ON THE ORDER AND TYPE OF INTEGRAL FUNCTIONS DEFINED BY DIRICHLET SERIES

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Certain relationships between two or more integral functions representable by DIRICHLET series are established. These results involve the order, lower order, type and lower type of integral functions.

1. Consider the DIRICHLET series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

(A) where $\lambda_{n+1} > \lambda_n, \quad \lambda_1 \geq 0, \quad \lim_{n\to\infty} \lambda_n = \infty,$

$$s = \sigma + it \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty.$$

Let σ_c and σ_a be the abscissa of convergence and abscissa of absolute convergence, respectively, of $f(s)$. Further, let $M(\sigma)$ be the l.u.b. of

$$|f(\sigma + it)| \; (-\infty < t < \infty)$$

where σ is a constant less than σ_a. If $\sigma_c = \sigma_a = \infty$, $f(s)$ defines an integral function.

It is known ([1], p. 67) that, if ϱ be the RITT order (R) and τ be the type of the order ϱ of $f(s)$, then

(1.1) $$\varrho = \limsup_{\sigma\to\infty} \frac{\log_2 M(\sigma)}{\sigma}, \quad (\log_2 x = \log \log x)$$

and

(1.2) $$\tau = \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{e^{\varrho\sigma}}.$$

Naturally the lower order λ and the lower type ν of the order ϱ of $f(s)$ can be given by

(1.3) $$\lambda = \liminf_{\sigma \to \infty} \frac{\log_2 M(\sigma)}{\sigma}$$

and

(1.4) $$\nu = \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\varrho\sigma}}.$$

Using these relations, we, in this paper, establish certain relationships between two or more integral functions representable by DIRICHLET series. The results involve the order, lower order, type and lower type of integral functions and have been given in the form of Theorems and their corollaries.

It will be supposed throughout that the functions are representable by DIRICHLET series, satisfy conditions (A), are integral and hence any of the numbers ϱ, τ, λ, ν of the function can be defined as before.

2. **Theorem 1.** *If $\Phi(s)$, $\Psi(s)$ are two functions of finite orders ϱ_1, ϱ_2 lower orders λ_1, λ_2 and of l. u. b's $M_1(\sigma)$, $M_2(\sigma)$ respectively, then, if*

$$\log_2 M(\sigma) \sim \log \{\log M_1(\sigma) \log M_2(\sigma)\},$$

the order ϱ and lower order λ of the function $f(s)$ of l. u. b. $M(\sigma)$ are such that

(2.1) $$\lambda_1 + \lambda_2 \leq \lambda \leq \varrho \leq \varrho_1 + \varrho_2$$

and, if

$$\log_2 M(\sigma) \sim | \sqrt{\log_2 M_1(\sigma) \log_2 M_2(\sigma)} | ,$$

then

(2.2) $$\sqrt{\lambda_1 \lambda_2} \leq \lambda \leq \varrho \leq \sqrt{\varrho_1 \varrho_2}.$$

Proof: In view of (1.1), for $\Phi(s)$ and $\Psi(s)$, we have

$$\limsup_{\sigma \to \infty} \frac{\log_2 M_1(\sigma)}{\sigma} = \varrho_1$$

and

$$\limsup_{\sigma \to \infty} \frac{\log_2 M_2(\sigma)}{\sigma} = \varrho_2.$$

Therefore, for any $\varepsilon > 0$ and sufficiently large σ,

(2.3) $$\frac{\log_2 M_1(\sigma)}{\sigma} < \left(\varrho_1 + \frac{\varepsilon}{2}\right)$$

and

$$\frac{\log_2 M_2(\sigma)}{\sigma} < \left(\varrho_2 + \frac{\varepsilon}{2}\right). \tag{2.4}$$

Adding the inequalities (2.3) and (2.4) we get

$$\frac{\log \{\log M_1(\sigma) \log M_2(\sigma)\}}{\sigma} < (\varrho_1 + \varrho_2 + \varepsilon). \tag{2.5}$$

Again, using (1.3) for $\Phi(s)$ and $\Psi(s)$, we get, for any $\varepsilon > 0$ and sufficiently large σ,

$$\frac{\log_2 M_1(\sigma)}{\sigma} > \left(\lambda_1 - \frac{\varepsilon}{2}\right) \tag{2.6}$$

and

$$\frac{\log_2 M_2(\sigma)}{\sigma} > \left(\lambda_2 - \frac{\varepsilon}{2}\right). \tag{2.7}$$

Therefore, on adding (2.6) and (2.7), we get

$$\frac{\log \{\log M_1(\sigma) \log M_2(\sigma)\}}{\sigma} > (\lambda_1 + \lambda_2 - \varepsilon). \tag{2.8}$$

From (2.5) and (2.8) we, therefore, get, for any $\varepsilon > 0$ and sufficiently large σ,

$$(\lambda_1 + \lambda_2 - \varepsilon) < \frac{\log \{\log M_1(\sigma) \log M_2(\sigma)\}}{\sigma} < (\varrho_1 + \varrho_2 + \varepsilon).$$

Hence, if

$$\log_2 M(\sigma) \sim \log \{\log M_1(\sigma) \log M_2(\sigma)\},$$

we have for sufficiently large σ,

$$(\lambda_1 + \lambda_2 - \varepsilon) < \frac{\log_2 M(\sigma)}{\sigma} < (\varrho_1 + \varrho_2 + \varepsilon).$$

Therefore on proceeding to limits we get

$$\lambda_1 + \lambda_2 \leq \liminf_{\sigma \to \infty} \frac{\log_2 M(\sigma)}{\sigma} \leq \limsup_{\sigma \to \infty} \frac{\log_2 M(\sigma)}{\sigma} \leq \varrho_1 + \varrho_2$$

or

$$\lambda_1 + \lambda_2 \leq \lambda \leq \varrho \leq \varrho_1 + \varrho_2$$

by the help of (1.1) and (1.3). This proves (2.1).

To establish (2.2) we multiply (2.3) and (2.4) getting

$$\frac{\log_2 M_1(\sigma) \log_2 M_2(\sigma)}{\sigma^2} < \left(\varrho_1 + \frac{\varepsilon}{2}\right)\left(\varrho_2 + \frac{\varepsilon}{2}\right). \tag{2.9}$$

Also multiplying (2.6) and (2.7) we get

$$\left(\lambda_1 - \frac{\varepsilon}{2}\right)\left(\lambda_2 - \frac{\varepsilon}{2}\right) < \frac{\log_2 M_1(\sigma) \log_2 M_2(\sigma)}{\sigma^2}. \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$\left(\lambda_1 - \frac{\varepsilon}{2}\right)\left(\lambda_2 - \frac{\varepsilon}{2}\right) < \frac{\log_2 M_1(\sigma) \log_2 M_2(\sigma)}{\sigma^2} < \left(\varrho_1 + \frac{\varepsilon}{2}\right)\left(\varrho_2 + \frac{\varepsilon}{2}\right) \tag{2.11}$$

for any $\varepsilon > 0$ and sufficiently large σ.

Thus, if

$$\log_2 M(\sigma) \sim | \sqrt{\log_2 M_1(\sigma) \log_2 M_2(\sigma)} | ,$$

from (2.11) we obtain, on proceeding to limits,

$$\sqrt{\lambda_1 \lambda_2} \leq \liminf_{\sigma \to \infty} \frac{\log_2 M(\sigma)}{\sigma} \leq \limsup_{\sigma \to \infty} \frac{\log_2 M(\sigma)}{\sigma} \leq \sqrt{\varrho_1 \varrho_2}.$$

or

$$\sqrt{\lambda_1 \lambda_2} \leq \lambda \leq \varrho \leq \sqrt{\varrho_1 \varrho_2}.$$

Corollary: *If*

$$f_\nu(s) \qquad (\nu = 1, 2, \ldots, m)$$

are m functions of finite orders

$$\varrho_1, \varrho_2, \ldots, \varrho_m$$

lower orders

$$\lambda_1, \lambda_2, \ldots, \lambda_m$$

and of l. u. b.'s

$$M_1(\sigma), \; M_2(\sigma), \; \ldots, \; M_m(\sigma),$$

respectively, then, if

$$\log_2 M(\sigma) \sim \log \{\log M_1(\sigma) \log M_2(\sigma) \ldots \log M_m(\sigma)\},$$

the order ϱ and lower order λ of the function $f(s)$ of l. u. b. $M(\sigma)$ are such that

$$\lambda_1+\lambda_2+\cdots+\lambda_m\leq\lambda\leq\varrho\leq\varrho_1+\varrho_2+\cdots+\varrho_m$$

and, if

$$\log_2 M(\sigma)\sim|\{\log_2 M_1(\sigma)\log_2 M_2(\sigma)\ldots\log_2 M_m(\sigma)\}^{1/m}|$$

then

$$(\lambda_1\lambda_2\ldots\lambda_m)^{1/m}\leq\lambda\leq\varrho\leq(\varrho_1\varrho_2\ldots\varrho_m)^{1/m}.$$

Remark. If $\Phi(s)$, $\Psi(s)$ are of linear regular growth and fulfil the conditions of Theorem 1, then so is $f(s)$.

Since for functions of linear regular growth

$$\lambda_1=\varrho_1,\quad \lambda_2=\varrho_2$$

we get, from (2.1),

$$\varrho_1+\varrho_2\leq\lambda\leq\varrho\leq\varrho_1+\varrho_2$$

which indicates that $\lambda=\varrho$.

This remark also applies to m functions.

Theorem 2. *If $\Phi(s)$, $\Psi(s)$ are two functions of the same finite order ϱ, finite types τ_1, τ_2, lower types ν_1, ν_2 and of l. u. b.'s*

$$M_1(\sigma),\ M_2(\sigma)$$

respectively, then if

$$\log_2 M(\sigma)\sim\log\{\log M_1(\sigma)\log M_2(\sigma)\},$$

the type τ and lower type ν of the order ϱ of the function $f(s)$ of l. u. b. $M(\sigma)$ are such that

$$\nu_1+\nu_2\leq\nu\leq\tau\leq\tau_1+\tau_2 \tag{2.12}$$

and if

$$\log_2 M(\sigma)\sim|\sqrt{\log_2 M_1(\sigma)\log_2 M_2(\sigma)}|,$$

then

$$\sqrt{\nu_1\nu_2}\leq\nu\leq\tau\leq\sqrt{\tau_1\tau_2}.$$

This theorem can be proved easily by following the same lines of proof as that of theorem 1, using (1.2) and (1.4) in place of (1.1) and (1.3). Hence the proof is omitted.

Corollary: The result of this theorem can also be extended to m functions.

Remark: If $\Phi(s)$, $\Psi(s)$ are of perfectly linear growth satisfying the conditions of theorem 2 then $f(s)$ is also of the same nature.

This follows from (2.12), since, for functions of perfectly linear growth,

$$\nu_1 = \tau_1, \quad \nu_2 = \tau_2$$

and hence

$$\tau_1 + \tau_2 \leq \nu \leq \tau \leq \tau_1 + \tau_2$$

which gives $\nu = \tau$.

This remark can be applied to m functions also.

Note: Similar results hold for integral functions represented by TAYLOR series.

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ÖZET

Bu araştırmada Dirichlet serileri ile gösterilebilen iki veya daha fazla integral fonksiyon arasında mevcut bazı bağıntılar elde edilmiştir. Bu bağıntılar bu integral fonksiyonların derecesi, alt derecesi, tip ve alt tiplerini ihtiva etmektedir.