(p, q, r, s, t = x, y, z, u, v;\quad p \neq q \neq r \neq s \neq t;$ $i, j, k, l, m = x', y', z', u', v';\ i \neq j \neq k \neq l \neq m)$ for the convenience of enumeration, generate the configuration considered here.

1.2. Definitions. The generating lines of the configuration are referred to as *g-lines*; *silm*-solids as *g-solids*; the fundamental solids $p=i$ as *f-solids*; the plane common to a pair of f-solids $p=i,\ q=j$ as *h-plane*. A pair of h-planes $p=i,\ q=j;\ p=j,\ q=i$ determine an *h-line* $p=i=q=j$ common to them and an *h-solid* $p+q=i+j$ which meets an f-solid $r=k$ in a *g-plane*, a *c-plane* $r=k=s$ in an *e-line* and a *c'-plane* $i=p=j$ in an *e'-line*.

The g-lines intersect in the following 4 types of points: A *g-point* is an intersection of an h-line $p=i=q=j$ with an f-solid $r=k$; an *h-point* is that of an h-plane $p=i,\ q=j$ with a c-plane $r=k=s$ which meets an f-solid $p=i$ in an *f-line*, and an *h'-point* with a *c'-plane* $k=r=l$ which meets $p=i$ in an *f-line*; a *G-point* is one common to 4 f-solids $p=i,\ q=j,\ r=k,\ s=l$. Obviously 24 G-points lie in a *tm-solid* $p+q+r+s=i+j+k+l$, called a *G-solid*, and 6 in a *G-plane* common to an f-solid $p=i$.

The e- (e'-) lines concur by threes in E- (E'-) *points* of intersection of h-solids $p+q=i+j$ with c- (c'-) *lines* $q=r=s=k\ (r=j=k=l)$; h-lines in H- (H'-) *points* $p=i=q=j=r\ (i=p=j=q=k)$; by 2 e- (e'-) lines and 1 h-line in D- (D'-) *points* as intersections of h-solids $p+q=i+j$ with h-lines

$$j=r=k=s\ (q=k=r=l).$$

The c- (c'-) planes lie by fives in p- (p'-) *solids* $p=q\ (i=j)$ which meet f-solids $r=k$ in p- (p'-) *planes* and the h-planes $r=k,\ s=l$ in p- (p'-) *lines* which again lie by threes in h-planes concurrent at S- (S'-) *points* of their intersection with S- (S'-) *planes* $p=q=t\ (i=j=m)$.

A 2-quadric will be referred as a *quadric* and 3-quadric as *quadratic*.

1.3. Incidences. We may now enumerate the various elements of the configuration, observe and record their incidences in a table given below. The figures below the diagonal show the number of subspaces in each space, those above, the number of spaces through each subspace.

2. QUADRICS

2.1. g-Solids. The 6 g-lines in an $stlm$-solid form the following 2 triads:

(i) $p=i,\ q=j,\ r=k;\ p=j,\ q=k,\ r=i;\ p=k,\ q=i,\ r=j$

(ii) $p=i,\ q=k,\ r=j;\ p=j,\ q=i,\ r=k;\ p=k,\ q=j,\ r=i.$

Every line of one triad meets that of the other and no two lines of one meet each other. Hence they generate a quadric denoted as *stlm-quadric*. Thus we have the following

Theorem 1. *The* 6 *g-lines of a g-solid form* 2 *triads of generators of the* 2 *opposite systems of a quadric and meet in* 9 *g-points. There are thus* 100 *such qaadrics, one in each g-solid.*

2.2. Quadratics. Every 2 of the 4 $stlm$-quadrics for 3 given values of t, l, m or s, t, m meet in a conic, in the plane common to their solids, determined by the 6 h- or h'-points of intersection of the 6 g-lines of one with the corresponding ones of the other. Hence they lie on a quadratic denoted as *tlm*- or *stm-quadratic* accordingly. Each such quadratic then contains 24 g-lines meeting in 36 g- and 36 h- or h'-points.

2.3. f-Solids. An f-solid $p=i$ meets a tlm-quadratic in a quadric denoted as *pi/tlm-quadric.* It contains the following 6 g-lines:

(i) $i=p,\ j=q,\ k=r$

(ii) $i=p,\ j=r,\ k=s$

(iii) $i=p,\ j=s,\ k=q$

(iv) $i=p,\ j=r,\ k=q$

(v) $i=p,\ j=s,\ k=r$

(vi) $i=p,\ j=q,\ k=s.$

The first 3 lines belong to one system of its generators and the last 3 to the other intersecting the former in 3 g- and 6 h-points.

Similarly behave the 6 g-lines of a pi/stm-quadric intersecting in 3 g- and 6 h'-points.

But the f-solid $p=i$ meets the 24 g-lines of a pij- or pqi-quadratic in 24 G-points, those of a $pjk\,(qri)$-quadratic in 24 h-$(h'$-$)$ points, and those of a $qij\,(pqj)$-quadratic in 6 G- and 18 h'-$(h$-$)$ points.

2.4. p-Solids. Further we observe that the p-$(p'$-$)$ solid $s=t\,(l=m)$ meets the 24 g-lines of a $ptm\,(pmi)$- and those of the $psm\,(pli)$-quadratic in the same 24 h-$(h'$-$)$ points which therefore lie on their common quadric section by this solid and by sixes on 4 conics in its 4 c-$(c'$-$)$planes other than $s=m=t\,(l=p=m)$. It is also seen to meet the 18 g-lines of a $tlm\,(stl)$- and those of the $slm\,(stm)$-

quadratic other than their common 6 g-lines on their common $stlm$-quadric in the same 18 h- (h'-) points which therefore lie on their common quadric section by this solid and by sixes on 3 conics in its 3 c- (c'-) planes

$$s=i=t,\ s=j=t,\ s=k=t\ (l=p=m,\ l=q=m,\ l=r=m).$$

It is also observed to meet the 24 g-lines of a pqi (pij)-quadratic, in pairs, in 12 g-points (see § 3.1) which therefore lie on its quadric section and by sixes on 4 conics in its 4 c- (c'-) planes other than $s=i=t\ (l=p=m)$, and of a pij (pqi)-quadratic in 6 g- and 6 h- (h'-) points which therefore lie on its quadric section and by 4 g- and 2 h- (h'-) points on 3 conics in its 3 c- (c'-) planes other than

$$s=i=t,\ s=j=t\ (l=p=m,\ l=q=m).$$

2.5. G-Solids. A tm-solid (§ 1.2) meets the 24 g-lines of every one of the tmi-, tmj-, tmk-, tml-, ptm-, qtm-, rtm- and stm-quadratics in the same 24 G-points which therefore lie on their common quadric section, by this solid, denoted as a *tm-qaadric.*

2.6. Definitions. An $stlm$-, a pi/tlm- and pi/stm-quadric are called respectively *g-, h- and h'-quadric* and tlm- and stm-quadraties as *h- and h'-quadratics.* A quadric through 24 G- or h- (h'-) points is referred to as *G- or h- (h'-) quadric* accordingly, one through 6 G- and 18 h- (h'-) points as *T- (T'-) quadric,* that through 18 h- (h'-) points only as *P- (P'-) quadric,* one through 12 g-points as *C-quadric* and that through 6 g- and 6 h- (h'-) points as *D- (D'-) quadric.* A conic through 6 G-, g-, h- or h'-points is called *G-, g-, h-, h'-conic* accordingly, one through 4 g- and 2 h- (h'-) points as *d- (d'-)conic* and that through 2 G- and 4 h- (h'-) points as *T- (T'-) conic.* We thus have the following theorems:

Theorem 2. *The 100 g-quadrics lie by fours on 50 h- and 50 h'-quadratics, each containing 24 g-lines meeting in 36 g- and 36 h- or h'-points. The 4 g-quadrtcs on each h- (h'-) quadratic meet by twos in 6 h- (h'-) contes giving rise to 300 such conies of either type which then lie by sixes on each g-quadric.*

Theorem 3. *The 72 g-lines tn an f-solid lie by sixes on 24 h- and 24 h'-qaadrics, the 6 g-lines on each h- (h'-) qaadrtc form 2 triads of its generators of opposite systems intersecting in 3 g- and 6 h- (h'-) points.*

Theorem 4. *There are:* (i) *150 H- (H'-) qaadrics, 6 in each f-solid and 3 on each h- (h'-) quadratic, and 150 others, 15 in each p- (p'-) solid and 6 on each h- (h'-) quadratic;* (ii) *400 T- (T'-) quadrics, 16 in each f-solid and 8 on each h'- (h-) quadratic;* (iii) *100 P- (P'-) quadrics, 10 in each p- (p'-) solid and 4 on each h- (h'-) quadratic such that every 2 h- (h'-) quadratics through a g-quadric meet again in a P- (P'-) qaadric;* (iv) *600 h- (h'-) conies, 4 on each H- (H'-) quadric in a p- (p'-)*

solid and 12 *in each* c- (c'-) *plane, and* 300 *others,* 3 *on each* P- (P'-) *qaadric and* 6 *in each* c- (c'-) *plane.*

Theorem 5. *There are*: (I) 300 *C-quadrics,* 15 *in each* p- *and* p'-*solid and* 3 *on each* h- *and* h'-*qaadratic, as sections of* h- (h'-) *quadratics by* p'- (p-) *solids*; (II) 300 D- (D'-) *quadrics,* 30 *in each* p- (p'-) *solid and* 6 *on each* h- (h'-) *quadratic.*

Theorem 6. *The* 600 *G-points lie by* 24*s on* 25 *G-quadrics, one in each G-solid and each common to* 4 h- *and* 4 h'-*quadratics sach that an* h- *and an* h'-*quadratic meet again in a* g-*quadric and two* h- (h'-) *quadratics in an* H'- (H-) *qaadric in a* p'- (p-) *solid. There are* 200 *other G-quadrics,* 8 *in each* f-*solid and* 2 *on each* h- *or* h'-*quadratic.*

3. CONICS

3.1. g-**Conics.** A c'-plane $i=p=j$ is observed to meet a tlm-quadratic in a g-conie determined by the the 6 g-poiats of intersection of its 6 g-lines in the f-solid $i=p$ with those in $p=j$ as follows:

(I) $i=p,\ j=q,\ k=r,\qquad i=q,\ j=p,\ k=r$

(II) $i=p,\ j=r,\ k=s,\qquad i=r,\ j=p,\ k=s$

(III) $i=p,\ j=s,\ k=q,\qquad i=s,\ j=p,\ k=q$

(IV) $i=p,\ j=q,\ k=s,\qquad i=q,\ j=p,\ k=s$

(V) $i=p,\ j=r,\ k=q,\qquad i=r,\ j=p,\ k=q$

(VI) $i=p,\ j=s,\ k=r,\qquad i=s,\ j=p,\ k=r.$

The 6 g-planes determined by the 6 pairs of g-lines are as follows:

(I) $p+q=i+j,\ r=k$ (IV) $p+q=i+j,\ s=k$

(II) $p+r=i+j,\ s=k$ (V) $p+r=i+j,\ q=k$

(III) $p+s=i+j,\ q=k$ (VI) $p+s=i+j,\ r=k.$

They obviously concur at the E-point of intersection of an h-solid

$$i+j=p+q\ (p+r\ \text{or}\ p+s)$$

with the e-line $k=q=r=s$ (§ 1.2).

Similarly the 6 g-planes determined by the 6 pairs of g-lines through the 6 g-points of the g-conic section of an stm-quadratic by a c-plane $p=i=q$ concur at the E'-point given by $r=j=k=l,\ p+q=i+j\ (i+k\ \text{or}\ i+l)$. Thus follows

Theorem 7. *The 900 g-points lie by sixes on 1200 g-conics, 12 in each c- (c′-) plane as 12 sections of 12 h′- (h-) quadratics, 2 on each h- and each h′-quadric through its 12 g-points other than the 3 g-points of intersection of its 3 pairs of g-lines, 12 on each h- and h′-quadratic, 96 in each f- and 60 in each p- (p′-) solid, 4 on each C-quadric and 8 through each g-point* (see Table 1 and § 2.4). *The 6 g-planes determined by the 6 pairs of g-lines through the 6 g-points of a g-conic concur at an E- or E′-point according as it lies on an h- or h′-quadratic.*

3.2. *d- (d′-)* **Conics.** Repeating the argument of the preceding proposition, we may further observe that a *c- (c′-)* plane $p = i = q$ $(t = p = f)$ cuts a *tlm (stm)*-quadratic in a *d- (d′-)* conic (§ 2. 6) through the 4 *g-* and 2 *h- (h′-)* points of intersection of its 6 *g*-lines in the *f*-solid $p = i$ with those in $q = i$ $(p = j)$, and have

Theorem 8. *The 36 g- and 36 h- (h′-) points in a c- (c′-) plane lie by 4 g- and 2 h- (h′-) points on 18 d- (d′-) conics as 18 sections of 18 h- (h′-) quadratics. The 4 g- and 2 h-planes determined by the 6 pairs of g-lines through the 4 g- and 2 h- (h′-) points of a d- (d′-) conic concur at a D- (D′-) point* (§ 1.2). *There are thus 900 such conics of either type, 72 in each f-solid, 90 in each p- (p′-) solid, 3 on each h- (h′-) quadric, 18 on each h- (h′-) quadratic and 4 through each g- and one through each h- (h′-) point and 3 on each D- (D′-) quadric* (see Table 1 and § 2.4).

3.3. *h- (h′-)* **Conics.** a. Besides the 1200 *h- (h′-)* conics enumerated above (Ths. 2, 4), we have 600 more of either type. For example, the *f*-solid $p = i$ meets the 6 *g*-lines of a *pqjk- (qrij-)* solid (§ 2.1) in 6 *h- (h′-)* points of an *h- (h′-)* conic section of the *g*-quadric in this *g*-solid (§§ 1.2, 2.6) by their common plane. It therefore lies on the *h- (h′-)* quadric section of the *qjk (qrj)*-quadratic, on the *T- (T′-)* quadric section of the *pqj (qij)-* or *pqk (rij)*-quadratic as well as on the *H- (H′-)* quadric section of the *pjk (qri)*-quadratic by $p = i$ (§ 2.3).

b. The *h- (h′-)* conic (Th. 2) of a *ptli*-quadric common with a *psli- (ptmi-)* quadric is seen to lie in the *p- (p′-)* solid $s = t$ $(l = m)$ on the *P- (P′-)* quadric (§ 2.4) common to the *tli (ptl)-* and *sli (ptm)*-quadratics.

c. The *h- (h′-)* conic (Th. 4) in the *c- (c′-)* plane $s = i = t$ $(l = q = m)$ and on the *H- (H′-)* quadric (§ 2.4), common to *ptm (pmi)-* and *psm (pli)* quadratics obviously lies on the *h′- (h-)* quadric section of the *ptm (pmi)* quadratic by the *f*-solid $s = i$ $(l = q)$ and on that of the *psm (pli)* quadratic by $i = t$ $(q = m)$ or on the *T- (T′-)* quadric section of the later (former) quadratic by the former (later) solid (§ 2.3).

d. The (*h- (h′-)* conic (Th. 4) in the *c- (c′-)* plane $s = i = t$ $(l = q = m)$ aud on the *P- (P′-)* quadric (§ 2.4), common to the *tlm (stl)-* and *slm (stm)*-quadratics, obviously lies on the *h- (h′-)* quadric section of the *tlm (stl)*-quadratic by the *f*-solid $s = i$ $(q = m)$ and on that of the *slm (stm)*-quadratic by $i = t$ $(l = q)$ or on the

H- (H'-) quadric section of the former (later) quadratic by the later (former) solid (§ 2.3).

e. Following the argument of § 3.1, we can now prove that the 6 h-planes determined by the 6 pairs of g-lines through the 6 h- (h'-) points of an h- (h'-) conic also concur at an H- or H'-point (§ 1.2). Here we notice that 6 such concurrent planes meet 9 h- and 9 h'-conics, each meeting each h-plane in an h- or h'-point through which pass a pair of g-lines determining the h-plane. For example, consider those which concur at an H-point $p=i=q=j=r$. They are as follows:

(I) $p=i,\ q=j$ (II) $q=i,\ r=j$ (III) $r=i,\ p=j$

(IV) $p=j,\ q=i$ (V) $q=j,\ r=i$ (VI) $r=j,\ p=i$.

The 9 h- and h'-conics meeting them are given by the following matrix scheme of 3 g- and 6 h-quadrics such that the 3 quadrics in a row (column) meet one another

$$\begin{pmatrix} stlm & stmk & stkl \\ skl/tlm & sl/tmk & sm/tkl \\ tk/slm & tl/smk & tm/skl \end{pmatrix}$$

iu 3 h'- (h-) conics.

f. But there are 2 more varieties of h- as well h'-conics too which are quite distinct from the four just discussed. For example, the section of a pkl (rsi)-quadratic by an h-plane $p=i,\ q=j$ is one such h- (h'-) conic (§ 2.3), and by a p'- (p-) plane $q=j,\ m=k$ ($s=t$) is another such h'- (h-) conic. The 6 h-planes through the 6 pairs of g-lines through the 6 h- or h'-points of one such conic reduce to 3 only iu the f-solid $q=j$ such that 2 pairs of g-lines lie in each plane. In the former case, they are the 3 h-planes of $q=j$ common with the 3 f-solids

$$m=r,\ s,\ t\ (t=k,\ l,\ m)$$

and meet $p=i$ in 3 g-lines, each containing 2 of the 6 h- (h'-) points. That is, the 2 pairs of g-lines in each h-plane have a g-iine in common or reduce to 3 g-lines only. In the later case, they are the 3 h-planes of $q=j$ common with

$$i=r,\ s,\ t\ (p=k,\ l,\ m)$$

and meet the p'- (p-) solid $m=k$ ($s=t$) in 3 p'- (p-) lines, each containing 2 of the 6 h'- (h-) points. Thus we obtain the following

Theorem 9. *There are*: (I) 600 *h- (h'-) conics, one on each h- (h'-) quadric,* 24 *in each f-solid,* 6 *on each g-quadric,* 3 *on each T- (T'-) qaadric,* 4 *on H- (H'-) qaadric in an f-solid and* 24 *on each h- and h'-quadratic*; (II) 300 *others* (Th. 2), 3 *on each P- (P'-) quadric and* 30 *in each p- (p'-) solid*; (III) 600 *more* (Th. 4), 2 *on each h'- (h-) and* 3 *on each T- (T'-) qaadric*, 48 *in each f-solid*, 60 *in each p- (p'-) solid and* 24 *on each h'- (h-) quadratic*; (IV) *on* 300 *others* (Th. 4), 4 *on each H- (H'-) quadric in an f-solid, one on each h- (h'-) qaadric*, 24 *in each f-solid*, 30 *in each p- (p'-) solid and* 12 *on each h- (h'-) quadratic*; (V) 1200 *more, each cutting* 3 *g-lines in* 3 *pairs of h- (h'-) points*, 6 *in each h-plane through its* 36 *h- (h'-) points other than the* 9 *of intersection of its* 9 *g-lines*, 8 *on each H- (H'-) quadric in an f-solid*, 2 *on each h- (h'-) quadric*, 96 *in each f- and* 12 *in each h-solid and* 24 *on each h- (h'-) quadratic*; (VI) 1200 *others, each cutting* 3 *p- (p'-) lines in* 3 *pairs of h- (h'-) points*, 8 *in each p- (p'-) plane*, 120 *in each p- (p'-) solid*, 8 *on each H- (H'-) quadric in a p- (p'-) solid*, 4 *on each h'- (h-) quadric*, 48 *in each f-solid and* 48 *on each h'- (h-) quadratic. The* 6 *h-planes determined by the* 6 *pairs of g-lines through the* 6 *h- (h'-) points of an h- (h'-) conic, not in an h- or p- (p'-) plane, concur at an H- or H'-point and are the same for* 9 *h- and* 9 *h'-conics. The* 6 *h-planes determined similarly for one such conic in an h- or p- (p'-) plane reduce to* 3 *only lying in an f-solid such that each plane contains* 2 *pairs of g-lines which reduce to* 3 *g-lines only for the conic in an h-plane.*

3.4. *G*-Conics. The *G*-solid (§ 1.2) of a *tm*-quadric (§ 2.5), an *f*-solid $p = i$ and the *ptmi*-solid obviously meet in a *G*-plane (§ 1.2). Hence a *tm*-, a *ptmi*- the *pi/tmi*- and *pi/ptm*-quadrics (§ 2.3) have a *G*-conic common through the 6 *G*-points in the *G*-plane common to $p = i$ and *ptmi*-solid. Or, a *tmi*- and a *ptm*-quadratic meet in the *tm*- and *pimi*-quadrics which then meet in a *G*-conic. Obviously this *G*-plane cuts the *pti*-, *qtm*-, *rtm*-, *stm*-, *pmi*-, *tmj*-, *tmk*- and *tml*-quadratics too in the same *G*-conic.

Again an *h*-plane $p = i$, $q = j$ is observed to meet the *pri*- or *qrj*- *psi*- or *qsj*-, *pti*- or *qtj*-, *pki*- or *qjk*-, *pli*- or *qlj*- and *pmi*- or *qmj*-quadratics in 6 *G*-conics. Thus we find the following

Theorem 10. *There are*: (I) 400 *G-conics, one on each T- and each R'-quadric*, 2 *on each h- and h'-quadric*, 4 *on each G-quadric in an f-solid*, 4 *on each g-quadric*, 16 *in each f-solid*, 16 *on each G-qaadric in a G-solid and* 40 *on each h- and each h'-quadratic such that an h- and an h'-quadratic through a g-qaadric meet again in a G-quadric which meets the g-quadric in a G-conic*; (II) 1200 *others*, 6 *in each h-plane*, 12 *in each h- and* 96 *in each f-solid*, 2 *on each h- and each h'-quadric*, 12 *on each G-quadric in an f-solid and* 24 *on each h- and h'-quadratic.*

3.5. *T*- (*T'*-) Conics. The section of an *rik* (*prk*)-quadratic by an *h*-plane $p = i$, $q = j$ is seen to be a *T*- (*T'*-) conic through 2 *G*- and 4 *h'*- (*h*-) points (§§ 1.2, 2.3, 2.6). Now follows

Theorem 11. *There are* 3600 *T- (T'-) conics*, 18 *in each h-plane*, 288 *in each f- and* 36 *in each h-solid*, 6 *on each h'- (h-) and* 9 *on each T- (T'-) quadric and* 72 *on each h'- (h-) quadratic.*

4. HYPERCONES

4.1. Definitions. The 6 g-planes concurrent at an E- (E'-) point are observed to form 2 triads of generating planes of the 2 opposite systems of a *quadric point-cone* [1], called an *E- (E'-) cone*. For example, the first 3 g-planes of § 3.1 form one triad and the last 3 the other such that every plane of one meets that of the other in a line, both lying in a g-, an f-, an f- or h-solid which forms a *tangent solid* of the *hypercone* with their common line as its *line of contact*, and every two of one triad meet only in the *vertex* of this E-cone at the E-point.

Similarly, we may define *H- (H'-)* and *D- (D'-) cone* generated respectively by the 6 h-planes through an H- (H'-) point and the 4 g- and 2 h-planes through a D- (D'-) point. The H-cone with vertex at an H-point $p=i=q=j=r$ and the H'-cone with vertex at the H'-point $k=s=l=t=m$ are said to form a *complementary pair of cones*. We thus have the following theorems:

Theorem **12**. *There are* 600 *E- (E'-) cones, each corresponding to a g-conic* (Th. 7). *Each such cone contains*: 6 *g-planes lying in pairs in* 3 *f-,* 3 *g- and* 3 *h-solids*; 12 *g-lines,* 2 *in each generating g-plane*; 48 *G-,* 24 *g-,* 60 *h- (h'-) and* 72 *h'- (h-) points*; *a pair of h- (h'-) quadrics as its intersections with an h- (h'-) quadratic, say Q, or as its sections by the pair of f-solids determining the c'- (c-) plane of its corresponding g-conic common to them* (§ 3.1); *a pair of other g- and a pair of h- (h'-) conics lying respectively on a third h- (h'-) and a g-quadric of Q* (Th. 7, § 3.3a); 4 *pairs of G-conics as its intersections with* 4 *G-quadrics of Q,* 2 *in* 2*f- and* 2 *in* 2 *G-solids* (Th. 10); 3 *pairs of h- (h'-) conics common with* 3*H- (H'-) quadrics of Q in* 3 *f-solids and one pair with a P- (P'-) quadric of Q sach that the conics of this pair, lying in* 2 *c- (c'-) planes of its p- (p'-) solid, are not new, bat belong to* 2 *of the* 3 *pairs, onc to each, while the other* 4 *conics lie in* 4 *h-planes* (§§ 2.3, 2.4); 6 *pairs of h'- (h-) conics common with* 6*H'- (H-) quadrics of Q in* 6*p'- (p-) solids, and* 2 *pairs with* 2*T'- (T-) quadrics of Q sach that the conics of either pair, lying in* 2*c'- (c-) planes of an f-solid, are not new, but belong to* 2 *of the* 6 *pairs, one to each, while the other* 8 *conics lie in* 8 *p'- (p-) planes* (§§ 2.3, 2.4); 6 *pairs of T'- (T-) conics common with* 6 *T'- (T-) quadrics of Q* (Th. 11); 3 *pairs of d- (d'-) conics common with* 3 *D- (D'-) quadrics of Q* (Th. 8).

Theorem **13**. *There are* 100 *H- (H'-) cones, each corresponding to* 9 *h- and* 9 *h'-conics* (Th. 9). *Each such cone contains*: 6 *h-planes lying in pair in* 6 *f- and* 3 *h-solids*; 54 *g-lines,* 9 *in each generating h-plane*; 108 *G-,* 117 *g-,* 234 *h- (b'-) and* 270 *h'- (h-) points*; 3 *g-quadrics as well as* 3 *pairs of h- (h'-) quadrics as its farther intersections with* 3 *pairs of h- (h'-) quadratics through the* 3 *g-quadrics such that these* 9 *quadrics meet one another in* 18 *G-conics besides its corresponding* 9 *h- and* 9 *h'-conics* (§ 3.3e); *the other* 36 *G-,* 36 *h- and* 36 *h'-conics lying by sixes and* 108 *T- and* 108 *T'-conics lying by* 18*s in the* 6 *generating h-planes* (Ths. 9, 10, 11); *the other* 30 *h-*

and 30 h'-*conics lying by tens on the g-quadrics* (Th. 9); 12 g-, 18 d- (d'-), 12 h- (h'-) *in* 12 h-*planes*, 24 h'- (h-) *in* 24 p'- (p-) *planes and* 36 T'- (T-) *conics lying respectively by* 2 g-, 3 d- (d'-), 2 h- (h'-), 4 h'- (h-) *and* 6 T'- (T-) *conics on the* 6 h- (h'-) *quadrics* (Ths. 7, 8, 9, 11).

Corollary. *Twelve of the* 18 G-*conics common to the* 9 *quadrics of an* H- (H'-) *cone lie in* 12 G-*planes, by fours on the* 3 g-*quadrics and in pairs on the* 6 h- (h'-) *quadrics and on* 6 G-*quadrics in* 6 G-*solids* (Th. 6); *the other six lie in* 6 h-*planes and in pairs on the* 6 h- (h'-) *quadrics* (Th. 10) Hence: *The* 6 h-*planes determined by the* 6 *outside pairs of g-lines through the* 6 G-*points of a* G-*conic in an* h-*plane and common to a pair of* h- (h'-) *quadrics also concur at an* H- (H'-) *point, form* 2 *triads of generating planes of the* 2 *opposite systems of an* H- (H'-) *cone and are the same for* 6 G-*conics in* 6 h-*planes generating similarly its complementary* H'- (H-) *cone.*

Theorem 14. *There are* 900 D- (D'-) *cones, each corresponding to a* d- (d'-) *conic* (Th. 8). *Each such cone contains*: 4 g- *and* 2 h-*planes lying, in pairs, in* 4 f-, 2 g- *and* 2 h-*solids*; 26 g-*lines*, 2 *in each generating* g- *and* 9 *in each generating* h-*plane* (see Table 1); 68 G-, 61 g-, 126 h- (h'-) *and* 138 h'- (h-) *points*; *a pair of* h- (h'-) *quadrics as its intersection with an* h- (h'-) *quadratic, say* W, *or as its sections by the pair of* f-*solids determining the* c- (c'-) *plane of its corresponding conic common to them* (§ 3.2); 12 G-, 12 h- *and* 12 h'-*conics lying by sixes and* 36 T- *and* 36 T'-*conics lying by* 18 *s in each generating* h-*plane* (Ths. 9, 10, 11); 2 *pairs of other* d- (d'-) *conics common with* 2 *other* h- (h'-) *quadrics of* W (Th. 8), 2 *pairs of g-conics with* 2 C-*quadrics* (Th. 7), *a pair of* h- (h'-) *conics with a g-quadric* (Th. 9), 2 *pairs of* h'- (h-) *conics in* 2 *pairs of* c'- (c-) *planes and* 4 *pairs in* 4 *pairs of* p'- (p-) *planes with* 6 H'- (H-) *quadrics in* 6 p'- (p-) *solids* (§ 2.4), *a pair of* h- (h'-) *conics in a pair of* c- (c'-) *planes and* 2 *pairs in* 2 *pairs of* h-*planes with* 3H- (H'-) *quadrics in* 3 f-*solids* (§ 2.3), 2 *pairs of* G-*conics with* 2 G-*quadrics in* 2 G-*solids and* 2 *pairs with those in* 2f-*solids* (Th. 10), 4 *pairs of* T'- (T-) *conics with* 4 T'- (T-) *quadrics and* 4 *more* T'- (T-) *conics,* 2 *on each* h- (h'-) *quadric of the hypercone under consideration* (§ 2.3, Th. 11).

4.2. **Summary.** To summarise the incidences of the points on the g-lines, conics, quadrics, quadratics and hypercones of various types through them in a tabular form (see Table 2) like the Table 1, we may introduce some new notations. The same are explained in its blank squares at the appropriate places.

But before we proceed with the planned table, we may note the distribution of the G-, g-, h- and h'-points on different conics in different planes and solids as follows:

(i) Besides the 96 G- and 36 g-points, the 72 g-lines in an f-solid $p = i$ meet in 144 h- and 144 h'-points which lie on 96 h-, 96 h'-, 144 T- and 144 T'-conics such that 48 h-conics lie in its 6 p-planes, 48 h'-conics in its 6 p'-planes and the rest in its

16 h-planes, the other 144 g-, 144 h- and 144 h'-points of this solid lie on 96 g-, 144 h-, 144 h'-, 144 T-, 144 T'-, 72 d- and 72 d'-conics such that 48 g-, 72 h- and 72 d-conics lie in its 4 c-planes, 48 g-, 72 h'- and 72 d'-conics in its 4 c-planes, 24 h-conics as its 24 sections of the 24 $pqjk$-quadrics in its 24 planes common with their 24 $pqjk$-solids, 24 h'-conics as its 24 sections of the 24 $qrij$-quadrics in its 24 planes common with their 24 $qrij$-solids, and the rest in its 16 h-planes; the 96 G-points lie on 16 G-conics in its 16 G-planes and on 96 others in its 16 h-planes besides their 144 T- and 144 T'-conics (see Table 1; §§ 2.3—3.5; Ths. 7—11).

(II) The 36 h- and 36 h'-points other than the 9 h- and 9 h'-points of intersection of the 9 g-lines in an h-plane and the 18 G-points therein lie on 6 h-, 6 h'-, 6 G-, 18 T- and 18 T'-conics (Ths. 9—11) and thus the situation in an h-solid is obvious.

(III) The 24 G-, 36 h- and 36 h'-points in a g-solid lie on 4 G-, 12 h- and 12 h'-conics (Ths. 9 (I) and (II), 10).

(IV) The 9 g- and 180 h- (h'-) points of a p- (p'-) solid $p = q$ ($i = j$) lie on 60 g- and 90 d- (d'-) conics in its 5 c- (c'-) planes and 240 h- (h'-) conics, 90 in its 5 c- (e'-) planes, 120 in its 15 p- (p'-) planes and 30 common to the 30 pairs of $prij$- and $qrij$- ($pqik$- and $pqjk$-) quadrics (Ths. 8—9).

5. QUARTIC PRIMALS

The 24 g-lines of a tlm-quadratic obviously lie on a quartic primal given by the equation

$$a(i-p)(i-q)(i-r)(i-s)\ b(j-p)(j-q)(j-r)(j-s)$$
$$+ c(k-p)(k-q)(k-r)(k-s) = 0$$

for any arbitrary values of a, b, c. Thus follows the following

Theorem 15. *The 24 g-lines of an h- or h'-quadratic lie on the quartic primals of a net determined by any two of them and therefore on the octic surfaces of its intersection with them. There are thus 100 such nets, each corresponding to an h- or h'-quadratic.*

6. QUADRILATERALS

6.1. Definitions. An h-solid $p + q = i + j$ meets a c-plane $q = k = r$ in a *d'-line*; a c'-plane $j = r = k$ in a *d-line*; an h-plane $q = l$, $r = k$ in an *a'-line* and $j = s$, $r = k$ in an *a-line*; a p-plane $r = k$, $q = s$ in a *b-line*; a p'-plane $r = k$, $j = l$ in a *b'-line* (cf. § 1.2).

The 2 pairs of g-lines through the pair of h- (h'-) points of an e- (e'-) line are seen to form a skew quadrilateral, called an e- (e'-) *quadrilateral*. An a- (a'-)

or a b- (b'-) line too passes through a pair of h- (h'-) points and also similarly determine *an a- (a'-) or a b- (b'-) quadrilateral* accordingly. A d- or d'-line passes through a pair of g-points such that the 2 pair of g-lines through them form a skew quadrilateral, called *g-quadrilateral.*

6.2. e- (e'-) Quadrilaterals. Consider a tlm-quadratic and its section by an h-solid $p+q=i+j$ which has the 4 g-lines

$$\text{(I)}\quad p=i,\ q=j,\ r=k \qquad \text{(III)}\quad p=j,\ q=i,\ r=k$$

$$\text{(II)}\quad p=i,\ q=j,\ s=k \qquad \text{(IV)}\quad p=j,\ q=i,\ s=k$$

common with it. They form an e-quadrilateral with one diagonal along the e-line through the pair of h-points of intersection of the 2 pairs of g-lines (I), (II) and (III), (IV) and second along the h-line of the h-solid through the pair of g-points of intersection of (I), (III) and (II), (IV). There are 18 such solid sections of the h-quadratic under consideration such that every g-line on it meets 6 others on it, 3 in $3g$- and 3 in $3h$-points, and through every h-point passes one and only one e-line (see Table 1).

Similar is the case of an h'-quadratic. Thus follows the following

Theorem 16. *The 36 g- and 36 h- (h'-) points of intersection of the 24 g-lines of an h- (h'-) quadratic distribute uniquely into 18 tetrads as the vertices of 18 e- (e'-) quadrilaterals, lying in 18 h-solids, such that every g- or h- (h'-) point belongs to one and only one such quadrilateral and every g-line to 3 such quadrilaterals, and to every g-point of a g-line on it correspond an h- (h'-) point on the same line and another on the second g-line through this g-point. Hence: There are 900 e- and 900 e'-quadrilaterals arising respectively from the 50 h- and 50 h'-quadratics.*

6.3. a- (a'-) and b- (b'-) Quadrilaterals. It is easy to see that the second diagonal of an a- or a b'-quadrilateral is an f-line (§ 1.2), the first being an a- or a b'-line, and that of an a'- or a b-quadrilateral is an f'-line, the first being an a'- or a b-line. The 6 g-lines of an h- (h'-) quadric are seen to form $6a$- (a'-) and $3\,b$- (b'-) quadrilaterals with vertices at their $3\,g$- and $6\,h$- (h'-) points of intersection. We may also observe that each a- (a'-) line passes through a D- (D'-) point and each b- (b'-) line through an E- (E'-) point. Thus follows the following

Theorem 17. *There are 3600 a- (a'-) and 1800 b- (b'-) quadrilaterals, 6 a- (a'-) and 3 b- (b'-) quadrilaterals on each h- (h'-) quadric, such that each f- (f'-) line forms a diagonal of 6 a- (a'-) and 3 b'- (b-) quadrilaterals, their second diagonals being 6 a- (a'-) and 3 h'- (h-) lines. There pass 4 a- (a'-) lines through each D- (D'-) point and 3 b- (b'-) lines through each E- (E'-) point. There lie 4 a-, 4 a'-, 4 b- and 4 b'-lines in each g-plane, one through each of its 4 D-, 4 D'-, 4 E- and 4 E'-points respectively.*

6.4. **g-Quadrilaterals.** The $6g$-lines of a g-quadric are seen to form $9g$-quadrilaterals with verices at their $9g$-points of intersection and diagonals along $9d$- and 9 d'-lines. Each d-line is observed to contain 2 D- and 2 E'-points and each d'-line 2 D'- and 2 E-points. Thus follows

Theorem 18. *There are* 900 *g-qaadrilaterals,* 9 *on each g-quadric, with* 600 *d- and* 900 *d'-lines as their diagonals. Each D- (D'-) point lies on* 2 *d- (d'-) lines and each E- (E'-) point on* 3 *d'- (d-) lines. Each g-plane contains* 2 *d- and* 2 *d'-lines meeting its* 2 *e- and* 2 *e'-lines in its* 4 *D-,* 4 *D'-,* 4 *E- and* 4 *E'-points* (see Table 1).

6.5. Quadrilaterals on hypercones. As an immediate consequence of the 3 preceding propositions follow the following

Theorem 19. *The* 12 *g-lines of an E- (E'-) cone form* 3 *e- (e'-),* 3 *g- and* 3 *b- (b'-) quadrilaterals such that their* 3 *e- (e'-),* 3 *d'- (d-) and* 3 *b- (b'-) line diagonals concur at its vertex as the* 9 *contact lines of* 3 *h-,* 3 *g- and* 3 *f-solids tangent to it, and their* 3 *h-,* 3 *d- (d'-) and* 3 *f'- (f-) line diagonals lie in the plane of its corresponding g-conic* (Th. 12).

Theorem 20. *The* 6 *pairs of g-lines through the* 4 *g- and* 2 *h- (h'-) points of a d- (d'-) conic form* 3 *e- (e'-),* 2 *g- and* 4 *a- (a'-) quadrilaterals such that their* 3 *h-,* 2 *d'- (d-) and* 4 *f- (f'-)line diagonals lie in its plane and their* 3 *e- (e'-),* 2 *d- (d'-) and* 4 *a- (a'-)line diagonals concur at the vertex of its corresponding D- (D'-)cone as the* 9 *contact lines of* 3 *h-,* 2 *g- and* 4 *f-solids tangent to it.*

Theorem 21. *The* 6 *pairs of g-lines through the* 6 *h- (h'-) points of an h- (h'-) conic, not in an h- or a p-plane, form* 3 *e- (e'-)qaadrilaterals such that their* 3 *e- (e'-) line diagonals lie in its plane and their* 3*h-line diagonals concur at the vertex H- or H' of its corresponding H- or H'-cone as the* 3 *contact lines of its* 3 *tangent h-solids. They also form* 6 *a- (a'-) or b- (b'-) quadrilaterals, if the conic lies in the plane of a g-solid common with an f- or another g-solid, such that their* 6 *a- (a'-) or b- (b'-)line diagonals lie in this plane and their* 6 *f- (f'-) or f'- (f-)line diagonals concur at H or H' as the* 6 *contact lines of its* 6 *tangent f-solids. They form* 6 *new skew quadrilaterals too, if the conic lies in a c- (c'-)plane as a section of an h- (h'-) or h'- (h-) quadratic, such that their* 6 *f- (f'-)line diagonals lie in its plane and their* 6 *f- (f'-) or f'- (f-)line diagonals concur at the vertex of the corresponding H- (H'-) or H'- (H-) cone as the* 6 *contact lines of its* 6 *tangent f-solids.*

Corallary 1. *Thus there arise* 7200 *new skew quadrilaterals formed of the* 600 *g-lines*: 1800 *with pairs of f- and* 1800 *with pairs of f'-lines as pairs of their respective diagonals,* 72 *of either kind in each f-solid and* 18 *of the first kind on each H- and* 18 *of the second on each H'-cone*; 3600 *with an f- and an f'-lines as pairs of their diagonals,* 144 *in each f-solid and* 36 *on each H- and each H'-cone.*

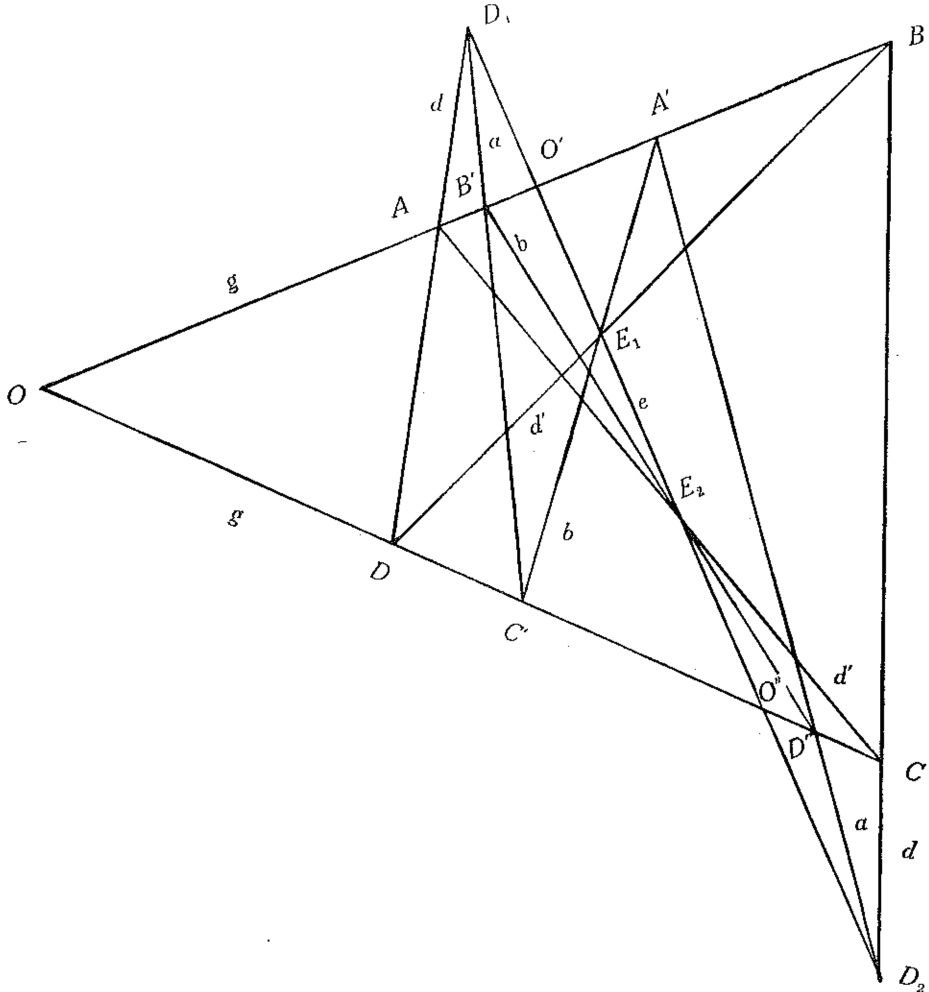

Fig. 1

The picture of the triangle formed by the pair of g-lines with an e-line and its 8 transversals as $2a$-, $2a'$-, $2b$- and $2b'$-lines in a g-plane.

Corollary 2. *Every g-, d-, d′-, h- or h′-conic determining a hypercone* (§ 4) *lies on* 2 *of the* 100 *g-*, 600 *h- and* 600 *h′-quadrics such that the* 9 *points of intersection of the* 6 *g-lines of one join those of the other to form the relevent* 9 *contact lines of the hypercone concurrent at its vertex.*

6.6. *g*-**plane picture.** Let OO', OO'' be the pair of *g*-lines of a *g*-plane; A, B and C, D be the other *g*-points on them besides their common *g*-point O; O', O'' be the pair of *h*-points on an *e*-line therein meeting the pair of *d*-lines AD, BC in the pair of *D*-points D_1, D_2 (Th. 18) and *d′*-lines BD, AC in the pair of *E*-points E_1, E_2; A', B' be the *h*-points of intersection of the *g*-lines of an *h*-quadratic on OO' corresponding to A, B and C', D' on OO'' corresponding to C, D (Th. 16). Now it is not difficult to see that $B'C'$, $A'D'$ are the pair of *a*-lines through D_1, D_2 and $C'A'$, $B'D'$ the pair of *h*-lines through E_1, E_2 (Th. 17). Again it is well known that any transversal meets the 3 pairs of opposite sides of a quadrangle in 3 pairs of points in an involution. Hence from the quadrangle $ABCD$ or $A'B'C'D'$ and its *e*-line transversal $O'O''$ (Fig. 1) follows the following

Theorem 22. *The pairs of h-, D- and E-points on an e-line belong to an involution. Similarly behave the pairs of h′-, D′- and E′-points on an e′-line.*

Further from the quadrangle $C'DD_1E_1$ or $CD'D_2E_2$ and its *g*-line transversal OO' follows the following

Theorem 23. *The* 3 *g-points on a g-line mate in an involution with their* 3 *corresponding h- or h′-points, on it, of intersection of the g-lines of an h- or h′-quadratic through it. The correspondence varies with the change of the h- or h′-quadratic.*

7. OBSERVATIONS

If now we study back the Table 1, we obtain a number of interesting subeonfigurations. Some of them are described below.

7.1. *g*-**points.** The 900 *g*-points, 600 *g*-, 600 *f*- and 600 *f′*-lines form a $(900_6, 1800_3)$ configuration such that 6 lines pass through each point and 3 points lie on each line.

The 18 *g*-points of an *h*-plane other than the 9 on its *h*-line lie by threes on its 6 *f*-lines, 2 through each of its 3 collinear *H*-points, and on 6 *f′*-liues, 2 through its 3 *H′*-points. It can be now proved that the 6 *f*- (*f′*-)lines in an *h*-plane form a pair of Desargues triangles with its *h*-line as their axis of perspectivity such that the 3 pairs of their corresponding vertices lie on its 3 *p*- (*p′*-)lines concurrent at its *S*- (*S′*-)point which is therefore their centre of perspectivity.

7.2. *H*- **and** *H′*-**points.** (1) The 10 *H*- (*H′*-)points, 10 *h*-lines and 5 *c′*- (*c*-) planes in a *p′*- (*p*-)solid form the well known configuration

$$10(\cdot, 3, 3) \quad 10(3, \cdot, 2) \quad 5(6, 4, \cdot)$$

such that 3 lines and 3 planes pass through each point, each line lies In 2 planes and contains 3 points, each plane contains 6 points as the vertices of the quadrilateral formed of the 4 lines therein.

(ii) The 100 H- (H'-)points, 100 h-lines and 50 c'- (c-)planes form 10 configurations of the type (i), one in each p'- (p-)solid.

(iii) The 10 H- (H'-)points in each S- (S'-)plane lie by fours on its 5 c- (c'-) lines.

(iv) The 100 H- (H'-)points and 150 p- (p'-)planes form a $(100_6, 150_4)$ configuration such that 6 planes pass through each point and 4 points lie in each plane, one on each of its 4 f- (f'-)lines.

7.3. c- (c'-)**lines and planes.** The 50 c- or c'-lines and 50 c- or c'-planes form a 50_3 configuration auch that 3 planes pass through each line and 3 lines lie in each plane.

If we take a section of it by a solid, we obtain a 50_3 configuration of 50 points and 50 lines such that 3 lines pass through each point and 3 points lie on each line.

7.4. S- (S'-) **planes and** p- (p'-) **solids.** The 10 S- or S'-planes and 10 p- or p'-solids form a 10_3 configuration such that 3 solids pass through each plane and 3 planes lie in each solid. In fact, they all *concur* at the *unit point* (1, 1, 1, 1, 1) of the respective simplex S or S' (§ 1.1) of reference, the 10 S- or S'-planes passing through its respective 10 edges and p- or p'-solids through its 10 plane faces.

The section of this configuration by a solid gives a 10_3 configuration of 10 lines and 10 planes such that 3 planes pass through each line and 3 lines lie in each plane, and that by a plane gives the famous Desargues 10_3 figure of 10 points and 10 lines.

7.5. S- (S'-) **point and** p- (p'-)**planes.** The 200 S- or S'-points and 150 p- or p'-planes form a $(200_6, 150_8)$ configuration such that 6 planes pass through each point and 8 points lie in each plane, one on each of its 8 p- or p'-lines.

7.6. An h-plane picture. The 9 g-lines in an h-plane $p=i$, $q=j$ are its 9 intersections by the 9 f-solids $r=k$, $s=k$, $t=k$, $r=l$, $s=l$, $t=l$, $r=m$, $s=m$, $t=m$. They may be respectively named as a, b, c, a', b', c', a'', b'', c''. The 9 h-points A, B, C, A', B', C', A'', B'', C'' of their intersection form a triad of triangles $ABC=abc$, $A'B'C'=a'b'c'$, $A''B''C''=a''b''c''$ perspective to one another from the same centre at its S-point O such that their corresponding vertices lie on its 3 p-lines concurrent thereat and their corresponding sides meet in its 9 h'-

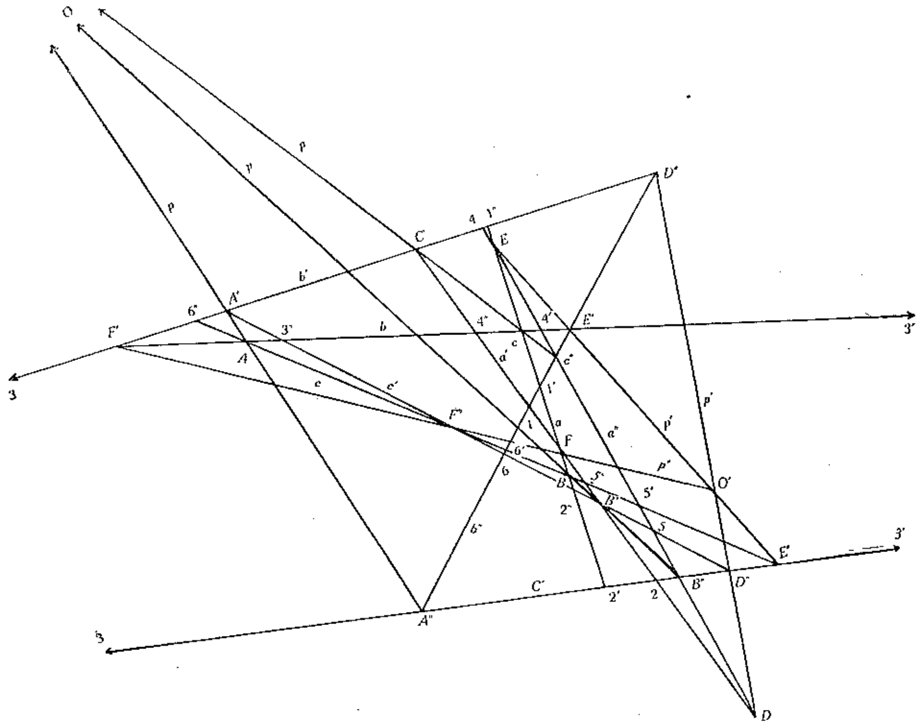

Fig. 2

The picture of 9 g-lines, 3 p-lines and 3 p'-lines, 9 h-points, 9 h'-points, 1 S-point, 1 S'-point and 18 G-points in an h-plane.

points D, D', D'', E, E', E'', F, F', F'' lying by threes on its 3 p'-lines as their 3 axes of perspectivity concurrent at its S'-point O' (*Fig.* 2).

The 9 h'-points of their intersection thus also form a triad of triangles $DEF = a\,a'a''$, $D'E'F' = b\,b'b''$, $D''E''F'' = c\,c'c''$ perspective to one another from the same centre at O' such that their corresponding vertices lie on its 3 p-lines and their corresponding sides meet at the vertices of former triad of triangles on its 3 p-lines as their 3 axes of perspectivity.

The non-corresponding sides of a pair of perspective triangles of either triad meet in 6 G-points of a G-conic. The 6 hexagrams formed of the 18 G-points inscribed in its 6 G-conics are 123456, 1'2'3'4'5'6', 1"2"3"4"5"6", 3 3'3"6 6'6", 2 2'2"5 5'5", 1 1'1"4 4'4". Thus we have the following

Theorem 24. *The 9 g-lines of an h-plane form 2 triads of triangles, triangles of one triad being perspective to one another from the same centre at its S-point and those of the other from its S'-point, such that the vertices of the former are its 9 h-points lying by threes on its 3 p-lines which form the 3 axes of perspectivity of the later and those of the later are its 9 h'-points lying by threes on its 3 p-lines which form the 3 axes of perspectivity of the former. Hence the 2 triads are mutually related such that one can be derived from the other giving rise to a* $(20_3, 15_4)$ *configuration of* 20 *points and* 15 *lines,* 3 *lines through each point and* 4 *points on each line* (BAKER, 1943, p. 351). *The non-corresponding sides of the* 6 *pairs of perspective triangles meet in* 18 *G-points which form* 6 *hexagrams inscribed respectively in its* 6 *G-conics having its p- and p'-lines as their* 6 *Pascal lines.*

7.7. E-(E'-) **Points.** The 600 E- or E'-points, 900 d'- or d- and 900 e-e'-lines, 900 g-planes and 100 g- and 100 h-solids form a configuration 600 (·, 6, 6, 6) 1800 (2, ·, 2, 3) 900 (4, 4, ·, 2) 200 (18, 27, 9, ·) such that 6 lines 6 planes and 6 solids pass through each point, each line contains 2 points and lies in 2 planes and 3 solids, each plane contains 4 points and 4 lines and lies in 2 solids, each solid contains 18 points, 27 lines and 9 planes (Th. 18).

8. GAUSS POINTS

Now we introduce a linear relation between the coordinates of a point referred to the 2 simplexes S, S' (§ 1.1) as

$$x + y + z + u + v = x' + y' + z' + u' + v'$$

or

$$p + q + r + s + t = i + j + k + l + m.$$

By introducing this relation we find that the polar solid of the unit point of S w. r. t. S' ([5], [6]) coincides with that of the unit point of S' w. r. t. S'.

As a result, the 600 G-points reduce to 120 GAUSS *points* [1]. For the 5 f-solids

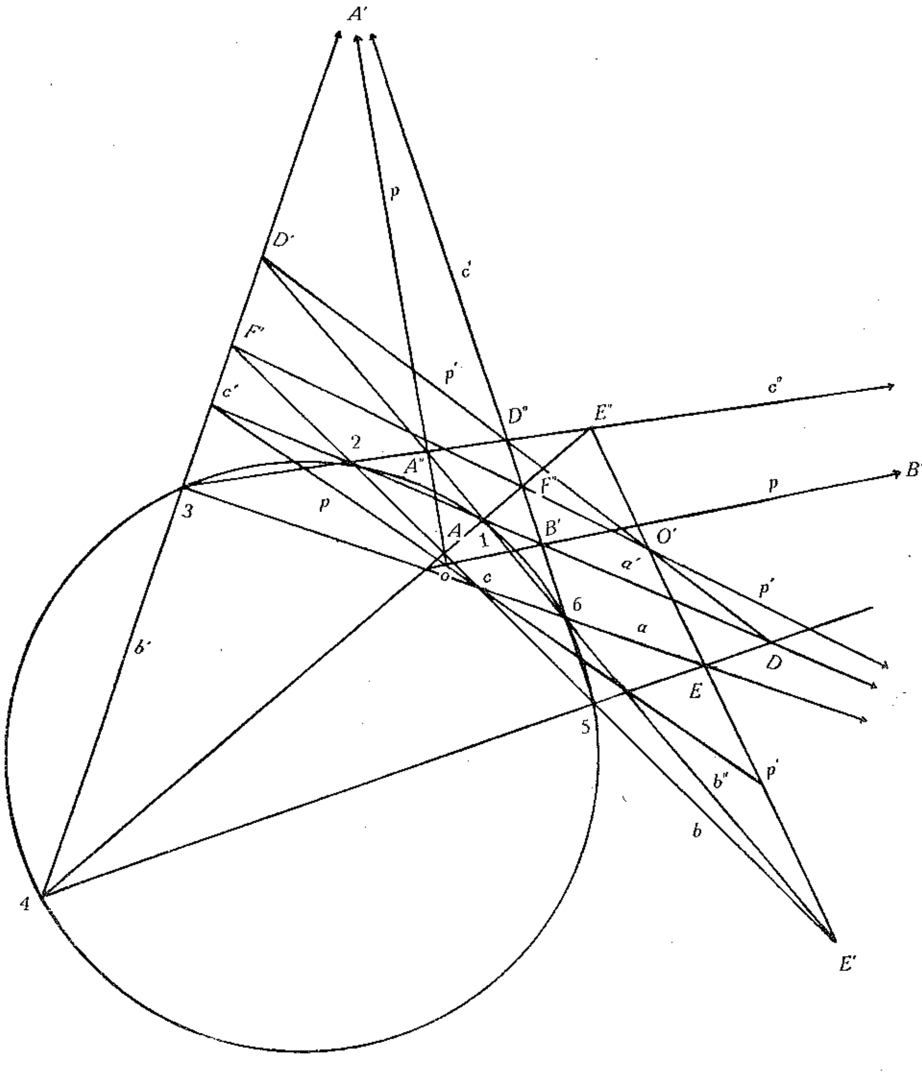

Fig. 3

The shape Fig. 2 takes when the h-plane there becomes specialised, the 18 G-points therein reducing to the 6 Gauss points here.

$p = i$, $q = j$, $r = k$, $s = l$, $t = m$ for given 5 values of p, q, r, s, t as well as of i, j, k, l, m *concur at a* GAUSS *point* where therefore coincide the 5 G-points determined by them (§ 1.2). Consequently :

(I) A G-solid $p + q + r + s = i + j + k + l$ coincides with the f-solid $t = m$, their 120 G-points reduce to 24 GAUSS points and therefore their 9 G-quadrics (Th. 6*) coincide into one.

(II) The pair of G-planes of a g-solid $p + q + r = i + j + k$ common with the pair of f-solids $s = l, t = m$ or $s = m, t = l$ coincide with their common h-plane, their 30 G-points reduce to 6 GAUSS points and therefore their 8 G-conics (Th. 10) coincide into one.

(III) The picture of an h-plane (Th. 24) is modified into the *dual of an adjoint pair of veronesian systems of triangles* [2] such that either system is determined by any 2 of the 3 triangles perspective to one another from the same centre and the non-corresponding sides of every one of the 6 pairs of perspective triangles meet in the same 6 GAUSS points 1, 2, 3, 4, 5, 6 (Fig. 3) forming 6 hexagrams 123456, 163254, 143652, 123654, 163452, 143256 inscribed in its unique G-conic with their 6 PASCAL lines along its 3 p'- and 3 p-lines which then *concur* respectively at a pair of STEINER *points* [2] or its S- and S'-points conjugate for the G-conic.

The 3 joins $a'' = 45$, $b'' = 61$, $c'' = 23$ are said to form, following COURT [3], the *veronesian triangle* $A'' B'' C''$ of the 2 triangles $ABC = abc$, $A' B' C' = a' b' c'$ perspective from O, where $4 = c \cdot b'$, $5 = b \cdot c'$, $6 = a \cdot c'$, $1 = c \cdot a'$, $2 = b \cdot a'$, $3 = a \cdot b'$ (Fig. 3). The 3 triangles are then said to form a *veronesian system* such that every one of them is the veronesian of the other two and they are mutually perspective from the same centre O. The 3 triangles $DEF = aa' a''$, $D' E' F' = bb' b''$, $D'' E'' F'' = cc' c''$ are also seen to form a veronesian system. The relation between the 2 systems is mutual such that one can be derived from the other by the same operations. Hence they are said to form a pair of *mutually adjoint veronesian systems*. Thus follow the following

Theorem 25. *The* 600 *G-points may reduce to* 120 GAUSS *points without affecting the main configuration except the following modifications*:

(I) *The* 25 *G-solids coincide with the* 25 *f-solids and the* 400 *G-planes, reducing to* 200 *only, with the* 200 *h-planes.*

(II) *The* 225 *G-quadrics reduce to* 25 *only and the* 1600 *G-conics to* 200 *only.*

(III) *The* 2 *triads of mutually perspective triangles forming a* $(20_3, 15_4)$ *configuration in an h-plane become a pair of mutually adjoint veronesian systems such that its S- and S'-points form a pair of conjugate* STEINER *points arising from the* 6 *hexagrams, formed of the common* 6 GAUSS *points of intersection of the non-corresponding sides of the* 6 *pairs of perspective triangles of the* 2 *systems, inscribed in its unique G-conic.*

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ÖZET

Şekli meydana getiren 600 doğuranı (g-çizgileri), ikişer ikişer 900 tane g-düzlemi ve dokuzar dokuzar da 200 tane h-düzlemi içinde bulunurlar; altışar doğru olarak 1300 tane kuadrik yüzey; onsekizer doğru olarak 100 tane h-uzayları içinde bulunurlar: bu uzaylar 100 h-çizgisi boyunca kesişen 100 tane h-düzlemi çifti tarafından belirtilmektedir. g-çizgileri ayrıca 72 şer takımlar olarak 25 tane f-uzayı arasında dağılabilmektedir; 24 lük takımlar şeklinde 100 adet 3 boyutlu kuadrikler üzerinde de bulunurlar: üstelik her bir 24 lük doğru takımı ayni zamanda bir huzmeye ait olan dördüncü dereceden 3 boyutlu varyetelerin hepsine ait olduğundan, bu hüzmeye ait herhangi iki varyetenin arakesiti olarak meydana gelen sekizinci dereceden yüzey üzerinde de bulunur.

Her g-çizgisi 3 tanesi 3 g-noktasında 0 tanesi 6 h-noktasında, 6 tanesi 6 h'-noktasında ve 12 tanesini de üçer üçer 4 G-noktasında olmak üzere tam 27 tane başka g-çizgisi ile kesişir. Bu surette belirtilen noktalardan 900 g-noktası, 1800 h-noktası ve 1800 h'-noktası 6 şar nokta takımları şeklinde 6600 tane konik üzerinde bulunurlar: bu konikler ise 156 şer takımlar şeklinde 100 tane 3 boyutlu kuadrikler üzerinde o tarzda dağıtılmış bulunmaktadırlar ki, her bir koniği belirtmeğe yarıyan 6 noktadan geçen 6 g-çizgisi çiftinin belirttikleri 6 düzlem tek bir noktada kesişip, bir kuadrik nokta konisinin 2 aykırı sisteminin 2 üçlü doğuran düzlem takımını teşkil ederler. Bu tarzda belirtilen 3200 tane hiperkoni mevcuttur.

Bir 3 boyutlu kuadrik üzerinde bulunan 24 g-çizgisi, tepeleri 36 g-noktası ve 36 h-noktası veya 36 h'-noktası içinden alınan 18 tane sapık dörtgen meydana getirirler: noktalar bu surette 18 tane dörtlü takımlara dağıtılmış bulunurlar ve her g-çizgisi üzerinde 3 g-noktası, 3 h-noktası veya 3 h'-noktası ile bir involüsyonunun eşleri olarak tekabül ettirilir.

Bir h düzleminde bulunan 9 g-çizgisi, biri diğerine perspektif olan her üçlü takımının perspektivite merkezi bir S veya S'-noktasında bulunan iki üçlü üçgen takımını meydana getirir: bu surette 6 perspectivite ekekseni olan malûm $(20_3\ 15_4)$ şekli elde edilir. Perspektif üçgenlerin birbirine tekabül etmeyen kenarları bir PASCAL şekli teşkil eden ve bir G-koniği üzerinde bulunan 6 tane G-noktasında kesişirler.

600 G-noktası 24 er takımlar olarak 225 adet kuadrik yüzey üzerinde bulunurlar: bu yüzeylerin sayısı ancak hususî bazı şartlar altında 25 e iner ve bu takdirde bu 600 G-noktası bir sempleks çiftine tekabül eden 120 tane GAUSS noktası olur. Bir h-düzleminde bulunan 2 birbirine perspektif üçlü üçgen takımları bu takdirde bir *birbirine ek* VERONESE *sisteminin* düalı olmuş olur, halbuki S ve S'-noktaları bunun içindeki 6 GAUSS noktasının teşkil ettikleri altıgenin STEINER *noktaları* çiftini teşkil ederler.

Bu şekilden çıkarılabilen diğer altşekiller ve bunların şekille bağıntıları de oldukça alaka çekicidir.