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On an iterative method without inverses of derivatives for solving equations

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Abstract

We present the semi-local convergence analysis of a Potra-type method to solve equations involving Banach space valued operators. The analysis is based on our ideas of recurrent functions and restricted convergence region. The study is completed using numerical examples.

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1. Introduction

Many problems in applied sciences can be reduced to the mathematical equation,

$$F(x) = 0, \quad (1)$$

where $F : D \subseteq \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a Fréchet-differentiable operator, \mathcal{B}_1 and \mathcal{B}_2 are Banach spaces and D is a nonempty open convex subset of \mathcal{B}_1 . Iterative methods are useful in solving many equations of the form (1). For example the following method is considered in [16, 17]

$$x_{n+3} = x_{n+2} - ([x_{n+2}, x_{n+1}; F] + [x_n, x_{n+1}; F] - [x_n, x_{n+1}; F])^{-1} F(x_n), \quad (2)$$

where x_0, x_1, x_2 are initial points.

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The convergence of iterative algorithms is analyzed in two categories: semi-local convergence analysis (i.e., based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms) and local convergence analysis (i.e., based on the information around a solution to find estimates of the computed radii of the convergence balls).

In this study, we introduce the following iterative method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned}x_{n+1} &= x_n - [x_n, y_n; F]^{-1}F(x_n) \\y_{n+1} &= x_{n+1} - A_n^{-1}F(x_{n+1}),\end{aligned}\tag{3}$$

where x_0, y_0 are an initial points, $A_n = [x_n, y_n; F] + [x_{n+1}, x_n; F] - [x_{n+1}, y_n; F]$, $[\cdot, \cdot; F] : D \times D \rightarrow \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is the finite difference of order one. We do not choose $x_0 = y_0$ in practice to avoid injecting derivatives in the method or study variants of Newton's method involving both divided differences and derivatives. But clearly our results can specialize to such methods if derivatives are allowed.

Using Lipschitz- type conditions we find computable radii of convergence as well as error bounds on the distances involved. The order of convergence is found using computable order of convergence (COC) or approximate computational order of convergence (ACOC) [24] (see Remark 2.4) that do not require usage of higher order derivatives. This way we expand the applicability of three step method (3) under weak conditions.

The rest of the study is organized as follows: Section 2 contains the local convergence of method (3), whereas in the concluding Section 3 applications and numerical examples can be found.

2. Semi-local convergence

The semi-local convergence of method (3) is given in this Section. We need two auxiliary results on majorizing sequences for method (3).

Lemma 2.1. *Let $L_0 > 0, L_1 \geq 0, L > 0, s_0 \geq 0$ and $t_1 \geq 0$ be given parameters. Denote by α the only root in the interval $(0, 1)$ of polynomial p defined by $p(t) = L_0t^3 + L_0t^2 + 2Lt - 2L$. Set $s_1 = t_1 + \frac{L(t_1+s_0)t_1}{1-(2L_0+L_1)s_0}$. Suppose that*

$$0 < \frac{L(t_1 + s_0)}{1 - (2L_0 + L_1)s_0} \leq \alpha \leq 1 - \frac{2L_0t_1}{1 - L_0s_0},\tag{1}$$

$$0 < \frac{L(t + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \leq \alpha\tag{2}$$

and

$$\frac{L_1}{L_0} - 1 \leq \alpha.\tag{3}$$

Then, scalar sequence $\{t_n\}$ defined for each $n = 0, 1, 2, \dots$, by

$$\begin{aligned}t_0 &= 0, \quad s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{1 - [L_0(t_n + s_n + s_0) + L_1(s_n - t_n)]} \\t_{n+2} &= t_{n+1} + \frac{L(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{1 - L_0(t_{n+1} + s_{n+1} + s_0)}\end{aligned}\tag{4}$$

is well defined, nondecreasing, bounded from above by

$$t^{**} = \frac{t_1}{1 - \alpha}\tag{5}$$

and converges to its unique least upper bound t^* , which satisfies

$$t^* \in [s_1, t^{**}].\tag{6}$$

Moreover, the following estimates hold for each $n = 1, 2, \dots$

$$0 < s_n - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0) \tag{7}$$

and

$$0 < t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0). \tag{8}$$

Proof. We have by the definition of polynomial p that $p(0) = -2L < 0$ and $p(1) = 2L_0 > 0$. Then, it follows from the intermediate value theorem that polynomial p has roots $\alpha \in (0, 1)$. By Descartes’s rule of signs α is the only root of polynomial p in $(0, 1)$. If $t_1 \leq s_0 \leq t_0$, then $t_n = s_n = 0$ for each $n = 0, 1, 2, \dots$ and (5)-(8) are satisfied. In what follows we suppose that $t_1 > 0$. It follows from (4) that estimates (7)– (8) are true, if

$$0 < \frac{L(t_{k+1} - t_k + s_k - t_k)}{1 - [L_0(t_{k+1} + s_{k+1} + s_0) + L_1(s_k - t_k)]} \leq \alpha \tag{9}$$

$$0 < \frac{L(t_{k+1} - t_k + s_k - t_k)}{1 - [L_0(t_{k+1} + s_{k+1} + s_0)]} \leq \alpha \tag{10}$$

and

$$t_{k+1} \leq s_{k+1}. \tag{11}$$

hold for each $k = 0, 1, 2, \dots$. Estimates (9)-(11) are true for $k = 0$ by (1) and (2). Then, we have that

$$\begin{aligned} s_1 - t_1 &\leq \alpha(t_1 - t_0), \quad t_2 - t_1 \leq \alpha(t_1 - t_0) \\ s_1 &\leq (1 + \alpha)(t_1 - t_0) = \frac{1 - \alpha^2}{1 - \alpha}(t_1 - t_0) < t^{**} \end{aligned}$$

and

$$t_2 \leq (1 + \alpha)(t_1 - t_0) = \frac{1 - \alpha^2}{1 - \alpha}(t_1 - t_0) < t^{**} \text{ (by (5)).} \tag{12}$$

Suppose these estimates hold for $k = 0, 1, 2, \dots, n$. Then, we get that

$$s_{k+1} - t_{k+1} \leq \alpha(t_{k+1} - t_k) \leq \alpha^k(t_1 - t_0), \tag{13}$$

$$t_{k+2} - t_{k+1} \leq \alpha(t_{k+1} - t_k) \leq \alpha^{k+1}(t_1 - t_0), \tag{14}$$

$$s_{k+1} \leq \frac{1 - \alpha^{k+2}}{1 - \alpha}(t_1 - t_0) < t^{**} \tag{15}$$

and

$$t_{k+2} \leq \frac{1 - \alpha^{k+2}}{1 - \alpha}(t_1 - t_0). \tag{16}$$

Using (3), we have in turn that

$$\begin{aligned} L_1 \leq L_0(1 + \alpha) &\implies L_1 \alpha^k \leq L_0(\alpha^k + \alpha^{k+1}) \\ &\implies L_0\left(\frac{1 - \alpha^k}{1 - \alpha}(t_1 - t_0) + \frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) + s_0\right) + L_1 \alpha^k(t_1 - t_0) \\ &\leq L_0\left(\frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) + \frac{1 - \alpha^{k+2}}{1 - \alpha}(t_1 - t_0) + s_0\right). \end{aligned} \tag{17}$$

In view of (13)-(16), estimates (9) and (10) are satisfied, if

$$\begin{aligned} &2L\alpha^k(t_1 - t_0)(t_1 - t_0) + \alpha\left[L_0\left(\frac{1 - \alpha^k}{1 - \alpha}(t_1 - t_0)\right.\right. \\ &\left.\left.+ \frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) + s_0 + L_1\alpha^k(t_1 - t_0)\right) - \alpha \leq 0 \end{aligned} \tag{18}$$

and

$$2L\alpha^k(t_1 - t_0) + \alpha L_0 \left[\frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) + \frac{1 - \alpha^{k+2}}{1 - \alpha}(t_1 - t_0) + s_0 \right] - \alpha \leq 0, \quad (19)$$

respectively. However, it follows from (17)-(19) that we must only show (22). Estimate (19) motivates us to introduce recurrent functions f_k on $[0, 1)$ by

$$f_k(t) = 2Lt^{k-1}(t_1 - t_0) + L_0 \left(\frac{1 - t^{k+1}}{1 - t} + \frac{1 - t^{k+2}}{1 - t} \right) (t_1 - t_0) + L_0 s_0 - 1. \quad (20)$$

We need a relationship between two consecutive functions f_k . Using (20) and the definition of polynomial p , we get that

$$f_{k+1}(t) = f_k(t) + p(t)t^{k-1}(t_1 - t_0). \quad (21)$$

In particular, we have that

$$f_{k+1}(\alpha) = f_k(\alpha), \quad (22)$$

by the definition of α . Hence, estimate (19) is true, if

$$f_k(\alpha) \leq 0 \text{ for each } k = 1, 2, \dots \quad (23)$$

Define function f_∞ on $(0, 1)$ by

$$f_\infty(t) = \lim_{k \rightarrow \infty} f_k(t). \quad (24)$$

Using (1), (20) and (24), we get that

$$f_\infty(\alpha) = \frac{2L_0(t_1 - t_0)}{1 - \alpha} + L_0 s_0 - 1. \quad (25)$$

But by (25) $f_k(\alpha) = f_\infty(\alpha) \leq 0$ for each $k = 1, 2, \dots$, which shows (23). Hence, the induction is complete and estimates (9)-(11) (i.e., estimates (7), and (8)) hold. It follows from (4), (7), (8), (15) and (16) that sequence $\{t_n\}$ is nondecreasing, bounded from above by t^{**} , and as such it converges to t^* which satisfies (6). □

In case (3) is not satisfied we have the alternative result.

Lemma 2.2. *Let $L_0 > 0, L_1 > 0, L > 0, s_0 \geq 0$ and $t_1 \geq 0$ be given parameters. Denote by β the only root in the interval $(0, 1)$ of polynomial q defined by $q(t) = (L_0 + L_1)t^2 + (2L + L_0 - L_1)t - 2L$. Suppose that*

$$0 < \frac{L(t_1 + s_0)}{1 - (2L_0 + L_1)s_0} \leq \beta \leq 1 - \frac{2L_0 t_2}{1 - L_0 s_0}, \quad (26)$$

$$0 < \frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \leq \beta \quad (27)$$

and

$$\beta \leq \frac{L_1}{L_0} - 1. \quad (28)$$

Then, the conclusions of Lemma 2.1 hold with β replacing α .

Proof. Notice that (1), (2) coincide with (25), (26), respectively, if we replace α by β . We have $q(0) = -2L < 0$ and $q(1) = 2L_0 > 0$. Hence, again β is the only root of polynomial q in $(0, 1)$. Then, the proof follows exactly as in Lemma 2.1 with β replacing α until (19) but (17) holds in reverse because of (28). Hence, this time we must show (18) instead of (19) leading to

$$g_k(t) = 2Lt^{k-1}(t_1 - t_0) + L_0 \left(\frac{1 - t^k}{1 - t} + \frac{1 - t^{k+1}}{1 - t} + L_1 t^k \right) (t_1 - t_0) + L_0 s_0 - 1,$$

instead of the definition of functions f_k . Then, we have that

$$g_{k+1}(t) = g_k(t) + q(t)t^{k-1}(t_1 - t_0)$$

$$g_{k+1}(\beta) = g_k(\beta) = g_\infty(\beta) = f_\infty(\beta) \leq 0,$$

where

$$g_\infty(t) = f_\infty(t).$$

□

Denote by $U(w, \xi), \bar{U}(w, \xi)$, the open and closed balls in \mathcal{B}_1 , respectively with center $w \in \mathcal{B}_1$ and of radius $\xi > 0$.

Next, we present the semi-local convergence analysis of method (3) using $\{t_n\}$ as a majorizing sequence.

Theorem 2.3. *Let $F : D \subseteq \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exists a divided difference $[\cdot, \cdot; F]$ of order one for operator F on $D \times D$. Moreover, suppose that there exist $x_0, y_0 \in D, L_0 > 0, L > 0, L_1 > 0, s_0 \geq 0$ and $t_1 \geq 0$ such that for each $x, y, w \in D$*

$$[x_0, y_0; F]^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \quad (29)$$

$$\|[x_0, y_0; F]^{-1}F(x_0)\| \leq t_1, \quad (30)$$

$$\|x_0 - y_0\| \leq s_0, \quad (31)$$

$$\|[x_0, y_0; F]^{-1}([x, y; F] - [x, w; F])\| \leq L_1\|y - w\| \quad (32)$$

$$\|[x_0, y_0; F]^{-1}([x, y; F] - [x_0, y_0; F])\| \leq L_0(\|x - x_0\| + \|y - x_0\|). \quad (33)$$

Let $D_0 = D \cap U(x_0, \frac{1}{3L_0+L_1})$.

$$\|[x_0, y_0; F]^{-1}([x, y; F] - [z, w; F])\| \leq L(\|x - z\| + \|y - w\|), \quad (34)$$

for each $x, y, z, \in D_0$

$$\bar{U}(x_0, t^*) \subseteq D \quad (35)$$

and hypotheses of Lemma 2.1 hold. Then, the sequence $\{x_n\}$ generated by method (3) is well defined, remains in $\bar{U}(x_0, t^*)$ and converges to a solution $x^* \in \bar{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$

$$\|x_n - x^*\| \leq t^* - t_n, \quad (36)$$

where $\{t_n\}$ and t^* are defined in the preceding Lemmas. Furthermore, if there exists $R > t^*$ such that

$$U(x_0, R) \subseteq D \quad (37)$$

and

$$L_0(t^* + R + s_0) \leq 1, \quad (38)$$

then, the point x^* is the only solution of equation $F(x) = 0$ in $U(x_0, R)$.

Proof. We shall show using induction that the following assertions hold.

$$(I_k) \quad \|x_{k+1} - x_k\| \leq t_{k+1} - t_k$$

and

$$(II_k) \quad \|y_k - x_k\| \leq s_k - t_k.$$

We have that (I_0) holds by (30). Hence, x_1 is well defined and $x_1 \in \bar{U}(x_0, t^*)$. Using (30), (31), (32), (33) and the proof of Lemma 2.1 we get in turn that

$$\begin{aligned} \|[x_0, y_0; F]^{-1}(A_0 - [x_0, y_0; F])\| &\leq \|[x_0, y_0; F]^{-1}([x_0, y_0; F] - [x_0, y_0; F])\| \\ &\quad + \|[x_0, y_0; F]^{-1}([x_1, x_0; F] - [x_1, y_0; F])\| \\ &\leq L_0(\|x_0 - x_0\| + \|y_0 - y_0\|) + L_1\|x_0 - y_0\| \\ &\leq L_1 s_0 < 1. \end{aligned} \quad (39)$$

It follows from (38) and the Banach lemma on invertible operators [1, 13] that A_0^{-1} exists and

$$\|A_0^{-1}[x_0, y_0; F]\| \leq \frac{1}{1 - L_0 s_0}. \quad (40)$$

Then, from the second substep of method (3) y_1 exists. We also have from method (3) for $n = 0$ that

$$F(x_1) = F(x_1) - F(x_0) - [x_0, y_0; F](x_1 - x_0) = ([x_1, x_0; F] - [x_0, x_0; F])(x_1 - x_0). \quad (41)$$

Using (40), (31), (34) and (41), we get that

$$\begin{aligned} \|[x_0, y_0; F]^{-1}F(x_1)\| &= \|[x_0, y_0; F]^{-1}([x_1, x_0; F] - [x_0, y_0; F])(x_1 - x_0)\| \\ &\leq L_0(\|x_1 - x_0\| + \|x_0 - y_0\|)\|x_1 - x_0\| \\ &\leq L_0(t_1 - t_0 + s_0 - t_0)(t_1 - t_0) \\ &\leq L(t_1 - t_0 + s_0 - t_0)(t_1 - t_0). \end{aligned} \quad (42)$$

By the second substep of method (3), (4), (40) and (42), we get that

$$\begin{aligned} \|y_1 - x_1\| &\leq \|A_0^{-1}[x_0, y_0; F]\| \|[x_0, y_0; F]^{-1}F(x_1)\| \\ &\leq \frac{L(t_1 - t_0 + s_0 - t_0)(t_1 - t_0)}{1 - L_1 s_0} \leq s_1 - t_1, \end{aligned} \quad (43)$$

which shows (II_0) . We also have that

$$\begin{aligned} \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq s_1 - t_1 + t_1 - t_0 \\ &= s_1 \leq t^*, \end{aligned}$$

which shows $y_1 \in \bar{U}(x_0, t^*)$. Suppose that (I_k) and (II_k) hold for $k = 0, 1, 2, \dots$. Using (32), (33), (I_k) , (II_k) and the proof of the Lemma we get in turn as in (39) that

$$\begin{aligned} &\|[x_0, y_0; F]^{-1}(A_k - [x_0, y_0; F])\| \\ &\leq \|[x_0, y_0; F]^{-1}([x_k, y_k; F] - [x_0, y_0; F])\| \\ &\quad + \|[x_0, y_0; F]^{-1}([x_{k+1}, x_k; F] - [x_{k+1}, y_k; F])\| \\ &\leq L_0(\|x_k - x_0\| + \|y_k - y_0\|) + L_1\|x_k - y_k\| \\ &\leq L_0(\|x_k - x_0\| + \|y_k - x_k\| \\ &\quad + \|x_k - x_0\| + \|x_0 - y_0\|) + L_1\|x_k - y_k\| \\ &\leq L_0(t_k - t_0 + s_k - t_k + t_k - t_0 + s_0 - t_0) + L_1(s_k - t_k) \\ &\leq L_0(t_k + s_k + s_0) + L_1(s_k - t_k) < 1. \end{aligned} \quad (44)$$

It follows from (44) and the Banach Lemma on invertible operators that A_k^{-1} exists and

$$\|A_k^{-1}[x_0, y_0; F]\| \leq \frac{1}{1 - [L_0(t_k + s_k + s_0) + L_1(s_k - t_k)]}. \quad (45)$$

As in (41), we get that

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - [x_k, y_k; F](x_{k+1} - x_k), \tag{46}$$

so, by (46) and (34), we get that

$$\begin{aligned} \|[x_0, y_0; F]^{-1}F(x_{k+1})\| &\leq L(\|x_{k+1} - x_k\| + \|x_k - y_k\|)\|x_{k+1} - x_k\| \\ &\leq L(t_{k+1} - t_k + s_k - t_k)(t_{k+1} - t_k). \end{aligned} \tag{47}$$

Then, from the second substep of method (3), (45), (46) and (47) we get that

$$\begin{aligned} \|y_{k+1} - x_{k+1}\| &\leq \|A_k^{-1}[x_0, y_0; F]\| \|[x_0, y_0; F]^{-1}F(x_{k+1})\| \\ &\leq \frac{L(t_{k+1} - t_k + s_k - t_k)(t_{k+1} - t_k)}{1 - (L_0(t_k + s_k + s_0) + L_1(s_k - t_k))} \\ &= s_{k+1} - t_{k+1} \end{aligned}$$

which shows (II_k) . We also have that

$$\begin{aligned} \|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \\ &\leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 \leq s_{k+1} \leq t^*. \end{aligned}$$

Hence, $y_{k+1} \in \bar{U}(x_0, t^*)$. We also have that

$$\begin{aligned} &\|[x_0, y_0; F]^{-1}([x_{k+1}, y_{k+1}; F] - [x_0, y_0; F])\| \\ &\leq L_0(\|x_{k+1} - x_0\| + \|y_{k+1} - y_0\|) \\ &\leq L_0(\|x_{k+1} - x_0\| + \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| + \|x_0 - y_0\|) \\ &\leq L_0(t_{k+1} - t_0 + s_{k+1} - t_{k+1} + t_{k+1} - t_0 + s_0 + t_0) \\ &= L_0(t_{k+1} + s_{k+1} + s_0) < 1. \end{aligned} \tag{48}$$

It follows from (48) and the Banach lemma on invertible operators that $[x_{k+1}, y_{k+1}; F]^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|[x_{k+1}, y_{k+1}; F]^{-1}[x_0, y_0; F]\| \leq \frac{1}{1 - L_0(t_{k+1} + s_{k+1} + s_0)}. \tag{49}$$

Then, using (47), (49) and (4), we get that

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &\leq \|[x_{k+1}, y_{k+1}; F]^{-1}[x_0, y_0; F]\| \|[x_0, y_0; F]^{-1}F(x_{k+1})\| \\ &\leq \frac{L(t_{k+1} - t_k + s_k - t_k)(t_{k+1} - t_k)}{1 - L_0(t_{k+1} + s_{k+1} + s_0)} \\ &= t_{k+2} - t_{k+1}, \end{aligned}$$

and

$$\begin{aligned} \|x_{k+2} - x_0\| &\leq \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - x_0\| \\ &\leq t_{k+2} - t_{k+1} + t_{k+1} - t_0 \leq t^*, \end{aligned}$$

so $x_{k+2} \in \bar{U}(x_0, t^*)$ which completes the induction. It follows that $\{x_k\}$ is a complete sequence (since $\{t_k\}$ is a complete sequence) in a Banach space \mathcal{B}_1 and as such it converges to some $x^* \in \bar{U}(x_0, t^*)$ (since $\bar{U}(x_0, t^*)$ is a closed set). By letting $k \rightarrow \infty$ in (47) we obtain $F(x^*) = 0$. Estimate (36) follows from (I_k) , (II_k) by using standard majorizing techniques [1, 2, 3, 13, 16].

To complete the proof we show the uniqueness of the solution in $\bar{U}(x_0, R)$. Let $w \in \bar{U}(x_0, R)$ be such that $F(w) = 0$. By (27), we have in turn that

$$\begin{aligned} \|A_0^{-1}([x^*, w; F] - A_0)\| &\leq L_0(\|x^* - x_0\| + \|w - y_0\|) \\ &\leq L_0(\|x^* - x_0\| + \|w - x_0\| + \|x_0 - y_0\|) \\ &\leq L_0(t^* + R + 2s_0) < 1 \end{aligned} \tag{50}$$

It follows that $[x^*, w; F]^{-1}$ exists. Then from the identity

$$[x^*, w; F](x^* - w) = F(x^*) - F(w) = 0,$$

we conclude that $x^* = w$. □

Remark 2.4. (a) The limit point t^* can be replaced by t^{**} given by (5) in Theorem 2.3.

(b) Notice that $L_0 \leq L$, $L_1 \leq L$, hold in general and $\frac{L}{L_0}, \frac{L}{L_1}$ can be arbitrarily large [1, 2, 3]. In the literature with the exception of ours $L_0 = L = L_1$ is used for the study of Secant-type methods [4], [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. However, the latter choice is leading to less precise error estimates and stronger sufficient convergence conditions (see also the numerical examples).

(c) In view of (4) it follows that L_0 and not L is needed in the computation of $\|[x_0, y_0; F]^{-1}F(x_1)\|$. Therefore, $\{t_n\}$ can be replaced by the tighter sequence $\{\bar{t}_n\}$ in Theorem 2.3 defined by

$$\begin{aligned}\bar{t}_0 = 0, \bar{s}_1 &= \bar{t}_1 + \frac{L_0(\bar{t}_1 - \bar{t}_0 + \bar{s}_0 - \bar{t}_0)(\bar{t}_1 - \bar{t}_0)}{1 - [L_0(\bar{t}_0 + 2\bar{s}_0) + L_1(\bar{s}_0 - \bar{t}_0)]}, \\ \bar{t}_1 = t_1, \bar{s}_0 &= s_0 \\ \bar{s}_{k+1} &= \bar{t}_{k+1} + \frac{L(\bar{t}_{k+1} - \bar{t}_k + \bar{s}_k - \bar{t}_k)(\bar{t}_{k+1} - \bar{t}_k)}{1 - [L(\bar{t}_k + \bar{s}_k + \bar{s}_0) + L_1(\bar{s}_k - \bar{t}_k)]}\end{aligned}$$

for each $n = 1, 2, \dots$,

$$\begin{aligned}\bar{t}_2 &= \bar{t}_1 + \frac{L_0(\bar{t}_1 - \bar{t}_0 + \bar{s}_0 - \bar{t}_0)(\bar{t}_1 - \bar{t}_0)}{1 - L_0(\bar{t}_1 + \bar{s}_1 + \bar{s}_0)} \\ \bar{t}_{k+2} &= \bar{t}_{k+1} + \frac{L(\bar{t}_{k+1} - \bar{t}_k + \bar{s}_k - \bar{t}_k)(\bar{t}_{k+1} - \bar{t}_k)}{1 - L_0(\bar{t}_{k+1} + \bar{s}_{k+1} + \bar{s}_0)}\end{aligned}$$

for each $n = 1, 2, \dots$

3. Numerical Examples

We shall use the divided difference given by $[x, y; F] = \frac{1}{2}(F'(x) + F'(y))$ in both examples.

Example 3.1. Let $D = \bar{U}(x_0, 1 - \gamma)$, $x_0 = 1$, $y_0 = x_0 + 10^{-3}$, $\gamma \in [0, 1)$. Define function F on D by

$$F(x) = x^3 - \gamma.$$

We have that

$$L = L_1 = \frac{2(2 - \gamma)}{x_0^2 + y_0^2} = 1.0479$$

$$L_0 = \frac{3 - \gamma}{x_0^2 + y_0^2} = 1.0230.$$

Next, we verify that all conditions of Lemma 2.1 hold. In fact, by the definition of polynomial p , we get that $\alpha \approx 0.6543$. We also have

$$0 < \frac{L(t_1 + s_0)}{1 - (2L_0 + L_1)s_0} = 0.0196 \leq \alpha \leq 1 - \frac{2L_0t_1}{1 - L_0s_0} = 0.9659,$$

$$0 < \frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} = 0.0203 \leq \alpha$$

and

$$0.0244 = \frac{L_1}{L_0} - 1 \leq \alpha.$$

We see by now that all conditions of Theorem 2.3 are satisfied. Hence, Theorem 2.3 applies.

Example 3.2. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$, $\Omega_0 = \Omega = (-1, 1)^3$ and define $F = (F_1, F_2, F_3)^T$ on Ω by

$$F(x) = F(x_1, x_2, x_3) = (e^{x_1} - 1, x_2^2 + x_2, x_3)^T. \tag{1}$$

We get for the points $u = (u_1, u_2, u_3)^T, v = (v_1, v_2, v_3)^T \in \Omega$,

$$[u, v; F] = \begin{pmatrix} \frac{e^{u_1} - e^{v_1}}{u_1 - v_1} & 0 & 0 \\ 0 & u_2 + v_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\bar{y}_0 = (0.01, 0.01, 0.01)^T, \bar{x}_0 = (0.011, 0.011, 0.011)^T$ be two initial points for the method (2). Here, we use \bar{x}_n instead of x_n to distinct iterative points with its component for some integer $n \geq 0$. Then, we have $t_0 = 0.01$,

$$A_0 \approx \begin{pmatrix} 1.0101 & 0 & 0 \\ 0 & 1.0200 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_0^{-1} \approx \begin{pmatrix} 0.9900 & 0 & 0 \\ 0 & 0.9804 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$t_1 = t_0 + \|A_0^{-1}F(\bar{x}_0)\| = 0.0172, \quad \bar{x}_1 \approx (0.0001, 0.0001, 0).$$

Note that, for any $x = (x_1, x_2, x_3)^T, y = (y_1, y_2, y_3)^T, z = (z_1, z_2, z_3)^T, v = (v_1, v_2, v_3)^T \in \Omega$, we have

$$[x, y; F] - [z, v; F] = \begin{pmatrix} \frac{e^{x_1} - e^{y_1}}{x_1 - y_1} - \frac{e^{z_1} - e^{v_1}}{z_1 - v_1} & 0 & 0 \\ 0 & x_2 + y_2 - z_2 - v_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

In view of

$$\begin{aligned} & \left| \frac{e^{x_1} - e^{y_1}}{x_1 - y_1} - \frac{e^{z_1} - e^{v_1}}{z_1 - v_1} \right| = \left| \int_0^1 (e^{y_1 + t(x_1 - y_1)} - e^{v_1 + t(z_1 - v_1)}) dt \right| \\ & = \left| \int_0^1 \int_0^1 e^{v_1 + t(z_1 - v_1) + \theta(y_1 + t(x_1 - y_1) - v_1 - t(z_1 - v_1))} (y_1 + t(x_1 - y_1) - v_1 - t(z_1 - v_1)) d\theta dt \right| \\ & \leq \int_0^1 \int_0^1 e^{|t(x_1 - z_1) + (1-t)(y_1 - v_1)|} d\theta dt \\ & \leq \frac{e}{2} (|x_1 - z_1| + |y_1 - v_1|), \end{aligned}$$

we get

$$\begin{aligned} & \|A_0^{-1}([x, y; F] - [z, v; F])\| \\ & \leq \max\left(\frac{e \times 0.9047}{2} (|x_1 - z_1| + |y_1 - v_1|), 0.8333(|x_2 - z_2| + |y_2 - v_2|)\right) \\ & \leq \max\left(\frac{e \times 0.9047}{2} (\|x - z\| + \|y - v\|), 0.8333(\|x - z\| + \|y - v\|)\right) \\ & = \frac{e \times 0.9047}{2} (\|x - z\| + \|y - v\|). \end{aligned} \tag{3}$$

In particular, set $z = \bar{x}_0$ and $v = \bar{x}_1$ in (3), we obtain

$$\|A_0^{-1}([x, y; F] - A_0)\| \leq \frac{e \times 0.9047}{2} (\|x - \bar{x}_0\| + \|y - \bar{x}_1\|). \tag{4}$$

That is, we can choose constants $L_0 = L_1 = L_2 = L_3 = \bar{L}_2 = \bar{L}_3 \approx \frac{e \times 0.9047}{2} \approx 1.3456$ in Theorem 2.3.

Next, we verify that all conditions of Lemma 2.1 hold. In fact, by the definition of polynomial p , we get that $\alpha \approx 0.6506$. We also have

$$0 < \frac{L(t_1 + s_0)}{1 - (2L_0 + L_1)s_0} = 0.0257 \leq \alpha \leq 1 - \frac{2L_0t_1}{1 - L_0s_0} = 0.9535,$$

$$0 < \frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} = 0.0268 \leq \alpha$$

and

$$0 = \frac{L_1}{L_0} - 1 \leq \alpha.$$

We see by now that all conditions of Theorem 2.3 are satisfied. Hence, Theorem 2.3 applies.

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