STANDARD ERROR FRACTIONAL FUNCTIONAL PROGRAMMING

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The object of this paper is to replace any standard error fractional functional programming differing from each other hy only a change in sign in the functional and in one constraint. Futhermore, it is shown that the two problems to which any programming problem is reduced are both convex programming problems. The paper consists of two different sections, the first of which deals with the transformation of the problem while in the second the main body of the paper is developed and a particular case of the problem is discussed.

Introduction. The problems we shall deal with may be called the Programming Problems with Standard Error Fractional Functional (S.E.F.F.). Members belonging to this class of mathematical programming have been found in various contexts (mostly in statistical problems). The mathematical model for S.E.F.F. programming can be stated as follows :

Maximize

$$R(X) = \frac{C'X - [X'BX]^{\frac{1}{2}} + \alpha}{D'X + \beta}$$

subject to

$$(A) \qquad AX \leq b\,, \qquad X \geq 0$$

where

C, D and X are $(n \times 1)$ column vectors

b is a $(m \times 1)$ column vector

A is a $(m \times n)$ matrix

B is a $(m \times n)$ positive semi definite matrix

$'$ denotes the prime of a vector

α and β are arbitrary scalar constants.

We assume that S, the set of feasible solutions is regular *i. e.*

$$S = \{ X \mid AX \leqq b, \; X \geqq 0 \}$$

is non empty and bounded.

The object of this paper is to replace any standard error fractional functional programming problems with at most two standard error programming problems [2]. These two problems differ from each other by only a change in sign in the functional and in one constraint. The two reduced problems are convex programming problems. This paper consists of two different sections. Section I deals with the transformation of the problem. In Section II the main body of the paper is developed. At the end of this section a particular case of the problem has been discussed.

SECTION I

Consider the following transformation of variables

$$Y = pX \tag{1.1}$$

where $p \geqq 0$ is to be chosen so that

$$D'Y + \beta p = \vartheta \tag{1.2}$$

where $\vartheta \neq 0$ is a specified number.

On multiplying the numerator, the denominator of the objective function and the system of inequalities in (A) by p and taking (1.2) into account, wc obtain the following standard error programming :

Maximize

$$C'Y - [Y'BY]^{\frac{1}{2}} + \alpha p$$

subject to

$$\begin{aligned} AY - bp &\leqq 0 , \\ D'Y + \beta p &= \vartheta , \\ (Y, p) &\geqq 0 . \end{aligned} \tag{1.3}$$

As in [2] we have that every (Y, p) satisfying constraints of (1.3) has $p > 0$.

Outline of the Proof. Suppose $(\hat{Y}, 0)$ satisfies the constraint of (1.3). Let $\hat{X}$ be any element of S: then

$$X_\nu = \hat{X} + \nu \hat{Y} \quad \text{is in } S \text{ for all } \nu > 0$$

Indeed

$$AX_\nu = A\,(\hat{X} + \nu\,\hat{Y}) = A\hat{X} + \nu\,A\hat{Y} \leq b$$

as

$$A\hat{X} \leq b$$

and

$$A\hat{Y} \leq 0$$

therefore X_ν is in S for all $\nu > 0$.

Therefore S is unbounded and contradicts the regularity hypothesis imposed on S.

SECTION II

Theorem 1. *If*

(*i*) $0 <$ Sign $\vartheta <$ Sign $(D'x^* + \beta)$ *for* X^* *and optimal solution of* (A) *and*

(*ii*) (Y^*, p^*) *is an optimal solution of* (1.3) *then* (Y^*/p^*) *is on optimal solution of* (A).

Proof. Suppose the theorem does not hold *i.e.* assume that there exists an optimal solution $X^* \in S$ such that

$$\frac{C'x^* - [X^{*\prime}\,BX^*]^{\frac{1}{2}} + \alpha}{D'X^* + \beta} > \frac{C'\,(Y^*/p^*) - [(Y^*/p^*)'\,B\,(Y^*/p^*)]^{\frac{1}{2}} + \alpha}{D'\,(Y^*/p^*) + \beta}\,. \tag{2.1}$$

By condition (*i*)

$$D'x^* + \beta = e\,\vartheta \tag{2.2}$$

for some $e > 0$.

Consider

$$\begin{cases} \hat{Y} = e^{-1}\,X^*, \\ \hat{p} = e^{-1}. \end{cases} \tag{2.3}$$

Then

$$e^{-1}\,(D'X^* + \beta) = D'\,\hat{Y} + \beta\,\hat{p} = \vartheta \tag{2.4}$$

and $(\hat{Y}, \hat{p})$ also satisfies $AY - bp \leq 0$ *i.e.* $A\hat{Y} - b\hat{p} \leq 0$

$$(\widehat{Y}, \widehat{p}) \geq 0$$

as

$$A\, x^* \leqq b,$$

X^* being the optimal solution of (A),

or

$$A\, e^{-1}\, X^* \leqq e^{-1}\, b,$$

or

$$A\, \widehat{Y} \leqq b\, \widehat{p},$$

or

(2.5) $$A\, \widehat{Y} - b\, \widehat{p} \leqq 0,$$

therefore $(\widehat{Y}, \widehat{p})$ is a solution of (1.3).

Now

$$\frac{C'X^* - [\,X^{*\prime}\, BX^*\,]^{\frac{1}{2}} + \alpha}{D'X^* + \beta} = \frac{e^{-1}\,\{\,C'X^* -]X^{*\prime}\ BX^*]^{\frac{1}{2}} + \alpha\,\}}{e^{-1}\,(D'X^* + \beta)}$$

(2.6) $$= \frac{C'\,\widehat{Y} - [\,\widehat{Y}'\,\beta\,\widehat{Y}\,[^{\frac{1}{2}} + \alpha\,\widehat{p}}{D'\,\widehat{Y} + \beta\,\widehat{p}}$$

$$= \frac{C'\,\widehat{Y} - [\,\widehat{Y}'\,\beta\,\widehat{Y}\,]^{\frac{1}{2}} + \alpha\,\widehat{p}}{\vartheta}\,.$$

Also

$$\frac{C'\,(Y^*/p^*) - [\,(Y^*/p^*)'\ B\,(Y^*/p^*)\,]^{\frac{1}{2}} + \alpha}{D'\,(Y^*/p^*) + \beta} = \frac{C'\,Y^* - [\,Y^{*\prime}\,B\,Y^*\,]^{\frac{1}{2}} + \alpha\,p^*}{D'\,Y^* + \beta\,p^*}$$

(2.7) $$= \frac{C'\,Y^* - [\,Y^{*\prime}\ B\ Y^*\,]^{\frac{1}{2}} + \alpha\,p^*}{\vartheta}\,.$$

But now

(2.8) $$\frac{C'\,X^* - [\,X^{*\prime}\,B\,X^*\,]^{\frac{1}{2}} + \alpha}{D'X^* + \beta} > \frac{C'\,(Y^*/p^*) - [\,(Y^*/p^*)'\ B(Y^*/p^*)\,]^{\frac{1}{2}} + \alpha}{D'\,(Y^*/p^*) + \beta}$$

since by hypothesis (i)

$$\vartheta \neq 0\,.$$

We have

$$C'\,\widehat{Y} + \alpha\,\widehat{p} > C'\,Y^* + \alpha\,p^*$$

vhich contradicts the assumption that (Y^*, p^*) is an optimal solution for (1.3). If sign $D'X^* + \beta < 0$ for X^* an optimal solution of (A), we multiply the numerator and denominator by -1 : the functional is unaltered and for the new denominator we shall have the sign $D'x^* + \beta > 0$.

Thus we may state: For any regular S, to solve the problem (A) it suffices to solve the following two ordinary standard error programming problems :

Maximize

$$C' Y - [Y' B Y]^{\frac{1}{2}} + \alpha p$$

subject to

$$AY - bp \leqq 0 ,$$

(2.9 *a*)
$$D' Y + \beta p = 1 ,$$

$$(Y, p) \geqq 0$$

and

Maximize

$$- C' Y + [Y' B Y]^{\frac{1}{2}} - \alpha p$$

subject to

$$AY - bp \leqq 0 ,$$

(2.9 *b*)
$$- D' Y - \beta p = 1 ,$$

$$(Y, p) \geqq 0 .$$

These two ordinary standard error programming problems can be solved by the available algorithm for quadratic programming as referred in [2].

Theorem 2. *If for all* $X \in S$

$$D'X + \beta = 0$$

then the problems (2.9 a) and (2.9 b) *are both inconsistent.*

Proof. If $D'X + \beta = 0$, it is impossible to obtain

$$\pm (D'X + \beta) = \pm (D'Y + \beta p) = 1.$$

Next we observe that if

$$D'X + \beta = 0 \quad \text{for} \quad X \in S_1 \subset S$$

and

$$D'X + \beta \neq 0 \quad \text{for} \quad X \in S_2 \subset S$$

where

$$S_1 \cup S_2 = S$$

then $D'X+\beta$ being convex, any point in $D'X+\beta=0$ is a limit of a sequence of points $[X^n]$ for which $D'X^n+\beta=\varepsilon_n \neq 0$ and $\varepsilon_n \neq 0$.

We notice that

$$\pm\, p_n (D'x^n + \beta) = \pm\, D'\, Y^n + \beta\, p_n = p_n\, \varepsilon_n = 1 \text{ and } \varepsilon_n \to 0.$$

Thus $p_n = \dfrac{1}{\varepsilon_n} \to \infty$ if an optimal of $R(x)$ [including $R(x)=\infty$] is approached by tending to a point X where $D'X+\beta=0$ then $p_n \to \infty$.

Finally, in particular if the matrix $B=0$ then the given standard error frational programming reduces to a linear fractional functional programming model given as follows :

Maximize

$$R(x) = \frac{C'X+\alpha}{D'X+\beta}$$

subject to

$$AX \leqq b\,,$$

$$X \geqq 0$$

which for its solution is equivalent to the solution of two linear programming problems *viz.*,

(*i*) *Maximize*

$$C'Y + \alpha\, p$$

subject to

$$AY - bp \leqq 0\,,$$

$$D'Y + \beta\, p = 1\,,$$

$$(Y, p) \geqq 0$$

and

(*ii*) *Maximize*

$$-\,C'Y - \alpha\, p$$

subject to

$$AY - bp \leqq 0\,,$$

$$-\,D'Y - \beta\, p = 1\,,$$

$$(Y, p) \geqq 0\,.$$

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ÖZET

Bu yazının gayesi herhangi bir stardard hatalı kesirli fonksiyonelli programlama probleminin yerine en çok iki standard hatalı programlama problemini daima ikame etmek mümkün olduğunu göstermektir. Bu iki problem arasındaki fark fonksiyoneldeki bir işaret ve koşulan şartlardan biridir; üstelik her ikisinin konveks programlama problem sınıfına dahil oldukları gösterilmektedir. Yazı iki kısımdan ibaret olup, birincisinde problemin dönüştürülmesi ele alınmakta, ve ikincisinde de ispatlar tamamen verilmektedir. Son kısımda ayrıca bir özel hal de incelenmiştir.