SOME THEOREMS IN A RIEMANNIAN SPACE WITH AN ADDED AFFINE CONNECTION

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Assuming that there exists in a Riemannian space with the metric tensor g_{ij} a given affine connection arbitrarily chosen R. N. SEN [1] has constructed an algebraic system of affine connections generated by the given one such that there are affine connections with respect to which the values of the covariant derivatives of g_{ij} are equal or opposite or connected in some other way. Later, M. C. CHAKI [2] studied in a general way properties connecting affine connections in a Riemannian space with respect to which the metric tensor g_{ij} of the space has equal covariant derivatives. In the present paper some theorems have been obtained by considering the case where the covariant derivatives of g_{ij} are symmetric tensors.

1. Let Γ^t_{ij} be the coefficients of an arbitrarily given affine connection in a Riemannian space with metric tensor g_{ij}, T^t_{ij} be a tensor and let the covariant derivatives of g_{ij} with respect to Γ^t_{ij} and $\Gamma^t_{ij}+T^t_{ij}$ be denoted by a comma and a solidus respectively.

Then

$$g_{ij|k}-g_{ik|j}=(g_{ij,k}-g_{ik,j})+g_{is}(T^s_{kj}-T^s_{jk})+g_{ks}T^s_{ij}-g_{js}T^s_{ik}. \tag{1.1}$$

A necessary and sufficient condition that $g_{ij,k}$ be a tensor symmetric in the indices may be expressed in various ways, e. g., it may be

$$g_{ij,k}-g_{ik,j}=0$$

or

$$g_{ij,k}=\frac{1}{2}(g_{ik,j}+g_{jk,i}), \text{ etc.}$$

Hence from (1.1) it follows that if $g_{ij|k}$ be a symmetric tensor, then $g_{ij,k}$ is so if and only if

$$g_{js}T^s_{ik}-g_{ks}T^s_{ij}-g_{is}(T^s_{kj}-T^s_{jk})=0.$$

It is easy to verify that if $g_{ij|k}$ be a symmetric tensor then $g_{ij,k}$ is so if

$$T^t_{ij}=g^{mt}g_{im,j}.$$

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The affine connection with coefficients $\Gamma^t_{ij} + T^t_{ij}$ where $T^t_{ij} = g^{mt} g_{im,j}$ belongs to SEN's sequence of affine connections which is defined as follows:

Put $$a = \Gamma^t_{ij}, \qquad a^* = \Gamma^t_{ij} + g^{mt} g_{im,j}, \qquad a' = \Gamma^t_{ji}.$$

Then for the affine connection given by a, there exist uniquely two others given by a^* and a' which are respectively called the *associate* and the *conjugate* of a having the property

$$a^{**} = a'' = a.$$

In particular, a is called *self-associate* if $a = a^*$ and *self-conjugate* if $a = a'$. If we now construct the sequence

$$a_1 = a, \quad a_2 = a^*, \quad a_3 = a^{*\prime}, \quad a_4 = a^{*\prime *}, \quad a_5 = a^{*\prime *\prime}, \ldots.$$

then it is seen that the sequence is a finite cyclic sequence of 12 terms (assuming that all are distinct): this is known as SEN's *sequence*. Denoting $\Gamma^t_{ij} + T^t_{ij}$ by d and applying the notions of associate (*) and conjugate (′) of a, as given above, M. C. CHAKI [3] constructed the sequence

$$d_1 = d, \; d_2 = d^*, \; d_3 = d^{*\prime}, \; d_4 = d^{*\prime *}, \ldots, d_{12} = d^{*\prime * \cdots *}$$

and putting

$$\alpha = g^{tm} g_{im,j}, \quad \alpha_c = g^{tm} g_{im,i}, \quad \lambda = g^{tm} g_{ij,m} = \lambda_c$$
$$\beta = g^{tm} g_{is} (\Gamma^s_{mj} - \Gamma^s_{jm}), \quad \beta_c = g^{tm} g_{js} (\Gamma^s_{mi} - \Gamma^s_{im})$$
$$\gamma = T^t_{ij}, \quad \gamma_c = T^t_{ji}, \quad \delta = g^{tm} g_{is} T^s_{mj}, \quad \delta_c = g^{tm} g_{js} T^s_{mi}$$
$$\varepsilon = g^{tm} g_{is} T^s_{im}, \quad \varepsilon_c = g^{tm} g_{js} T^s_{im}$$

obtained their values as follows:

$$\text{(I)} \qquad \begin{cases} d_1 = d = a + \gamma \\ d_2 = a + \alpha - \delta \\ d_3 = a' + \alpha_c - \delta_c \\ d_4 = a + \alpha - \lambda + \beta + \varepsilon_c \\ d_5 = a' + \alpha_c - \lambda + \beta_c + \varepsilon \\ d_6 = a + \alpha + \alpha_c - \lambda + \beta + \beta_c - \gamma_c \\ d_7 = a' + \alpha + \alpha_c - \lambda + \beta + \beta_c - \gamma \\ d_8 = a' + \alpha_c - \lambda + \beta + \beta_c + \delta \\ d_9 = a + \alpha - \lambda + \beta + \beta_c + \delta_c \\ d_{10} = a' + \alpha_c + \beta_c - \varepsilon_c \\ d_{11} = a + \alpha + \beta - \varepsilon \\ d_{12} = a' + \gamma_c \end{cases}$$

From (I) it is seen that the CHRISTOFFEL symbol which is both self-associate and self-conjugate is given by

$$(\text{I}')\qquad \left\{ {t \atop ij} \right\} = \frac{1}{2}(d_l + d_{l+6}), \qquad l = 1, 2, \ldots$$

If Γ^t_{ij} be self-conjugate and $g_{ij,k}$ be symmetric, then the members of the sequence (I) assume the following values:

$$\begin{array}{lll} d_1 = a + \gamma, & d_2 = a + \alpha - \delta, & d_3 = a + \alpha - \delta_c, \\ d_4 = a + \varepsilon_c, & d_5 = a + \varepsilon, & d_6 = a + \alpha - \gamma_c, \\ d_7 = a + \alpha - \gamma, & d_8 = a + \delta, & d_9 = a + \delta_c, \\ d_{10} = a + \alpha - \varepsilon_c, & d_{11} = a + \alpha - \varepsilon, & d_{12} = a + \gamma_c. \end{array}$$

Further, if T^t_{ij} be symmetric in i and j,

$$\begin{array}{lll} d_1 = d_{12}, & d_2 = d_{11}, & d_3 = d_{10}, \\ d_4 = d_9, & d_5 = d_8, & d_6 = d_7. \end{array}$$

Calculating the covariant derivatives of g_{ij} with respect to d_1, d_2, d_3, d_4, d_5 and d_6 for the case where T^t_{ij} is symmetric, Γ^t_{ij} is self-conjugate and $g_{ij,k}$ is symmetric, their values are obtained as

$$g_{tk}(\lambda - \delta - \delta_c), \quad -g_{tk}(\lambda - \delta - \delta_c), \quad g_{tk}(-\lambda + 2\gamma), \quad -g_{tk}(-\lambda + 2\gamma)$$

$$g_{tk}(\lambda - \delta - \delta_c) \text{ and } -g_{tk}(\lambda - \delta - \delta_c)$$

respectively.

Hence we have the following theorem:

Theorem 1. *If the affine connection with coefficients Γ^t_{ij} be self-conjugate, T^t_{ij} be symmetric and $g_{ij,k}$ be symmetric tensor, then the sequence (I) will have six distinct terms*

$$d_1 = d_{12}, \quad d_2 = d_{11}, \quad d_3 = d_{10}, \quad d_4 = d_9, \quad d_5 = d_8, \quad d_6 = d_7$$

such that the covariant derivatives of g_{ij} with respect to the members of each of the pairs (d_1, d_3), (d_2, d_6) have equal values while those with respect to the members of each of the pairs

$$(d_1, d_2), \quad (d_3, d_4), \quad (d_1, d_6), \quad (d_2, d_3), \quad (d_5, d)$$

have equal and opposite values.

Let us now suppose that Γ^t_{ij} is not self-conjugate and T^t_{ij} is symmetric. In this case the covariant derivatives of g_{ij} with respect to d_1, d_3, d_5, d_8, d_{10}, d_{12}

will have their values as follows: The covariant derivative of g_{ij} with respect to d_1 or d_8 is

$$g_{tk}(\lambda - \varepsilon - \varepsilon_c),$$

the covariant derivative of g_{ij} with respect to d_3 or d_{10} is

$$g_{tk}(\lambda - \alpha - \alpha_c - \beta - \beta_c + 2\gamma)$$

and the covariant derivative of g_{ij} with respect to d_5 or d_{12} is

$$g_{tk}(\lambda - \beta - \beta_c - \varepsilon - \varepsilon_c).$$

We now consider the following two properties for the sequence (I):

(I″)
- i) covariant derivatives of g_{ij} with respect to d_1 and d_{12} are equal;
- ii) covariant derivatives of g_{ij} with respect to d_2 and d_3 are equal.

If (i) holds, then

$$\beta + \beta_c = 0 \tag{1.2}$$

Further, if (ii) holds, then in virtue of (I′) we have

$$2\lambda - \alpha - \alpha_c = \beta + \beta_c + \varepsilon + \varepsilon_c - 2\gamma. \tag{1.3}$$

Therefore if (i) and (ii) hold together, then

$$2\lambda - \alpha - \alpha_c = \varepsilon + \varepsilon_c - 2\gamma. \tag{1.4}$$

Suppose that $g_{ij,k}$ is a symmetric tensor. Then the lefthand-side of (1.4) will vanish, so

$$\varepsilon + \varepsilon_c - 2\gamma = 0$$

or

$$T^t{}_{ij} = \frac{1}{2} g^{tm}(g_{is} T^s{}_{jm} + g_{js} T^s{}_{im}). \tag{1.5}$$

Again, if (1.5) holds, then it follows from (1.4) that

$$2\lambda - \alpha - \alpha_c = 0$$

i.e., $g_{ij,k}$ is a symmetric tensor.

We have therefore the following theorem:

Theorem 2. *If in the sequence* (I) *the properties* (i) *and* (ii) *hold simultaneously and $T^t{}_{ij}$ be symmetric, then $g_{ij,k}$ will be a symmetric tensor if and only if*

$$T^t{}_{ij} = \frac{1}{2} g^{tm} (g_{is} T^s{}_{jm} + g_{js} T^s{}_{im})$$

2. Let $\Gamma^t{}_{ijk}$ and $L^t{}_{ijk}$ be the curvature tensors formed with $\Gamma^t{}_{ij}$ and $L^t{}_{ij} = \Gamma^t{}_{ij} + T^t{}_{ij}$ and let $g_{ht} \Gamma^t{}_{ijk} = \Gamma_{hijk}$ and $g_{ht} L^t{}_{ijk} = L_{hijk}$.

If $\Gamma^t{}_{ij}$ is self-conjugate

$$\Gamma^t{}_{ijk} - L^t{}_{ijk} = T^t{}_{ij,k} - T^t{}_{ik,j} + T^t{}_{sk} T_{sij} - T^t{}_{sj} T^s{}_{ik}.$$

Therefore

$$\Gamma_{hijk} - L_{hijk} = g_{ht} (T^t{}_{ij,k} - T^t{}_{ik,j}) + (T^t{}_{sk} T^s{}_{ij} - T^t{}_{sj} T^s{}_{ik}),$$

hence

$$\begin{aligned} L_{hijk} + L_{ihkj} + L_{jkih} + L_{kjhi} &= (\Gamma_{hijk} + \Gamma_{ihkj} + \Gamma_{jkih} + \Gamma_{kjhi}) \\ &- g_{ht} [T^t{}_{ij,k} - T^t{}_{ik,j} - T^t{}_{sj} T^s{}_{ik} + T^t{}_{sk} T^s{}_{ij}] \\ &- g_{it} [T^t{}_{hk,j} - T^t{}_{hj,k} - T^t{}_{sk} T^s{}_{hj} + T^t{}_{sj} T^s{}_{hk}] \\ &- g_{jt} [T^t{}_{ki,h} - T^t{}_{kh,i} - T^t{}_{si} T^s{}_{kh} + T^t{}_{sh} T^s{}_{hk}] \\ &- g_{kt} [T^t{}_{jh,i} - T^t{}_{ji,h} - T^t{}_{sh} T^s{}_{ji} + T^t{}_{si} T^s{}_{jh}]. \end{aligned} \tag{2.1}$$

Let us denote

$$\Gamma_{hijk} + \Gamma_{hjki} + \Gamma_{hkij} \quad \text{by} \quad A_1(\Gamma_{hijk})$$

and

$$\Gamma_{ihjk} + \Gamma_{jhki} + \Gamma_{khij} \quad \text{by} \quad A_2(\Gamma_{hijk}).$$

Since $\Gamma^t{}_{ij}$ is self-conjugate we have by the generalized RICCI's identity [4]

$$g_{hi,jk} - g_{hi,kj} = \Gamma_{hijk} + \Gamma_{ihjk}$$

Therefore

$$A_1(\Gamma_{hijk}) + A_2(\Gamma_{hijk}) = (g_{hk,ij} - g_{hk,ji}) + (g_{hj,ki} - g_{hj,ik}) + (g_{hi,jk} - g_{hi,kj}). \tag{2.2}$$

If $g_{ij,k}$ be a symmetric tensor, the righthand-side of (2.2) vanishes and therefore

$$A_1(\Gamma_{hijk}) + A_2(\Gamma_{hijk}) = 0. \tag{2.3}$$

Since $\Gamma^t{}_{ij}$ is self-conjugate,

$$A_1(\Gamma_{hijk}) = 0 \tag{2.5}$$

hence from (2.3) we get

$$A_2(\Gamma_{hijk}) = 0 \tag{2.5}$$

From (2.4) and (2.5) it follows that

$$\Gamma_{hijk} + \Gamma_{ihkj} + \Gamma_{jkih} + \Gamma_{kjhi} = 0. \tag{2.6}$$

In virtue of (2.6) the equation (2.1) takes the form

$$\begin{aligned} & L_{hijk} + L_{ihkj} + L_{jkih} + L_{kjhi} \\ &= -g_{ht}[T^t{}_{ij,k} - T^t{}_{ik,j} - T^t{}_{sj} T^s{}_{ik} + T^t{}_{sk} T^s{}_{ij}] \\ &\quad - g_{it}[T^t{}_{hk,j} - T^t{}_{hj,k} - T^t{}_{sk} T^s{}_{hj} + T^t{}_{sj} T^s{}_{hk}] \\ &\quad - g_{it}[T^t{}_{ki,h} - T^t{}_{kh,i} - T^t{}_{si} T^s{}_{kh} + T^t{}_{sh} T^s{}_{ki}] \\ &\quad - g_{kt}[T^t{}_{jh,i} - T^t{}_{ji,h} - T^t{}_{sh} T^s{}_{ji} + T^t{}_{si} T^s{}_{jh}] \end{aligned} \tag{2.7}$$

If $T^t{}_{ij} = 0$ then (2.7) reduces to (2.6).

Hence we have the following theorem:

Theorem 3. *If the covariant derivatives of g_{ij} with respect to a self-conjugate affine connection with coefficients $\Gamma^t{}_{ij}$ be symmetric and L_{hijk} be the fully covariant curvature tensor formed with $\Gamma^t{}_{ij} + T^t{}_{ij}$ where $T^t{}_{ij}$ is a tensor, then the components of L_{hijk} satisfy* (2.7).

The identity of Bianchi for the affine connection with coefficients $\Gamma^t{}_{ij}$ is

$$\Gamma^t{}_{ijk,l} + \Gamma^t{}_{ikl,j} + \Gamma^t{}_{ilj,k} = \Gamma^t{}_{isl} V^s{}_{jk} + \Gamma^t{}_{isj} V^s{}_{kl} + \Gamma^t{}_{isk} V^s{}_{lj} \tag{2.8}$$

where

$$V^t{}_{ij} = \Gamma^t{}_{ij} - \Gamma^t{}_{ji}.$$

Multiplying both sides of (2.8) by g_{ht} and summing with respect to t we get

$$\begin{aligned} \Gamma_{hijk,l} + \Gamma_{hikl,j} + \Gamma_{hilj,k} &= g_{ht,l}\Gamma^t{}_{ijk} + g_{ht,j}\Gamma^t{}_{ikl} + g_{ht,k}\Gamma^t{}_{ilj} \\ &\quad + \Gamma_{hisl} V^s{}_{jk} + \Gamma_{hisj} V^s{}_{kl} + \Gamma_{hisk} V^s{}_{lj}. \end{aligned} \tag{2.9}$$

Denote

$$\Gamma_{hijk} - \Gamma_{jihk} - \Gamma_{kijh} \quad \text{by} \quad \Lambda_{hijk}. \tag{2.10}$$

Then

$$\begin{aligned} & \Lambda_{hijk,l} + \Lambda_{hikl,j} + \Lambda_{hilj,k} + \Lambda_{likj,h} = \\ & g_{ht,l}\Gamma^t{}_{ijk} + g_{ht,j}\Gamma^t{}_{ikl} + g_{ht,k}\Gamma^t{}_{ilj} + \Gamma_{hisl} V^s{}_{jk} + \Gamma_{hisj} V^s{}_{kl} + \Gamma_{hisk} V^s{}_{lj} \\ & - g_{lt,h}\Gamma^t{}_{ijk} - g_{lt,j}\Gamma^t{}_{ikh} - g_{lt,k}\Gamma^t{}_{ihj} - \Gamma_{lish} V^s{}_{jk} - \Gamma_{lisj} V^s{}_{kh} - \Gamma_{lisk} V^s{}_{hj} \\ & - g_{jt,l}\Gamma^t{}_{ihk} - g_{jt,h}\Gamma^t{}_{ikl} - g_{jt,k}\Gamma^t{}_{ilh} - \Gamma_{jisl} V^s{}_{hk} - \Gamma_{jish} V^s{}_{kl} - \Gamma_{jisk} V^s{}_{lh} \\ & - g_{kt,l}\Gamma^t{}_{ijh} - g_{kt,j}\Gamma^t{}_{ihl} - g_{kt,h}\Gamma^t{}_{ilj} - \Gamma_{kisl} V^s{}_{jh} - \Gamma_{kisj} V^s{}_{hl} - \Gamma_{kish} V^s{}_{lj}. \end{aligned} \tag{2.11}$$

If $g_{ij,k}$ be a symmetric tensor, then (2.11) reduces to

$$\begin{aligned} &A_{hijk,l} + A_{hikl,j} + A_{hilj,k} + A_{likj,h} \\ &= A_{hisl} V^s{}_{jk} + A_{hisj} V^s{}_{kl} + A_{hisk} V^s{}_{lj} + A_{lisj} V^s{}_{hk} \\ &+ A_{lisk} V^r{}_{jh} + A_{jisk} V^s{}_{hl} + \Gamma_{sihl} V^s{}_{jk} + \Gamma_{sihj} V^s{}_{kl} \\ &+ \Gamma_{sihk} V^s{}_{lj} + \Gamma_{silj} V^s{}_{hk} + \Gamma_{silk} V^s{}_{jh} + \Gamma_{sijk} V^s{}_{hl} \end{aligned} \tag{2.12}$$

If $\Gamma^t{}_{ij}$ is self-conjugate, $V^t{}_{ij} = 0$ and from (2.5) it follows that $A_{hijk} = 0$. Hence (2.12) reduces to an identity.

Therefore we have the following theorem:

Theorem 4. *If the covariant derivative of g_{ij} with respect to an affine connection with coefficients $\Gamma^t{}_{ij}$ be a symmetric tensor and A_{hijk} be defined by (2.10), then the relation (2.12) holds. If, in particular, the affine connection be self-conjugate, then the relation (2.12) reduces to an identity.*

We now refer to the sequence (I) and suppose that for it the properties (i) and (ii) mentioned in (I″) hold. Further we suppose that $g_{ij,k}$ is symmetric and $T^t{}_{ij}$ is so.

In virtue of (i)

$$\beta + \beta_c = 0$$

or

$$g_{ts} V^s{}_{jk} + g_{js} V^s{}_{ik} = 0 \quad \text{where} \quad V^t{}_{ij} = \Gamma^t{}_{ij} - \Gamma^t{}_{ji}.$$

Hence

$$V^t{}_{jk} = -g^{mt} g_{js} V^s{}_{mk}.$$

Therefore

$$V^t{}_{tk} = 0 \quad \text{or} \quad \Gamma^t{}_{tk} = \Gamma^t{}_{kt}. \tag{2.13}$$

Since (i) and (ii) hold, $g_{ij,k}$ is symmetric and so is $T^t{}_{ij}$ it follows from theorem 2 that

$$2T^t{}_{ij} = g^{mt} g_{is} T^s{}_{jm} + g^{tm} g_{js} T^s{}_{im}.$$

So

$$\begin{aligned} 2T^t{}_{tj} &= g^{tm} g_{ts} T^s{}_{jm} + g^{tm} g_{js} T^s{}_{tm} \\ &= T^t{}_{jt} + g^{tm} g_{js} T^s{}_{tm} \end{aligned}$$

whence

$$T^t{}_{tj} = g^{tm} g_{js} T^s{}_{tm} \tag{2.14}$$

Let the fully covariant curvature tensor with respect to d_i of sequence (I) be denoted by $L^{(i)}{}_{hijk}$. Then

$$g^{hi} L^{(1)}{}_{hijk} = \frac{\partial}{\partial x^j} (\Gamma^t{}_{tk} + T^t{}_{tk}) - \frac{\partial}{\partial x^k} (\Gamma^t{}_{tj} + T^t{}_{tj})$$

and

$$g_{hi} L^{(13)}{}_{hijk} = \frac{\partial}{\partial x^j} (\Gamma^t{}_{kt} + T^t{}_{kt}) - \frac{\partial}{\partial x^k} (\Gamma^t{}_{jt} + T^t{}_{jt}).$$

Since $T^t{}_{ij}$ is symmetric and (2.13) holds, we have

$$g^{hi} L^{(1)}{}_{hijk} = g^{hi} L^{(12)}{}_{hijk}. \tag{2.15}$$

Again

$$g^{hi} L^{(2)}{}_{hijk} = \frac{\partial}{\partial x^j} (\Gamma^t{}_{tk} + T'^t{}_{tk}) - \frac{\partial}{\partial x^k} (\Gamma^t{}_{tj} + T'^t{}_{tj})$$

and

$$g^{hi} L^{(3)}{}_{hijk} = \frac{\partial}{\partial x^j} (\Gamma^t{}_{kt} + T'^t{}_{kt}) - \frac{\partial}{\partial x^k} (\Gamma^t{}_{jt} + T'^t{}_{jt})$$

where

$$T'^t{}_{ij} = g^{tm} g_{im,j} - g^{tm} g_{is} T^s{}_{mj}. \tag{2.16}$$

From (2.16) we have

$$T'^t{}_{tk} = g^{tm} g_{tm,k} - T^t{}_{tk}$$

and

$$T'^t{}_{kt} = g^{tm} g_{km,t} - g^{tm} g_{ks} T^s{}_{mt}.$$

Since $T^t{}_{ij}$ is symmetric and (2.14) holds

$$T'^t{}_{tk} = T'^t{}_{kt}.$$

Therefore

$$g^{hi} L^{(2)}{}_{hijk} = g^{hi} L^{(3)}{}_{hijk}. \tag{2.17}$$

Similarly

$$g^{hi} L^{(1)}{}_{hijk} = g^{hi} L^{(4)}. \tag{2.18}$$

Hence taking into account the property (I′) of the sequence (I) we have

$$\begin{aligned} g^{hi} L^{(1)}{}_{hijk} &= g^{hi} L^{(3)}{}_{hijk} = g^{hi} L^{(5)}{}_{hijk} = - g^{hi} L^{(6)}{}_{hijk} \\ &= - g^{hi} L^{(7)}{}_{hijk} = - g^{hi} L^{(16)}{}_{hijk} = - g^{hi} L^{(12)}{}_{hijk} = g^{hi} L^{(12)}{}_{hijk} \\ \text{and} \quad g^{hi} L^{(1)}{}_{hijk} &= g^{hi} L^{(8)}{}_{hijk} = - g^{hi} L^{(2)}{}_{hijk} = - g^{hi} L^{(9)}{}_{hijk}. \end{aligned} \tag{2.19}$$

If, in particular, $T^t{}_{ij} = 0$ the sequence (I) reduces to SEN's sequence and $L^{(i)}{}_{hijk}$ becomes $\Gamma^{(i)}{}_{hijk}$. As in this case

$$g_{hi} \Gamma^{(1)}{}_{hijk} = - g^{hi} \Gamma^{(2)}{}_{hijk}$$

we have

$$\begin{aligned} g^{hi} \Gamma^{(1)}{}_{hijk} &= - g^{hi} \Gamma^{(2)}{}_{hijk} = - g^{hi} \Gamma^{(3)}{}_{hijk} = g_{hi} \Gamma^{(4)}{}_{hijk} \\ &= g_{hi} \Gamma^{(5)}{}_{hijk} = - g^{hi} \Gamma^{(6)}{}_{hijk} = - g^{hi} \Gamma^{(7)}{}_{hijk} = g^{hi} \Gamma^{(2)}{}_{hijk} \\ &= g^{hi} \Gamma^{(2)}{}_{hijk} = - g^{hi} \Gamma^{(10)}{}_{hij} = - g^{hi} \Gamma^{(11)}{}_{hijk} = g^{hi} \Gamma^{(12)}{}_{hijk} \end{aligned} \tag{2.20}$$

Hence we have the following theorem:

Theorem 5. *If in the sequence* (I) *the properties* (i) *and* (ii) *mentioned in* (II″) *hold,* $T^t{}_{ij}$ *is symmetric and further* $g_{ij,k}$ *is a symmetric tensor, then the fully covariant curvature tensor* $L^{(i)}{}_{hijk}$ *formed with* d_i *satisfies the relations* (2.19). *If, in particular, the sequence* (I) *is* SEN'*s sequence, then the relations* (2.20) *hold.*

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ÖZET

Metrik tansörü g_{ij} ile gösterilen bir RIEMANN uzayında keyfî bir afin koneksiyon verildiği takdirde, bunun tarafından doğurulan bir eebrik afin koneksiyon sistemi, sisteme ait koneksiyonlara nazaran g_{ij}' nin kovaryant türevleri aralarında eşit, veya mutlak değerce eşit veya başka herhangi bir eebrik bağıntı tahkik edecek tarzda, inşa edilebileceği R. N. SEN tarafından ispat edilmiştir [1]. Daha sonra, M. C. CHAKI umumî olarak, bir RIEMANN uzayının g_{ij} metrik tansörünün kovaryant türevleri eşit bırakan koneksiyonları bağlayan özelliklerini incelemiştir [2]. Bu yazıda ise g_{ij} tansörünün kovaryant türevleri simetrik olması halinde elde edilen bazı teoremler meydana çıkarılmıştır.