SPHERICAL AND CIRCULAR VISCOELASTIC INCLUSIONS

R. D. BHARGAVA AND C. B. SHARMA

Motivated by ALFREY's analysis for solving problems in linear viscoelasticity, LEE has introduced a concept of an associated elastic problem to which a viscoelastic problem reduces after removal of its time dependence by application of LAPLACE transform. Using this concept the problems of spherical and circular viscoelastic inclusions have been solved. The results of the purely elastic case can be deduced as a particular case of the above.

1. Introduction. Inclusion problems were initially studied by J. FRENKEL, N. F. MOTT and R. N. NABARRO. Later on J. D. ESHELBY applied the concept of point force. JASWON and BHARGAVA coupled it with complex variable techniques to obtain explicit solutions for some two-dimensional problems. These have been used by BHARGAVA and KAPUR and BHARGAVA and SHARMA to obtain the solution of some more problems. BHARGAVA gave a still simpler approach using the so-called theorem of minimum complementary energy. This theorem states: «The displacement which satisfies the differential equations of equilibrium, as well as the conditions at the boundary surface, yields a smaller value for the potential energy of deformation than any other displacement which satisfies the same conditions at the boundary surface».

The method consists in taking an arbitrary position of equilibrium for the inclusion and the matrix. The sum of energies in both matrix and inclusion is found out and is minimized with respect to the parameters of equilibrium position to obtain the correct position of equilibrium, stresses and the elastic field. BHARGAVA and RADHAKRISHNA applied these concepts. As indicated by E. H. LEE [8] there is a close analogy between viscoelastic and elastic problems. This analogy has been made use of in solving the viscoelastic problems for spherical and circular inclusions.

2. Visco-elastic Elastic Analogy.

In the case of a MAXWELL material the stress strain relation is given by

$$\text{(i)} \qquad \dot{e}_{ij} = \frac{1}{2\mu}\,\dot{s}_{ij} + \frac{1}{2\eta}\,s_{ij}$$

where s_{ij} and e_{ij} are deviatoric stress and strain components, μ is the shear modulus and η is the coefficient of viscosity, whereas a dot signifies differentiation with respect to time. We can very easily obtain the correspondence principle. In elastic the solution replace the dependent variables and the boundary conditions by their LAPLACE transforms and the elastic moduli by the corresponding s-varying moduli. Inversion of the expression so obtained for the transform of dependent variables gives the viscoslastic solutions for these variables. By making use of equilibrium equations and boundary conditions, we can see by comparing with the elastic case, that corresponding expressions for LAME's constants λ and μ are

$$\mu s / [s + (1/\tau)] \quad \text{and} \quad [\lambda s + (K/\tau)] / [s + (1/\tau)]$$

where $\tau = \eta/\mu$ is called the relaxation time, and s is the LAPLACE transform parameter.

3. Spherical Inclusion.

Suppose we have a sphere of radius a undergoing spontaneous dimensional changes to a concentric sphere of radius $a(1 + \delta)$ in the absence of a matrix. On symmetry considerations the equilibrium shape of the inclusion would be a sphere of radius $a(1 + \varepsilon)$, $0 \leqq \varepsilon \leqq \delta$. Let λ' and μ' be LAME's constants of the sphere and λ, μ those of the matrix. The radial, hoop and shear strains in the inclusion are (Superscript ν refers to viscoeiastic case)

$$E^{\nu}_{\gamma\gamma} = E^{\nu}_{\vartheta\vartheta} = -(\delta - \varepsilon), \quad E^{\nu}_{\gamma\vartheta} = 0. \tag{2}$$

The stress components are therefore

$$P^{\nu}_{\gamma\gamma} = P^{\nu}_{\vartheta\vartheta} = 3\,k'(\delta - \varepsilon) \tag{3}$$

where k' is the bulk modulus of the inclusion (It appears at first sight that

$$P^{\nu}_{\gamma\gamma} \quad \text{and} \quad P^{\nu}_{\vartheta\vartheta}$$

are independent of the time factor. It is however not so, as ε would come out to be a function of time). The total strain energy in the inclusion is

$$W^{\nu}_{1} = \frac{9}{2}\,k'(\delta - \varepsilon)^2\,\frac{4}{3}\,\pi a^3 . \tag{4}$$

With respect to the matrix, the internal boundary has undergone a radial displacement equal to $a\varepsilon$. From elementary elasticity theory, the radial, hoop and shear strains are (denoted by small letters)

$$e^{\nu}_{\gamma\gamma} = -\frac{2a^3\varepsilon}{\gamma}, \quad e^{\nu}_{\vartheta\vartheta} = \frac{a^3\varepsilon}{\gamma}, \quad e^{\nu}_{\gamma\vartheta} = 0, \tag{5}$$

and corresponding stresses are therefore given by

$$P^{\nu}_{\gamma\gamma} = -4\mu\varepsilon\,\frac{a^3}{\gamma^3}\,e^{-t/\tau}$$

$$P^{\nu}_{\vartheta\vartheta} = 2\mu\varepsilon\,\frac{a^3}{\gamma^3}\,e^{-t/\tau} \qquad P^{\nu}_{\gamma\vartheta} = 0, \tag{6}$$

where $\tau = \eta/\mu$ denotes relaxation time.

The strain energy in the matrix is

$$W_2^{\nu} = 6\,\mu\,\varepsilon^2\,\frac{4}{3}\,\pi\,a^3\,e^{-t/\tau}. \tag{7}$$

Total energy is

$$W^{\nu} = W_1^{\nu} + W_2^{\nu} = \left[\frac{9}{2}\,k'\,(\delta - \varepsilon)^2 + 6\,\mu\,\varepsilon^2\,e^{-t/\tau}\right]\frac{4}{3}\,\pi\,a^3. \tag{8}$$

By the principle of minimum complementary energy the above expression should be minimum. Putting $\partial W^{\nu}/\partial\varepsilon = 0$, we find that

$$\varepsilon = \frac{3\,k'\,\delta}{3\,k' + 4\quad e^{-t/\cdot}}. \tag{9}$$

It is obvious that $\frac{\partial^2 W^{\nu}}{\partial\,\varepsilon^2} > 0$.

The stresses, strains and energy etc. may be obtained by giving the value of ε in appropriate expressions.

It may be seen that the boundary stresses are

$$P_{\gamma\gamma}^{\nu(b)} = p_{\gamma\gamma}^{\nu(b)} = -\frac{12\,\mu\,k'\,\delta\,e^{-t/\tau}}{3\,k' + 4\,\mu\,e^{-t/\tau}},$$

$$P_{\gamma\vartheta}^{\nu(b)} = p_{\gamma\vartheta}^{\nu(b)} = 0,$$

$$P_{\vartheta\vartheta}^{\nu(b)} = -2\,p_{\vartheta\vartheta}^{\nu(b)} = -\frac{12\,\mu\,k'\,\delta\,e^{-t/\tau}}{3\,k' + 4\,\mu\,e^{-t/\tau}},$$

where (b) denotes the value at the equilibrium interface.

4. Circular Inclusion.

We shall take up the case of circular cylindrical inclusion under conditions of plane strain wherefrom we can get to generalized plane stress case simply by replacing λ by $\frac{2\lambda\,\mu}{\lambda + 2\mu}$.

The circle of radius a undergoing deformations goes to a concentric circle of radius $a\,(1 + \delta)$ in case of sorrounding material is absent. But the presence of matrix checks the free motion of inclusion and restricts it to a circle of radius $a\,(1 + \varepsilon)$ where $0 \leq \varepsilon \leq \delta$. Here the radial displacement is

$$U_{\gamma}^{\nu} = -(\delta - \varepsilon)\,\gamma$$

whence the radial, hoop and shear strains are

$$E_{\gamma\gamma}^{\nu} = E_{\vartheta\vartheta}^{\nu} = -(\delta - \varepsilon), \quad E_{\gamma\vartheta}^{\nu} = 0. \tag{10}$$

If λ', μ' be the LAMB's constants and η' be the coefficient of viscosity, we have after some simplification, for the stresses in the inclusion

(11) $$P^{v}_{\gamma\gamma} = P^{v}_{\vartheta\vartheta} = -2\,\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'}\}$$

and the strain energy is

(12) $$W^{v}_{1} = 2\,\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'}\}\,(\delta - \varepsilon)^2\,\pi\,a^2\,.$$

With respect to the matrix since its boundary has undergone the displacement $a\,\varepsilon$, its radial hoop and shear strains are therefore,

(13) $$e^{v}_{\gamma} = -\left(\frac{a^2 t}{\gamma^2}\right), \quad e^{v}_{\vartheta\vartheta} = \frac{a^2\varepsilon}{\gamma^2}, \quad e^{v}_{\gamma\vartheta} = 0.$$

Then stresses obviously are,

$$p^{v}_{\gamma\gamma} = -\,2\mu\,\frac{a^2\varepsilon}{\gamma^2}\,e^{-t/\tau}\,,$$

(14) $$p^{v}_{\vartheta\vartheta} = 2\mu\,\frac{a^2\varepsilon}{\gamma^2}\,e^{-t/\tau}, \quad p^{v}_{\gamma\vartheta} = 0$$

and the matrix strain energy is

(15) $$W^{v}_{2} = 2\mu\varepsilon^2\,\pi a^2\,e^{-t/\tau}.$$

The total strain energy in the matrix and inclusion is

(16) $$W^{v} = W^{v}_{1} + W^{v}_{2} = 2\,[\,\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'}\}\,(\delta - \varepsilon)^2 + \mu\varepsilon^2\,e^{-t/\tau}\,]\,\pi a^2.$$

The appropriate value of ε is that for which the above expression is minimum. Putting $\frac{\partial W^{v}}{\partial \varepsilon} = 0$, we get

(17) $$\varepsilon = \frac{\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'}\}\,\delta}{\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'} + \mu\,e^{-t/\tau}\}}.$$

Substituting the above value of ε we can get the stresses, strains and strain energy.

The boundary stresses in case of circular inclusion may be seen to be

$$P^{v(b)}_{\gamma\gamma} = p^{v(b)}_{\gamma\gamma} = \frac{-2\,\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'}\}\,\mu e^{-t/\tau}\,\delta}{\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'} + \mu e^{-t/\tau}\}}\,,$$

$$P^{v(b)}_{\gamma\vartheta} = p^{v(b)}_{\gamma\vartheta} = 0,$$

$$P^{v(b)}_{\vartheta\vartheta} = -\,p^{v(b)}_{\vartheta\vartheta} = \frac{-2\,\{k' + (\lambda' + \mu' - k')\,e^{-/\tau'}\}\,\mu e^{-t/\tau}\,\delta}{\{k' + (\lambda' + \mu' - k')\,e^{-t/\tau'} + \mu e^{-t/\tau}\}}\,.$$

The pure elastic case may be deduced from the above case by making the relaxation time approach infinity. The above case reduces to that of BHARGAVA [1].

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INDIAN INSTITUTE OF TECHNOLOGY
KANPUR, INDIA

(Manuscript received May 11, 1965)

ÖZET

ALFREY'in lineer viskoelastisite problemlerinin çözümleri hakkındaki analizinden hareket eden LEE, viskoelâstik probleme bağlı elastik problem kavramını ortaya atmıştır: LAPLACE dönüşümü kullanılmak suretiyle zamana bağlılığın kaldırılması halinde viskoelâstik problem ona bağlı elâstik probleme irca olunur. Bu kavram kullanılmak suretiyle küresel ve dairesel viskoelâstik «inclusion» problemleri çözülmüştür. Elâstik hal için benzer problemlerin çözümleri yukardakilerin özel halleridir.