CONVEX FUNCTIONS AND THEIR APPLICATIONS

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The function $M(r)$, convex with respect to a particular function $\Phi(r)$ and such that $M(a) = 0$, and the non decreasing function $n(t)$, defined by means of the equation

$$M(r) = \int_a^r n(t)\ \Phi'(t)\ dt \qquad (a \geq 0)$$

are considered and some new properties concerning the growth of these functions are obtained.

1. Let $M(r)$ be a convex function with respect to the function $\Phi(r)$, where :

(*i*) $\Phi(r)$ is an absolutely continuous function for $0 < r < \infty$ ($r = 0$ is an admissible value in some cases).

(*ii*) $\Phi(r) \to \infty$ with r. Obviously $\Phi'(r)$ exists and is greater than zero.

Now if $M(a) = 0$, then it is known that (KAMTHAN [9]) *

$$(A) \qquad M(r) = \int_a^r n(t)\ \Phi'(t)\ dt \quad ; \quad a \geq 0$$

where $n(t)$ is a non-decreasing function tending to ∞ with t, having only enumerable discontinuities on the left. Obviously then $n'(t)$ exists almost everywhere.

Now let us introduce a function $\varrho(r)$ which according to KAMTHAN [*K*] is assumed to satisfy the following conditions :

* As this reference occurs frequently in the context, we refer to it as [*K*] in the results that follow.

(1.1) $$\varrho(r) \to \varrho \quad \text{as} \quad r \to \infty\,; \quad \text{where} \quad 0 < \varrho < \infty\,.$$

(1.2) $$\frac{\varrho'(r)\,\Phi(r)}{\Phi'(r)} \to 0 \quad \text{uniformly as} \quad r \to \infty\,.$$

(1.3) $$\overline{\lim_{r\to\infty}} \frac{M(r)}{\exp.\,\{\varrho(r)\,\Phi(r)\}} = 1.$$

Making use of (A), KAMTHAN has found a number of results depending on the growth of $M(r)$ and $n(r)$ with respect to exp. $\{\varrho(r)\,\Phi(r)\}$, which as a particular case reduce to already existing results on Entire Functions respresented by DIRICHLET and TAYLOR-series.

Here we wish to find certain other results on the growth of $M(r)$ and $n(r)$ and prove the following theorems :

2. **Theorem** 1. Let

$$n(r) \sim \exp.\,\{\varrho\,\Phi(r)\}, \quad r \to \infty$$

then the function

$$\frac{\log M(r)}{\Phi(r)}$$

behaves like the function $\varrho(r)$ defined as above satisfying the conditions (1.1) and (1.2).

Proof. Let

$$\varrho(r) = \frac{\log M(r)}{\Phi(r)}\,,$$

then

$$\varrho'(r) = \frac{M'(r)}{M(r)\,\Phi(r)} - \frac{\Phi'(r)}{[\Phi(r)]^2} \cdot \log M(r).$$

By using (A), we find that

(2.1) $$\frac{\varrho'(r)\,\Phi(r)}{\Phi'(r)} = \frac{n(r)}{M(r)} - \frac{\log M(r)}{\Phi(r)}\,;$$

for almost all values of r.

But from the hypothesis,

$$n(r) \sim \exp.\,\{\varrho\,\Phi(r)\} \text{ for large } r,$$

we have

$$M(r) = \int_a^r n(t)\,\Phi'(t)\,dt$$

$$\sim \int_a^r \exp.\,\{\varrho\,\Phi(t)\}\,\Phi'(t)\,dt$$

$$= \frac{1}{\varrho}\,[\exp.\,\{\varrho\,\Phi(t)\}]_a^r$$

$$= \frac{1}{\varrho}\,[e^{\varrho\,\Phi(r)} - e^{\varrho\,\Phi(a)}]$$

$$= \frac{1}{\varrho}\,[n(r) - e^{\varrho\,\Phi(a)}].$$

Hence

(2.2.) $$\frac{M(r)}{n(r)} \sim \frac{1}{\varrho} \quad \text{as} \quad r \to \infty.$$

Again

$$\lim_{r\to\infty} \frac{M(r)}{\exp.\{\varrho \Phi(r)\}} = \frac{1}{\varrho} \cdot$$

Therefore we have for every arbitrarily small $\varepsilon > 0$ and $r \geq r_0$.

$$\frac{1}{\varrho} - \varepsilon < \frac{M(r)}{\exp.\{\varrho \Phi(r)\}} < \frac{1}{\varrho} + \varepsilon$$

i.e. $$\log\left(\frac{1}{\varrho} - \varepsilon\right) + \varrho \Phi(r) < \log M(r) < \log\left(\frac{1}{\partial} + \varepsilon\right) + \varrho \Phi(r).$$

Hence

(2.3) $$\lim_{r\to\infty} \frac{\log M(r)}{\Phi(r)} = \varrho.$$

Thus making use of (2.2) and (2.3) in (2.1) we see that the condition (1.2) for the function $\frac{\log M(r)}{\Phi(r)}$ is satisfied and obviously the condition (1.1) for this function follows from (2.3).

Theorem 2. If

$$r_2 > r_1 > 0, \quad \text{then}$$

(2.4) $$n(r_1) \leq \frac{M(r_2) - M(r_1)}{\Phi(r_2) - \Phi(r_1)} \leq n(r_2).$$

Remark. This generalises the results of SRIVASTAVA ([4] p. 140), KAMTHAN ([1] th. 1).

Proof. Since

$$M(r_2) - M(r_1) = \int_{r_1}^{r_2} n(t)\, \Phi'(t)\, dt,$$

therefore

$$n(r_1)\,[\Phi(r_2) - \Phi(r_1)] \leq M(r_2) - M(r_1) \leq n(r_2)\,[\Phi(r_2) - \Phi(r_2)].$$

Hence

$$n(r_1) \leq \frac{M(r_2) - M(r_1)}{\Phi(r_2) - \Phi(r_1)} \leq n(r_2).$$

Corollary. If

$$R > r > 0 \quad \text{and} \quad R = R(r, k),$$

where k is some postive constant such that

$$\Phi(R) - \Phi(r) \to \Phi(k) \geqq 0 \quad \text{as} \quad r \to \infty, \tag{2.5}$$

which is always possible for a proper choice of $\Phi(r)$ and then of R (for the construction of such functions see KAMTHAN [K]. Then we have

$$\lim_{r\to\infty} [M(R) - M(r)] = \infty . \tag{2.6}$$

Teorem 3. If

$$M_i(r) \ (i = 1,2)$$

be two convex functions with respect to the function $\Phi(r)$ defined as

$$M_i(r) = \int_a^r n_i(t)\ \Phi'(t)\, dt \quad , \quad a > 0 \tag{2.7}$$

where

$$n_2(t) \geqq n_1(t)$$

and also

$$\overline{\lim_{r\to\infty}} \, [n_2(r) - n_1(r)] = \frac{A}{B} \ , \tag{2.8}$$

then

$$B \leqq \underline{\lim_{r\to\infty}} \ \frac{M_2(r) - M_1(r)}{\Phi(r)} \leqq \overline{\lim_{r\to\infty}} \ \frac{M_2(r) - M_1(r)}{\Phi(r)} \leqq A.$$

Proof. From (2.8), for every arbitrarily choosen small number $\varepsilon > 0$ and $r \geqq r_0$, we have

$$n_2(r) - n_1(r) < A + \varepsilon.$$

Also

$$M_2(r) - M_1(r) = \int_a^r [n_2(t) - n_1(t)]\, \Phi'(t)\, dt$$

$$< \int_a^{r_0} [n_2(t) - n_1(t)]\, \Phi'(t)\, dt + \int_{r_0}^r (A + \varepsilon)\ \Phi'(t)\, dt$$

and so

$$\overline{\lim_{r\to\infty}} \frac{M_2(r) - M_1(r)}{\Phi(r)} \leq A. \tag{2.9}$$

Similarly by considering the inequality

$$n_2(r) - n_1(r) > B - \varepsilon \quad \text{for} \quad \varepsilon > 0 \quad \text{and} \quad r \geq r_0$$

it can be proved that

$$\underline{\lim_{r\to\infty}} \frac{M_2(r) - M_1(r)}{\Phi(r)} \geq B. \tag{2.10}$$

Thus the result follows from (2.9) and (2.10).

Corollary. If $A = B$, then

$$[M_2(r) - M_1(r)] \sim A\,\Phi(r).$$

3. Theorem 4. If

$$\frac{\overline{\lim}}{\underline{\lim}}_{r\to\infty} \frac{\log M(r)}{\Phi(r)} = \frac{D}{C}$$

and

$$\frac{\overline{\lim}}{\underline{\lim}}_{r\to\infty} \frac{M(r)}{n(r)} = \frac{A}{B};$$

then

$$B \leq \frac{1}{D} \leq \frac{1}{C} \leq A,$$

where

$$0 < C \leq D < \infty.$$

Proof. We have

$$\underline{\lim_{r\to\infty}} \frac{\log M(r)}{\Phi(r)} = C.$$

Firstly we have to show that $A \geq \frac{1}{C}$; suppose that this statement is not true: then $A < \frac{1}{C}$ and therefore

$$\overline{\lim_{r\to\infty}} \frac{M(r)}{n(r)} < \frac{1}{C};$$

and so

$$M(r) \leq \alpha\, n(r) \quad \text{for} \quad r \geq r_0\,,$$

where

$$\alpha < \frac{1}{C},$$

But $M'(r)$ exists almost everywhere and so

$$M'(r) = n(r)\, d\{\Phi(t)\} \quad \text{for almost all} \quad r \geq r_0\,.$$

Thus

$$\frac{M'(r)}{M(r)} \geq \frac{d\,\Phi(r)}{\alpha}\,. \tag{3.1}$$

Integrating the inequality (3.1) over (r_0, r) we obtain

$$\log M(r) - \log M(r_0) \geq \frac{\Phi(r) - \Phi(r_0)}{\alpha}\,.$$

But $\Phi(r)$ increases with r and so

$$\varliminf_{r\to\infty} \frac{\log M(r)}{\Phi(r)} \geq \frac{1}{\alpha} > C.$$

Hence

$$\varliminf_{r\to\infty} \frac{\log M(r)}{\Phi(r)} > C,$$

which contradicts the hypothesis. Hence

$$A \geq \frac{1}{C}\,.$$

Similarly assuming that $B \leq \frac{1}{D}$ does not hold, we can arrive at a contradiction and therefore the theorem is completly proved.

Corollary. If
$$A = B\,,$$
then
$$C = D = \frac{1}{A}\,.$$

The proof is straightforward.

Remark. For an alternative proof see ([*K*], Theorem 3).

Theorem 5. If $M(r) > 0$, then the convergence (or divergence) of the first integral, given below, implies the convergence (or divergence) of the other and vice-versa, the integrals being given by

$$\left.\begin{aligned} J_1 &= \int_{r_0}^{\infty} \frac{M(n)}{e^{\alpha\,\Phi(n)}}\,\Phi'(n)\,dn \\ J_2 &= \int_{r_0}^{\infty} \frac{n(n)}{e^{\alpha\,\Phi(n)}}\,\Phi'(n)\,dn \end{aligned}\right\}\ \alpha > 0$$

Proof. We already know that

$$M(r) = M(r_0) + \int_{r_0}^{r} n(t)\,\Phi'(t)\,dt.$$

Hence

$$\int_{r_0}^{r} \frac{\Phi'(n)\,dn}{e^{\alpha\,\Phi(n)}} \int_{r_0}^{n} n(t)\,\Phi'(t)\,dt = \int_{r_0}^{r} [\,M(n) - M(r_0)\,]\,\frac{\Phi'(n)}{e^{\alpha\,\Phi(n)}}\,dn$$

$$= \left[\frac{M(n) - M(r_0)}{-\alpha\,e^{\alpha\,\Phi(n)}}\right]_{r_0}^{r} + \frac{1}{\alpha}\int_{r_0}^{r} \frac{n(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\,dn. \tag{3.2}$$

Also

$$\int_{r_0}^{r} \frac{\Phi'(n)}{e^{\alpha\,\Phi(n)}}\,dn \int_{r_0}^{n} n(t)\,\Phi'(t)\,dt = \int_{r_0}^{r} \frac{M(n)\,\Phi'(n)\,dn}{e^{\alpha\,\Phi(n)}} + \frac{M(r_0)}{\alpha}\,[\,e^{-\alpha\,\Phi(r)} - e^{-\alpha\,\Phi(r_0)}\,]. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\int_{r_0}^{r} \frac{M(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\,dn - \frac{M(r_0)}{\alpha\,e^{\alpha\,\Phi(n)}} + \frac{M(r)}{\alpha\,e^{\alpha\,\Phi(r)}} = \frac{1}{\alpha}\int_{r_0}^{r} \frac{N(n)\,\Phi'(n)\,dn}{c^{\alpha\,\Phi(n)}}. \tag{3.4}$$

Proof for convergence (i). During the proof of convergence or divergence of the integrals J_1 and J_2, K shall be used to denote an arbitrarily large positive number and ε will be an arbitrarily small positive number, both being not necesarily the same at each occurrence.

Suppose J_1 is convergent. Then

$$\varepsilon > \int_{r}^{R} \frac{M(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\,dn > \frac{M(r)}{\alpha\,e^{\alpha\,\Phi(r)}}\,[\,1 - e^{-\alpha\{\Phi(R) - \Phi(r)\}}\,]$$

where $R = R(r, k)$, $k > 0$ so that

$$\Phi(R) - \Phi(r) \to \Phi(k) \geq 0 \quad \text{as} \quad r \to \infty. \tag{3.5}$$

and so

$$\frac{M(r)}{e^{\alpha\,\Phi(r)}} \to 0, \quad J \to \infty. \tag{3.6}$$

Hence from (3.4) and (3.6) we have

$$\alpha\, J_1 + H = J_2 \tag{3.7}$$

where H is some finite number less than zero. From (3.7) it follows that J_2 is convergent, since J_1 is assumed to be convergent.

(*ii*) Let now J_2 be convergent and hence from (3.4) we have for large r,

$$\alpha \int_{r_0}^{r} \frac{M(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\, dn + \frac{M(r)}{e^{\alpha\,\Phi(r)}} < A; \tag{3.8}$$

A being some constant, and as

$$\int_{r_0}^{r} \frac{M(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\, dn > [e^{-\alpha\,\Phi(r_0)} - e^{-\alpha\,\Phi(r)}]\ \frac{M(r_0)}{\alpha}$$

both the terms on the left hand side of (3.8) are positive and this gives the convergence of J_1.

Proof for divergence. (*iii*) Suppose now that J_1 is divergent. Then for large r

$$K < \int_{r}^{R} \frac{M(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\, dn < \frac{M(R)}{\alpha\, e^{\alpha\,\Phi(R)}}\ [e^{\alpha\{\Phi(R)-\Phi(r)\}} - 1]$$

and hence for large r

$$\frac{M(r)}{e^{\alpha\,\Phi(r)}} > K$$

and so from (3.4) we fond that J_2 is divergent.

(*iv*) Finally, suppose J_2 is divergent. Then from (3.4) we have for large r

$$\alpha \int_{r_0}^{r} \frac{M(n)\,\Phi'(n)}{e^{\alpha\,\Phi(n)}}\, dn + \frac{M(r)}{e^{\alpha\,\Phi(r)}} > K, \tag{3.9}$$

we now say that J_1 is divergent if J_2 is divergent, for if J_1 is covergent then

$$\alpha \int_{r_0}^{r} \frac{M(n)}{e^{\alpha \Phi(n)}} \Phi'(n)\, dn > \infty$$

and

$$\frac{M(r)}{e^{\alpha \Phi(r)}} \tag{3.10}$$

is arbitrarily small (for (3.10) see (3.6)), for large r and then (3.9) gives a contradition. Hence J_1 is divergent.

Combining (i), (ii), (iii) and (iv) the theorem is completely established.

Remark. This generalises the result of Kamthan ([1]. Th. 11 and th 15, p. 139 [2]).

4. Kamthan [K] has proved the following results:

$$A \leq \frac{C}{\varrho} \tag{4.1}$$

$$B \leq \frac{D}{\varrho} \left\{1 + \log\left(\frac{C}{D}\right)\right\} \tag{4.2}$$

$$A \geq \frac{C}{\varrho e} c^{D/C} \tag{4.3}$$

$$B \geq \frac{D}{\varrho} \tag{4.4}$$

where

$$\overline{\lim_{r\to\infty}} \frac{M(r)}{g(r)} = \frac{A}{B} \quad ; \quad \overline{\lim_{r\to\infty}} \frac{n(r)}{g(r)} = \frac{C}{D} \; ;$$

and

$$g(r) = \exp. \int_{0}^{r} \varrho(n)\, d\,\Phi(n)\,.$$

Then from (4.4) and the fact that $B \leq A$, we have

$$D \leq \varrho A \tag{4.5}$$

and from (4.5)

$$C \leq A \varrho e e^{-D/C}$$

$$\leq A \varrho e\,. \tag{4.6}$$

Remark. It follows from (4.5) and (4.6) that

$$C + D \leqq \varrho A (e + 1)$$

but it is included in the following, for we find from (4.3) that

$$A \geqq \frac{C}{\varrho e} \left[1 + \frac{D}{C} + 0 \left(\frac{D}{C} \right)^2 \right]$$

$$\geqq \frac{C}{\varrho e} \quad \frac{C + D}{C} .$$

Thus

(4.7) $$C + D \leqq A \varrho c$$

and this completes our assertion.

Theorem 6. Equality cannot hold simultaneously in (4.4) and (4.7).

Proof. Let

$$D = \varrho A .$$

Then from (see [K] theorem 1 (9))

$$A \geqq \frac{e^{-\varrho \Phi(k)}}{\varrho} \quad [D + \varrho C \Phi (k)],$$

we have

$$A \geqq e^{-\varrho \Phi(k)} \; [A + C \Phi (k)$$

or

$$C \leqq A \frac{(e^{\varrho \Phi(k)} - 1)}{\Phi (k)} .$$

Now let

$$\Phi (k) = \frac{1}{\varrho} \log (1 + \eta), \quad \eta \to 0 .$$

Hence

$$C \leqq \frac{\varrho A \eta}{\eta + 0 (\eta^2)} \leqq \varrho A.$$

Further

$$D \leqq C .$$

Hence

$$C = \varrho A$$

or

$$C + D = 2 \varrho A < e \varrho A .$$

Next suppose that $C + D = e \varrho A$, then D will be less than ϱA, for, if it were equal to ϱA then by above theorem $C + D$ will have to be less than $e \varrho A$.

Remark. A similar result appears in ([1], theorem 8 (*ii*)).

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ÖZET

Bazı özellikleri haiz $\Phi(r)$ fonksiyonuna nazaran konveks olan ve $M(a) = 0$ şartını sağlayan bit $M(r)$ fonksiyonu ile bu fonksiyona bağlı olarak

$$M(r) = \int_a^r n(t)\ \Phi'(t)\ dt \qquad (a \geq 0)$$

bağıntısı ile tanımlanmış $n(t)$ fonksiyonları göz önüne alınmakta ve bu fonksiyonların «büyüme» leri ile ilgili yeni bazı sonuçlar elde edilmektedir.