A STEADY TEMPERATURE DISTRIBUTION FOR A LINE VORTEX

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A steady temperature distribution is considered for a vortex flow. It is assumed that temperature distribution depends only on the distance from the vortex axis. Supposing $T = T_1$ and $T = T_0$ for the temperatures respectively at $r = r_1$ and $r = o$, the temperature distribution can be expressed in terms of the exponential integrals $Ei\,(-x)$. It is found that, under some conditions, $T_1 > T_0$ for $Pr < 1$ and $T_1 < T_0$ for $Pr > 1$.

1. Introduction. The vortex flows have been investigated by many authors, because of vortical storms occuring in nature [1] *, [2], of advanced space propulsion [3], [4] and of power generation [5], [6] systems.

In spite of a large number of analytical and experimental investigations of vortices, ** there still exists a great deal of uncertainty concerning this complex flow pattern. It is difficult to find the exact solutions of the full Navier-Stokes equation, except in some simple cases [2], [7], [8]; so that approximate techniques have been used [9],]10], [11] to interpret experimental results.

In the previous studies, the influence of temperature variations has been neglected. But it may be important for some phenomena, such as vortical storms occuring in nature. For this, in the present paper, we have investigated the temperature distribution for a vortex flow given by Rott [7].

Supposing a temperature distribution in the form of $T = T(r)$ *** and taking into consideration the dissipation, we solve the energy equation for the vortex flow mentioned above. The solution has two arbitrary constants. One of them we have defined by means of $T = T_1$ for $r = r_1$ and the other, of regularity, at $r = 0$. The temperature distribution can be given exactly in terms of exponential integrals $Ei\,(-x)$ and T_1 which is the value of the temperature at $r = r_1$.

For sink flows it is found that under some conditions, when $Pr < 1$, the temperature at $r = r_1$ is higher than that which is at $r = 0$, and when $P_r > 1$, it is lower. For source flows, the result is the opposite of that mentioned above. Vortical storms occuring in nature correspond to the second case.

2. Basic equations. For an incompressible viscous fluid, the governing equations are [12]:

$$\mathrm{v} \cdot \nabla \mathrm{v} = -\frac{1}{\varrho} \nabla p + \nu \nabla^2 \mathrm{v}, \tag{2.1}$$

* Numbers in brackets refer to references at the end of the paper.

** See references in [14].

*** (r, ϑ, z) are cylindrical polar coordinates.

(2.2) $$\nabla \cdot \mathbf{v} = 0 \,,$$

(2.3) $$\varrho\, c_{\mathbf{v}}\, \mathbf{v} \cdot \nabla T = \varkappa\, \nabla^2 T + \Phi .$$

The conventional notations are used in equations (2.1) - (2.3). The above equations are respectively the NAVIER-STOKES equation, the continuity equation and the energy equation.

A common solution to (2.1) and (2.2), given by ROTT [7] for the flow of a line vortex, is of the following form:

(2.4) $$v_r = -\,a r, \qquad v_\vartheta = \frac{\Gamma_\infty}{2\,\pi\, r}\left(1 - e^{-\frac{a}{2\,\nu} r^2}\right), \qquad v_z = 2\, a z,$$

where a is a constant depending on flow parameter, such as the flow rate, and $2\,\pi\,\Gamma_\infty$ is the circulation. Substituting (2.4) into (2.3) and supposing $T = T(r)$, we get:

(2.5) $$\frac{d^2 T}{dr^2} + \left(\frac{1}{r} + \lambda\, r\right)\frac{d\,T}{dr} = g\,(r),$$

with

(2.6) $$g\,(r) = -\frac{\varrho\,\nu}{\varkappa}\left[12\, a^2 + \left(\frac{\Gamma_\infty\, a}{2\,\pi\,\nu}\right)^2 e^{-\frac{a}{\nu} r^2}\right],$$

where

$$\lambda = \frac{\varrho\, c_{\mathbf{v}}\, a}{\varkappa} \, .$$

3. **Solution.** The general solution of equation (2.5) is:

(3.1) $$T = B \int \frac{e^{-\frac{\lambda}{2} r^2}}{r}\, dr - \frac{\varrho\,\nu}{\varkappa}\left[\frac{12\, a^2}{\lambda}\int \frac{dr}{r} + \varepsilon \int \frac{e^{-\frac{a}{\nu} r^2}}{r}\, dr\right] + C,$$

where B and C are integration constants and

$$\varepsilon = \left(\frac{\Gamma_\infty\, a}{2\,\pi\,\nu}\right)^2 \Big/ \left[2\left(\frac{\lambda}{2} - \frac{a}{\nu}\right)\right] .$$

It is possible to express the first and the third integrals in terms of exponential integrals $E\,i\,(-x)$: *

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* $$Ei\,(-x) = \int_\infty^x \frac{e^{-t}}{t}\, dt\,, \qquad 0 < x < \infty\,.$$

$$T=\frac{B}{2}\left[Ei\left(-\frac{\lambda}{2}r^2\right)-Ei\left(-\frac{\lambda}{2}\alpha^2\right)\right]-\frac{\varrho\nu}{\varkappa}\left\{\frac{12a^2}{\lambda}\log\frac{r}{r_1}+\right.$$

$$\left.+\frac{\varepsilon}{2}\left[Ei\left(-\frac{a}{\nu}r^2\right)-Ei\left(-\frac{\lambda}{2}\beta^2\right)\right]\right\}+D, \tag{3.2}$$

where α, β, r_1 and D are not independent. Supposing $T=T_1$ for $r=r_1$, we find from equation (3.2):

$$T-T_1=\frac{B}{2}\left[Ei\left(-\frac{\lambda}{2}r^2\right)-Ei\left(-\frac{\lambda}{2}r_1^2\right)\right]-\frac{\varrho\nu}{\varkappa}\left\{\frac{12a^2}{\lambda}\log\frac{r}{r_1}+\right.$$

$$\left.+\frac{\varepsilon}{2}\left[Ei\left(-\frac{a}{\nu}r^2\right)-Ei\left(-\frac{a}{\nu}r_1^2\right)\right]\right\}. \tag{3.3}$$

The temperature at $r=0$ must be finite; using this condition in equation (3.3) and then equating to zero the coefficients of ali of the logarithmic terms, we get: *

$$B=\frac{\varrho\nu}{\varkappa}\,\frac{12a^2}{\lambda}+\varepsilon. \tag{3.4}$$

Substituting (3.4) into (3.3), for the temperature distribution, we have the following form:

$$T-T_1=\left(\frac{6\varrho\nu a^2}{\varkappa\lambda}+\frac{\varepsilon}{2}\right)\left[Ei\left(-\frac{\lambda}{2}r^2\right)-Ei\left(-\frac{\lambda}{2}r_1^2\right)\right]-$$

$$-\frac{\varrho\nu}{\varkappa}\left\{\frac{12a^2}{\lambda}\log\frac{r}{r_1}+\frac{\varepsilon}{2}\left[Ei\left(-\frac{a}{\nu}r^2\right)-Ei\left(-\frac{a}{\nu}r_1^2\right)\right]\right\}. \tag{3.5}$$

This is the required result for the temperature distribution.

4. **Discussion.** In order to find quantitative or qualitative results, one must introduce dimensionless quantities. For convenience, the following dimensionless quantities are introduced:

$$\xi=\frac{r^2}{r_1^2},\qquad \tau=\frac{c_v r_1^2}{\Gamma_\infty^2}\,T. \tag{4.1}$$

Since the radial flow rate is $2\pi Q$, supposing $Q_1=-2Q$, we find $a=Q_1/r_1^2$; using equation (4.1) and $a=Q_1/r_1^2$, from equation (3.3) we get:

$$\tau-\tau_1=\frac{\bar{B}}{2}\left[Ei\left(-\frac{Pe}{2}\xi\right)-Ei\left(-\frac{Pe}{2}\right)\right]-\left\{6\frac{\delta^2}{Re}\log\xi+\right.$$

$$\left.+\frac{Pe}{8\pi^2(Pr-2)}\left[Ei(-Re\,\xi)-Ei(-Re)\right]\right\}, \tag{4.2}$$

where $\bar{B}$ is an arbitrary constant, $Re=Q_1/\nu$ is the Reynolds number, $Pr=\varrho c_v\nu/\varkappa$ is the modified Prandtl number, $Pe=(Pr)(Re)$ is the Peclet number and $\delta=Q_1/\Gamma_\infty$. For regularity at $\xi=0$, we find:

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* $Ei(-x)=\gamma+\log x+\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n.n!}$ where $\gamma=0{,}5772......$ is Euler's constant [18].

$$\bar{B} = \frac{Pe}{4\pi^2(Pr-2)} + \frac{6\delta^2}{Re}. \tag{4.3}$$

Substituting (4.3) into (4.2) we get:

$$\begin{aligned}\tau - \tau_1 = \left[6\frac{\delta^2}{Re} + \frac{Pe}{8\pi^2(Pr-2)}\right]\left[Ei\left(-\frac{Pe}{2}\xi\right) - Ei\left(-\frac{Pe}{2}\right)\right] - \\ - \left\{6\frac{\delta^2}{Re}\log\xi + \frac{Pe}{8\pi^2(Pr-2)}\left[Ei(-Re\,\xi) - Ei(-Re)\right]\right\}.\end{aligned} \tag{4.4}$$

The temperature distribution for large values of ξ, takes the following form:

$$\begin{aligned}\tau - \tau_1 = -\left[6\frac{\delta^2}{Re} + \frac{Pe}{8\pi^2(Pr-2)}\right]Ei\left(-\frac{Pe}{2}\right) - \\ - \frac{Pe}{8\pi^2(Pr-2)}Ei(-Re) - 6\frac{\delta^2}{Re}\log\xi.\end{aligned} \tag{4.5}$$

Thus, the temperature distribution has a logarithmic form for the large values of ξ. If $\tau = \tau_0$ for $\xi = 0$, from equation (4.2), we have:

$$\begin{aligned}\tau_0 - \tau_1 = \left[6\frac{\delta^2}{Re} + \frac{Pe}{8\pi^2(Pr-2)}\right]\left[\gamma + \log\frac{Pe}{2} - Ei\left(-\frac{Pe}{2}\right)\right] - \\ - \frac{Pe}{8\pi^2(Pr-2)}\left[\gamma + \log Re - Ei(-Re)\right].\end{aligned} \tag{4.6}$$

For many physical flows of interest, $\delta^2 < 10^{-4}$, therefore δ^2 / Re can be neglected. For some liquids $Pr > 1$, hence $Pe > Re$ and $\log\frac{Pr}{2} > 0$, and $\tau_0 - \tau_1 > 0$; *i. e.* the temperature on the axis is higher than that at $r = r_1$. But this result is the opposite in the case of $Pr < 1$. Note that this result is true for a sink flow, namely that the flow rate is negative; but for the vortical storms occuring in nature, the flow rate is positive. Thus, we have to expect an opposite result when $Pr = 0.7$ (air) is supposed for vortical storms.

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ÖZET

Bir girdap hareketi için zamana bağlı olmıyan temperatür dağılımı tetkik edilmiştir. Temperatür dağılımının yılmz girdap ekseninden olan uzaklığa bağlı olduğu kabul edilmiştir. Temperatür dağılımı, $r=r_1$ ve $r=o$ daki temperatürleri sırasiyle $T=T_1$ ve $T=T_0$ farzederek, $Ei(-x)$ eksponensiel integralleri cinsinden ifade edilebilmiştir. Bazı şartlar altında, $Pr<1$ için $T_1>T_0$ ve $Pr>1$ için $T_1<T_0$ bulunmuştur.