THE PROBLEM OF THE STRIP COMPOSED OF TWO DIFFERENT MATERIALS IN PLANE ELASTOSTATICS

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In this paper, a method is given for the solution of the problem of a strip composed of two different materials in plane elastostatics, by using the analytic continuation technique. The problem is reduced, to a differential-difference equation, and its solution is found by using the FOURIER integral method.

The problem of a strip composed of only one material was solved by V. T. BUCHWALD [1] by using an analytic continuation method. In the problem for two different materials considered here, besides the analytic continuation method, another idea given in a previous paper [2] is also used.

1. **Introduction.** The strip problems are usually solved by using two different methods. One of them is the FOURIER integral method, and the other is the eigenfunction expansion method. In the classic work by FILON (1902), the FOURIER integral method is used. Later on, many authors following FILON, including HAWLAND (1929), GREEN (1939), HOPKINS (1950) and SNEDDON (1951) have obtained FOURIER integral solutions of the infinite strip problems. The eigenfunction expansion method has been used by SMITH (1952), KOITER (1954), FRIEDMAN (1956) and MORLEY (1963).

We know that, in an isotropic, homogeneous medium, the stress components of the two-dimensional theory of elasticity r_{xx}, r_{yy}, r_{xy} in cartesian, and r_{nn}, r_{ss}, r_{ns} in the curvilinear coordinates are given by the formulae:

$$\vartheta = r_{xx} + r_{yy} = 2\{\Omega'(z) + \bar{\Omega}'(\bar{z})\}, \tag{1.1}$$

$$F = r_{xx} - r_{yy} + 2i\, r_{xy} = -2\{r\,\bar{\Omega}''(\bar{z}) + \bar{\Psi}'(\bar{z})\}, \tag{1.2}$$

$$\vartheta' = r_{nn} + r_{ss} = \vartheta, \tag{1.3}$$

$$F' = r_{nn} - r_{ss} + 2i\, r_{ns} = e^{-2i\alpha} F, \tag{1.4}$$

where the functions $\Omega(z)$ and $\psi(z)$ are analytic in the region occupied by the material, except for isolated singularities which correspond to any point loads. α is the angle between the normal and the real axis. The bars indicate complex conjugate functions and variables in the usual way. The notation is based on that of GREEN-ZERNA (2). The complex displacement $D = u + iv$ is given by

(1.5) $$2\mu D = k\,\Omega(z) - z\,\bar{\Omega}'(\bar{z}) - \bar{\Psi}(\bar{z})$$

where μ is LAMÉ's constant, and $k = 3 - 4\sigma$, σ being POISSON's ratio. From (1.1) and (1.2) we get,

(1.6) $$\Phi(z, \bar{z}) = r_{yy} - i\,r_{xy} = \Omega'(z) + \bar{\Omega}'(\bar{z}) + z\,\bar{\Omega}''(\bar{z}) + \bar{\Psi}'(\bar{z}).$$

2. **The infinite strip.** Assume that there are two mediums, one of which occupies the region A, and the other the region B.

The region A is $(\ 0 \leq y \leq 1,\ -\infty \leq x \leq +\infty)$.

The region B is $(-1 \leq y \leq 0,\ -\infty \leq x \leq +\infty)$.

The region S is $(-1 \leq y \leq +1,\ -\infty \leq x \leq +\infty)$.

In the region A, the state of elasticity may be expressed in terms of two arbitrary functions of the complex variable z, $\Omega_1(z)$ and $\psi_1(z)$ and two elastic constants μ_1, k_1.

In the region B, the state may be expressed by two arbitrary functions $\Omega_2(z)$ and $\psi_2(z)$ and two elastic constant μ_2, k_2. But the functions $\Omega_1(z)$, $\psi_1(z)$, $\Omega_2(z)$ and $\psi_2(z)$ are analytic in the region S occupied by the materials from $y = -1$ to $y = +1$, except for isolated singularities which correspond to any point loads. In this paper, we consider the simple case $\mu_1 = \mu_2 = \mu$.

Denote the strip $1 < y < 3$ by P and the strip $-3 < y < -1$ by Q.

A. An analytic continuation from region A to region P.

We follow the same steps as in [1]: In the region A, the condition on the boundary $(y = 1)$ is

(2.1) $$\Phi_1 = f(x) \quad , \quad (y = 1)$$

where $f(x)$ is a prescribed function, and the stresses relation from (1.6) is

(2.2) $$\Phi_1(z, \bar{z}) = r_{yy} - i\,r_{xy} = \Omega_1'(z) + \bar{\Omega}_1'(\bar{z}) + z\,\bar{\Omega}_1''(\bar{z}) + \bar{\Psi}_1'(\bar{z})..$$

The function $\Omega_1'(z)$ may be continued analytically into the region P by the definition

(2.3) $$\Omega_{1P}'(z) = -z\,\bar{\Omega}_1''(z - 2i) - \bar{\Omega}_1'(z-2i) - \bar{\Psi}_1'(z-2i) \qquad (\text{for } z \text{ in } P).$$

Because $\Omega_1'(z)$, $\psi_1'(z)$ are analytic in the region S, the functions $\bar{\Omega}_1'(z-2i)$, $\bar{\Psi}'(z-2i)$ are analytic for z in P, and the function $\Omega_{1P}'(z)$ as defined in (2.3) is analytic in P. Thus taking the complex conjugate of (2.3), and replacing $\bar{z}$ by $z - 2i$, we obtain, for z in S

(2.4) $$\psi_1'(z) = -(z - 2i)\,\Omega_1''(z) - \Omega_1'(z) - \bar{\Omega}_{1P}'(z - 2i).$$

Since $\Omega_{1P}'(z)$ is analytic for z in P, by reflexion in $y = 1$, $\bar{\Omega}_{1P}'(z - 2i)$ is analytic in S, and, therefore, $\psi_1'(z)$ as defined in (2.4) is analytic for z in S. Substituting the complex conjugate of (2.4) in (2.2), we get

(2.5) $$\Phi_1(z, \bar{z}) = \Omega_1'(z) - \Omega_{1_P}'(\bar{z} + 2i) + (z - \bar{z} - 2i)\,\bar{\Omega}_1''(\bar{z}),$$

for z in S. Thus, for $z = x + i$, we find, using (2.1), that

(2.6) $$\Omega_1'(x + i) - \Omega_{1_P}'(x + i) = f(x).$$

B. An analytic continuation from region B to region Q.

Here also we follow the same steps as in [1]. Similarly in the region B the condition on the boundary ($y = 1$) is

(2.7) $$\Phi_2 = g(x) \quad , \quad (y = -1)$$

where $g(x)$ is a prescribed function, and the stresses relation from (1.6) is

(2.8) $$\Phi_2(z, \bar{z}) = r_{yy} - i\,r_{xy} = \Omega_2'(z) + \Omega_2'(\bar{z}) + z\,\bar{\Omega}_2''(\bar{z}) + \bar{\Psi}_2'(\bar{z}).$$

The function $\Omega_2(z)$ may be continued analytically into the region Q by

(2.9) $$\Omega_{2_Q}'(z) = -z\,\bar{\Omega}_2''(z + 2i) - \bar{\Omega}_2'(z + 2i) - \bar{\Psi}_2'(z + 2i).$$

Thus, for z in S,

(2.10) $$\psi_2'(z) = -(z + 2i)\,\Omega_2''(z) - \Omega_2'(z) - \Omega_{2_Q}'(z + 2i).$$

Taking the complex conjugate of (2.10), and substituting in (2.8), we have

(2.11) $$\Phi_2(z, \bar{z}) = \Omega_2'(z) - \Omega_{2_Q}'(\bar{z} - 2i) + (z - \bar{z} + 2i)\,\bar{\Omega}_2''(\bar{z})$$

for z in S. Hence for $z = x - i$, we find, using the (2.7) that

(2.12) $$\Omega_2'(x - i) - \Omega_{2_Q}'(x - i) = g(x).$$

C. An analytic continuation from region A to region B.

We first recall some physical principles :

a. On the common boundary of the two different materials, the normal and the shearing stresses are the same :

(2.13) $$(r_{nn} + i\,r_{ns})_A \equiv (r_{nn} + i\,r_{ns})_B.$$

b. On the same boundary the displacements are the same :

(2.14) $$D_A \equiv D_B.$$

In our problem the common boundary of the regions A and B is the real axis. The derivative of (1.5) is

(2.15) $$2\mu D' = k\,\Omega'(z) - \bar{\Omega}'(\bar{z}) - z\,\bar{\Omega}''(\bar{z})\,\frac{d\bar{z}}{dz} - \bar{\Psi}'(\bar{z})\,\frac{d\bar{z}}{dz}\,.$$

Along the real axis we have

(2.16) $$z = x\ ,\quad \frac{d\bar{z}}{dz} = 1\ ,\quad \alpha = \frac{\pi}{2}\quad \text{and}\quad F' = -F\,.$$

The derivative of the displacement along the real axis from (2.15) and (2.16) is

(2.17) $$2\mu D' = k\,\Omega'(x) - \bar{\Omega}'(x) - x\,\bar{\Omega}''(x) - \bar{\Psi}'(x)\,.$$

The sum $(\vartheta' + F' + 4\mu D')$ from (1.1), (1.2), (1.3), (1.4) and (2.17) along the real axis is

(2.18) $$\vartheta' + F' + 4\mu D' = 2\,(k+1)\,\Omega'(x),$$

or

(2.19) $$2\,(r_{nn} + i\,r_{ns}) + 4\mu D' = 2\,(k+1)\,\Omega'(x)\,.$$

From (2.13), (2.14), (2.18) and $\mu_1 = \mu_2 = \mu$, along the real axis we have

(2.20) $$2\,(k_1+1)\,\Omega_1'(x) = 2\,(k_2+1)\,\Omega_2'(x)\,.$$

By the use of an analytic continuation, (2.20) gives

(2.21) $$\Omega_2(z) = q\,\Omega_1(z)\,,$$

where

$$q = \frac{k_1+1}{k_2+1}\,.$$

From (1.5) in the region A and B we have

(2.22) $$2\mu D_1 = k_1\,\Omega_1(z) - z\,\bar{\Omega}_1'(\bar{z}) - \bar{\Psi}_1(\bar{z})$$

(2.23) $$2\mu D_2 = k_2\,\Omega_2(z) - z\,\Omega_2'(\bar{z}) - \bar{\Psi}_2(\bar{z})\,.$$

From the condition (2.14), along the real axis, the relations (2.22) and (2.23) must coincide. The equations (2.22) and (2.23) give

(2.24) $$\bar{\Psi}_2(\bar{z}) = \bar{\Psi}_1(\bar{z}) + (k_2\,q - k_1)\,\Omega_1(z) + (1-q)\,z\,\bar{\Omega}_1'(\bar{z})\,.$$

Thus taking the complex conjugate of (2.24), we obtain,

(2.25) $$\psi_2(z) = \psi_1(z) + c\,\bar{\Omega}_1(\bar{z}) + c\,\bar{z}\,\Omega_1'(z)$$

where

$$k_2 q - k_1 = 1 - q = c.$$

(2.21) and (2.25) are the expressions of an analytic continuation between the region A and the region B.

D. Exact forms of the differential-difference equation.

The derivative of (2.25) is

$$\psi_2'(z) = \psi_1'(z) + c\frac{d\bar{z}}{dz}\,\bar{\Omega}_1'(\bar{z}) + c\frac{d\bar{z}}{dz}\,\Omega_1'(z) + c\,\bar{z}\,\Omega_1''(z). \tag{2.26}$$

Subtituting the values of (2.4) and (2.10) into the equation (2.26) we have

$$\begin{aligned}-(z+2i)\,\Omega_2''(z) - \Omega_2'(z) - \Omega_{2_Q}'(z+2i) = \\ = -(z-2i)\,\Omega_1''(z) - \Omega_1'(z) - \bar{\Omega}_{1_p}'(z-2i) + \\ + c\frac{d\bar{z}}{dz}\,\bar{\Omega}_1'(\bar{z}) + c\frac{d\bar{z}}{dz}\,\Omega_1'(z) + c\,\bar{z}\,\Omega_1''(z).\end{aligned} \tag{2.27}$$

For $z = x$ from (2.16) we obtain

$$-2i(q+1)\,\Omega_1''(x) + \bar{\Omega}_{1_P}'(x-2i) - \Omega_{2_Q}'(x+2i) - c\,\Omega_1'(x) = 0. \tag{2.28}$$

Taking the complex conjugate of (2.28), we have

$$2i(q+1)\,\bar{\Omega}_1''(x) + \Omega_{1_P}'(x+2i) - \Omega_{2_Q}'(x-2i) - c\,\Omega_1'(x) = 0. \tag{2.29}$$

This is our differential-difference equation.

E. Solution.

Equations (2.6), (2.12) and (2.29) are sufficient to determine $\Omega_1'(z)$. The solution may be expressed as the sum of a complementary function and a particular integral. To obtain the latter, $\Omega_1'(z)$ is expressed as the FOURIER integral

$$\Omega_1'(z) = \int_{-\infty}^{+\infty} \Phi(t)\, e^{-izt}\, dt. \tag{2.30}$$

Let $\Phi_P(t)$, $\Phi_Q(t)$, $F(t)$ and $G(t)$ be the FOURIER transforms of $\Omega_{1_P}(z)$, $\Omega_{2_Q}(z)$, $f(t)$ and $g(x)$, respectively ; taking the transform of (2.6), we have

$$\Phi(t) - \Phi_P(t) = e^{-t}\, F(t), \tag{2.31}$$

where

$$2\pi F(t) = \int_{-\infty}^{+\infty} f(x)\, e^{ixt}\, dx.$$

Similarly, from (2.12) we obtain

$$q\,\Phi(t) - \Phi_Q(t) = e^t\,G(t)\,. \tag{2.32}$$

Finally, transforming (2.29), we have

$$2(q+1)\,t\,\bar{\Phi}(-t) - c\,\Phi(t) + e^{2t}\,\Phi_P(t) - e^{-2t}\,\Phi_Q(t) = 0\,. \tag{2.33}$$

Eliminating Φ_P and Φ_Q from (2.31), (2.32) and (2.33), we have

$$(e^{2t} - q\,e^{-2t} - c)\,\Phi(t) + 2(q+1)\,t\,\bar{\Phi}(-t) = 2\,H(t), \tag{2.34}$$

where

$$e^t\,F(t) - e^{-t}\,G(t) = 2\,H(t)\ ^{1)}.$$

Replacing t by $-t$ into (2.34), taking the complex conjugate of (2.34) and eliminating $\bar{\Phi}(-t)$, we obtain

$$\{\,(e^{-2t} - q\,e^{2t} - c)\,(e^{2t} - q\,e^{-2t} - c) + 4\,(q+1)^2\,t^2\,\}\,\Phi(t) =$$

$$2\,(e^{-2t} - q\,c^{2t} - c)\,H(t) - 4\,(q+1)\,t\,\bar{H}(-t)\ ^{2)}. \tag{2.35}$$

This expression for $\Phi(t)$ when substituted in (2.30), gives a particular integral for $\Omega_1'(z)$.

To obtain the complementary function, we suppose that $f(x)$ and $g(x)$ are idendically zero. We may drop the subscripts P and Q, then $\Omega_1'(z)$ becomes continuous across $y = \pm 1$. Our three equations (2.6), (2.12) and (2.29) reduce to the one simple homogeneous equation,

$$2i\,(q+1)\,\bar{\Omega}_1''(x) + \Omega_1'(x+2i) - q\,\Omega_1'(x-2i) - c\,\Omega_1'(x) = 0. \tag{2.36}$$

By the use of an analytic continuation (2.36) gives

$$2i\,(q+1)\,\bar{\Omega}_1''(z) + \Omega_1'(z+2i) - q\,\Omega_1'(z-2i) - c\,\Omega_1'(z) = 0. \tag{2.37}$$

Assume that $\Omega_1'(z) = 0\,(e^{c\,|x|})$ for large $|x|$, where c is a positive constant. Thus $\Omega_1'(z)$ has no FOURIER transform but we may define

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1) Taking $k_1 = k_2$ we have

$$sh\ 2t\ \Phi(t) + 2t\ \bar{\Phi}(-t) = H(t),$$

which is in accordance with V. T. BUCHWALD's result [1].

2) Taking $k_1 = k_2$ we have

$$\{\,sh^2\ 2t - 4\,t^2\,\}\ \Phi(t) = sh\ 2t\ H(t) + 2t\ \overline{H}(-t),$$

which is also in accordance with V. T. BUCHWALD's result [1].

$$2\pi\,\Phi_+(t)=\int_0^{\infty}\Omega_1'(z)\,e^{izt}\,dz,$$

(2.38)

$$2\,\pi\,\Phi_-(t)=\int_{-\infty}^{0}\Omega_1'(z)\,e^{izt}\,dz,$$

so that

(2.39) $$\Omega_1'(z)=\int_{ic-\infty}^{ic+\infty}\Phi_+(t)\,e^{-iet}\,dt+\int_{-ic-\infty}^{-ic+\infty}\Phi_-(t)\,e^{-iet}\,dt\,.$$

Substitution of (2.39) in (2.37) gives

(2.40) $$\int_{ic-\infty}^{ic+\infty}\{2\,(q+1)\,t\,\bar{\Phi}_+(-t)+(e^{2t}-q\,e^{-2t}-c)\,\Phi_+(t)\}\,e^{-izt}\,dt+$$

$$+\int_{-ic-\infty}^{-tc+\infty}\{2\,(q+1)\,t\,\bar{\Phi}\_(-t)+(e^{2t}-q\,e^{-2t}-e)\,\Phi\_(t)\}\,e^{-izt}\,dt=0.$$

In accordance with TITCHMARSH [1], the necessary and sufficient conditions for (2.40) are

(2.41) $$2\,(q+1)\,t\,\bar{\Phi}_+(-t)+(e^{2t}-q\,e^{-2t}-c)\,\Phi_+(t)=-\,2\varkappa_R\,(t),$$

$$2\,(q+1)\,t\,\bar{\Phi}\_(-t)+(e^{2t}-q\,e^{-2t}-c)\,\Phi\_(x)=2\varkappa_R\,(t),$$

where $\varkappa_R(t)$ is any analytic function in the strip R, $-c<Z(t)<c$. Taking the conjugate of (2.41), and eliminating $\bar{\Phi}_+(-t)$, $\bar{\Phi}_-(-t)$

(2.42) $$-\Phi_+(t)=+\Phi_-(t)=\frac{2\,(q\,c^{2t}-e^{-2t}+c)\,\varkappa_R\,(t)+4\,(q+1)\,t\bar{\varkappa}_R(-t)}{-\,(e^{2t}-q\,e^{-2t}-c)\,(e^{-2t}-q\,e^{2t}-c)-4\,(q+1)^2\,t^2}\,\cdot$$

These expressions for $\Phi_+(t)$ and $\Phi_-(t)$, when substituted in (2.39), give the comlementary function,

(2.43) $$\Omega_1'(z)=\oint\frac{2\,(q\,e^{2t}-e^{-2t}+c)\,\varkappa_R\,(t)+4\,(q+1)\,t\bar{\varkappa}_R(-t)}{-\,(e^{2t}-q\,e^{-2t}-c)\,(e^{-2t}-q\,e^{2t}-c)-4\,(q+1)^2\,t^2}\,e^{-izt}\,dt.$$

The complete general solution is obtained as the sum of the particular integral, given by (2.35) and (2.30) and the complementary function given by (2.43).

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ÖZET

Bu yazıda, farklı iki malzemeden müteşekkil çubuk (strip) problemi için, analitik devam tekniği kullanılarak bir çözüm metodu verildi. Bu metodla differensial - differens denklemlere gidilmekte, FOURIER İntegral metodu kullanılarak çözüm elde edilmektedir.