ON THE GROWTH OF A CLASS OF ENTIRE FUNCTIONS

S. K. Singh and K. Manjanathaiah

Let $f(z)$ be an entire function, $M(r, f)$ its maximum modulus on $|z| = r$ and

$$N(r, a) = \int_0^r \frac{n(t, a)}{t} dt.$$

M is compared to both N and n.

0. Let $f(z)$ be an entire function, let $M(r, f)$ be the maximum modulus of $f(z)$ on $|z| = r$ and let

$$N(r, a) = \int_0^r \frac{n(t, a)}{t} dt.$$

In section 1 we compare $\log M(r, f)$ with $N(r, a)$ and in section 2, we compare it with $n(r, a)$.

1. Theorem 1. *Let $f(z)$ be an entire function of order ϱ such that*

$$A_1 \{l_1(r)\}^{\alpha_1 - c} < \log M(r, f) < A_2 \{l_1 r\}^{\alpha_1} \{l_2(r)\}^{\alpha_2} \cdots \{l_m(r)\}^{\alpha_m} \tag{1}$$

where

$$l_1(r) = \log r, \quad l_m(r) = \log l_{m-1}(r);$$

$\alpha_1 \geq 1$ *and* $\alpha_2, \alpha_3 \cdots \alpha_m$ *are non negative and* $0 < c < 1$; *then*

$$\log M(r, f) \sim N(r, a)$$

for all a.

Proof: Without any loss of generality we can take $a = 0$. Put

$$N(r) = N(r, 0); \quad n(r) = n(r, 0).$$

Then

$$N(r^2) > A + \int_r^{r^2} \frac{n(t)}{t} dt > n(r) \log r.$$

So

$$n(r)\log r < N(r^2) < \log M(r^2, f) < A_8\{l_1(r)\}^{\alpha_1}\{l_2(r)\}^{\alpha_2}\cdots\{l_m(r)\}^{\alpha_m}$$

Hence

(1.1)
$$n(r) = 0\{l_1(r)\}^{\alpha_1-1}\{l_2(r)\}^{\alpha_2}\cdots\{l_m(r)\}^{\alpha_m}$$

Further from the right hand of the inequality of (1) it follows that $f(z)$ is of order zero.

Hence

(1.2)
$$\log M(r, f) < \int_0^r \frac{n(x)}{x}\,dx + r\int_r^\infty \frac{n(x)}{x^2}\,dx = N(r) + r\int_r^\infty \frac{n(x)}{x^2}\,dx.$$

Now

$$r\int_r^\infty \frac{n(x)}{x^2}\,dx < rA\int_r^\infty \frac{\{l_1(x)\}^{\alpha_1-1}\{l_2(x)\}^{\alpha_2}\cdots\{l_m(x)\}^{\alpha_m}\,dx}{x^2}$$

$$= Ar\int_r^\infty \frac{\{l_1(x)\}^{\alpha_1-1}\{l_2(x)\}^{\alpha_2}\cdots\{l_m(x)\}^{\alpha_m}}{x^k}\left(\frac{dx}{x^{2-k}}\right),$$

where we take $0 < K < 1$.

Since

$$\frac{\{l_1(x)\}^{\alpha_1-1}\{l_2(x)\}^{\alpha_2}\cdots\{l_m(x)\}^{\alpha_m}}{x^k}$$

is a decreasing function of x for $r \geq r_0$, so (*)

$$r\int_r^\infty \frac{n(x)}{x^2}\,dx < \frac{A_1 r\{l_1(r)\}^{\alpha_1-1}\{l_2(r)\}^{\alpha_2}\cdots\{l_m(r)\}^{\alpha_m}}{r^k}\int_r^\infty \frac{dx}{x^{2-k}}$$

$$\leq A\{l_1(r)\}^{\alpha_1-1}\{l_2(r)\}^{\alpha_2}\cdots\{l_m(r)\}^{\alpha_m}.$$

From (1.2) we then get,

$$\log M(r, f) < N(r) + A\{l_1(r)\}^{\alpha_1-1}\{l_2(r)\}^{\alpha_2}\cdots\{l_m(r)\}^{\alpha_m},$$

hence

$$1 < \frac{N(r)}{\log M(r, f)} + \frac{A\{l_1(r)\}^{\alpha_1-1}\{l_2(r)\}^{\alpha_2}\cdots\{l_m(r)\}^{\alpha_m}}{\log M(r, f)}.$$

But

$$\log M(r, f) > A_1\{l_1(r)\}^{\alpha_1-\varepsilon}$$

hence

(*) A denotes a constant which is not necessarily the same at each occurrence.

$$1 < \frac{N(r)}{\log M(r, f)} + 0(1)$$

so

$$\liminf_{r\to\infty} \frac{N(r)}{\log M(r, f)} \geq 1.$$

But, always

$$\limsup_{r\to\infty} \frac{N(r)}{\log M(r, f)} \leq 1,$$

hence

$$\log M(r, f) \sim N(r).$$

2. Theorem 2. *Let $f(z)$ be on entire function of order ϱ $(0 < \varrho < 1)$, then*

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{n(r)} \leq \frac{1}{\varrho(1-\varrho)} \tag{2.1}$$

Proof: Let ϱ_1 be the exponent of convergence of the zeros of $f(z)$. Then,

$$\varrho_1 = \limsup_{r\to\infty} \frac{\log n(r)}{\log r}$$

Hence there exists a proximate order $\varrho_1(r)$ having the following properties:

(I) $\varrho_1(r)$ is differentiable for $r > r_0$ except at isolated points at which $\varrho_1'(r-0)$ and $\varrho_1'(r+0)$ exist;

(II) $$\lim_{r\to\infty} \varrho_1(r) = \varrho_1;$$

(III) $$\lim_{r\to\infty} r\,\varrho_1'(r) \log r = 0;$$

(IV) $$n(r) \leq r^{\varrho_1(r)} \quad \text{for} \quad r \geq r_0;$$

$$n(r) = r^{\varrho_1(r)} \quad \text{for an infinity of values of } r.$$

The proof of the existence of $\varrho_1(r)$ is similar to that of the existence of $\varrho(r)$ with respect to $\log M(r, f)$.

Now

$$\log M(r, f) < \int_0^r \frac{n(x)}{x}\,dx + r\int_r^\infty \frac{n(x)}{x^2}\,dx$$

$$= N(r) + r\int_r^\infty \frac{N'(x)}{x}\,dx$$

$$= N(r) + r\left[\frac{N(x)}{x}\right]_r^{\infty} + r\int_r^{\infty}\frac{N(x)}{x^2}\,dx$$

$$= r\int_r^{\infty}\frac{N(x)}{x^2}\,dx\,,$$

because

$$N(x) = 0\,(x^{\varrho+\varepsilon})\,.$$

Further,

$$N(x) = A + \int_{r_0}^{x}\frac{n(t)}{t}\,dt < A + \int_{r_0}^{x} t^{\varrho_1(t)-1}\,dt$$

$$\sim \frac{x^{\varrho_1(x)}}{\varrho_1}$$

for x sufficiently large.

Hence

$$\log M(r, f) < \frac{r}{\varrho_1}\int_r^{\infty} x^{\varrho_1(x)-2}\,dx$$

$$\sim \frac{r}{\varrho_1}\,\frac{r^{\varrho_1(r)-1}}{1-\varrho_1}\,.$$

Thus

$$\log M(r, f) < \frac{r^{\varrho_1(r)}}{\varrho_1(1-\varrho_1)}$$

$$= \frac{n(r)}{\varrho_1(1-\varrho_1)}$$

for a sequence of values of r.

Hence

$$\liminf_{r\to\infty}\frac{\log M(r, f)}{n(r)} \leq \frac{1}{\varrho_1(1-\varrho_1)}$$

and the result follows because $\varrho = \varrho_1$ as $\varrho < 1$.

3. Polya [1] has proved that *if $f(z)$ is an entire function of non-integral order ϱ ($\varrho > 0$) then*

$$\liminf_{r\to\infty}\frac{\log M(r, f)}{n(r)} < \infty\,.$$

We give here an alternative proof of this theorem using the properties of proximate order.

Proof: From [2] we have the inequality,

$$\log M(r, f) < A r^q + K r^q \int_{r_0}^{r} \frac{n(t)}{t^{q+1}} dt + K r^{q+1} \int_{r}^{\infty} \frac{n(t)}{t^{q+2}} dt$$

$$< A r^q + K r^q \int_{r_0}^{r} {}^{\varrho}t^{(t)-q-1} dt + K r^{q+1} \int_{r}^{\infty} t^{\varrho(t)-q-2} dt$$

$$\sim A r^q + K r^q \frac{r^{\varrho(r)-q}}{\varrho-q} + K r^{q+1} \frac{r^{\varrho(r)-q-1}}{q+1-\varrho} .$$

Since $f(z)$ is of nonintegral order, so $q < \varrho(r)$ henee

$$\log M(r, f) < 0\,(r^{\varrho(r)}) + A\, r^{\varrho(r)}$$

$$= [A + 0\,(1)]\, n(r)$$

for a sequeuce of values of r tending to ∞.

Hence the result follows.

REFERENCES

[1] G. Polya : Math. Ann., **88**, Pp. 169-183, (1929).

[2] R. Nevanlinna : *Le Theoreme de* Picard-Borel *et la theorie des functions meromorphes*, P. 42, Paris (1929).

Mathematics Department
Karnatak University
Dharwar — India

(Received February 28, 1962)

ÖZET

$f(z)$ bir tam fonksiyon, $M(r,f)$ bu fonksiyonun $|z|=r$ üzerindeki maksimum modülü ve

$$N(r,a) = \int_0^r \frac{n(t,a)}{t} dt$$

olsun. M gerek N gerekse n ile mukayese edilmiştir.