LIE-DERIVATIVES OF VARIOUS GEOMETRIC ENTITIES IN FINSLER SPACE

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The LIE-derivative was introduced by DAVIES [1] [1]) in the theory of infinitesimal deformations of a FINSLER space. The vector along which the deformation is considered is taken to be independent of directions. In the present paper the infinitesimal transformation is studied in the general form where the above mentioned vector is taken to be dependent both on position and direction.

1. **Introduction.** Let F_n be an n-dimensional FINSLER space equipped with the symmetric tensor

$$g_{ij}(x, \dot{x}) \underline{\underline{\text{def}}} \tfrac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \tag{1.1a}$$

where

$$\dot{\partial}_i \equiv \frac{\partial}{\partial \dot{x}^i} \cdot \tag{1.2a}$$

Since the metric function $F(x, \dot{x})$ is assumed to be positively homogeneous of degree one in the $\dot{x}^i$'s, the metric tensor is a homogeneous function of degree zero in the $\dot{x}^i$'s. The contravariant components of the metric tensor are given by

$$g^{ij} g_{ih} = \delta^i_h = \begin{cases} 1 & \text{if} \quad h = i, \\ 0 & \text{if} \quad h \neq i. \end{cases} \tag{1.1b}$$

CARTAN's covariant derivative of a tensor $T^i_j(x, \dot{x})$ with respect to x^k is given by [3]

$$T^i_{j\,|\,k}(x, \dot{x}) = \partial_k T^i_j - (\dot{\partial}_l T^i_j) G^l_k + T^l_j \Gamma^{*i}_{lk} - T^i_l \Gamma^{*l}_{jk}\,, \tag{1.3}$$

1) The numbers in the square brackets refer to the references given at the end of the paper.

where

(1.2b) $$\partial_k \equiv \frac{\partial}{\partial x^k},$$

and

(1.4) $$G^l_k(x, \dot{x}) \underset{=}{\text{def}} \dot{\partial}_k G^l = \Gamma^{*l}_{mk}(x, \dot{x})\, \dot{x}^m .$$

The functions $G^m(x, \dot{x})$ are homogeneous of degree two in the $\dot{x}^i$'s [3] and $\Gamma^{*l}_{mk}(x, \dot{x})$ are CARTAN's connection coefficients. The following identities result from (1.3)

(1.5) $$F_{|k} = 0, \qquad g_{ij|k} = 0 .$$

The completely symmetric and skew-symmetric parts of a geometric object Ω_{ij} are given by

(1.6a) $$\Omega_{(ij)} \underset{=}{\text{def}} \tfrac{1}{2}(\Omega_{ij} + \Omega_{ji})$$

and

(1.6b) $$\Omega_{[ij]} \underset{=}{\text{def}} \tfrac{1}{2}(\Omega_{ij} - \Omega_{ji})$$

respectively.

2. **Infinitesimal transformation.** We consider the infinitesimal transformation

(2.1) $$\bar{x}^i = x^i + v^i(x, \dot{x})\, d\tau$$

in the space F_n. The entities $v^i(x, \dot{x})$ are the contravariant components of a vector-field and $d\tau$ is an infinitesimal constant. The corresponding variations in the variables $\dot{x}^i$ are represented by

(2.2) $$\dot{\bar{x}}^i = \dot{x}^i + \{(\partial_j v^i)\, \dot{x}^j + (\dot{\partial}_j v^i)\, \ddot{x}^j\}\, d\tau.$$

Differentiating (2.1) with respect to x^j we obtain

(2.3) $$\partial_j \bar{x}^i = \delta^i_j + (\partial_j v^i)\, d\tau .$$

The LIE-differential $\Delta\Omega(x, \dot{x})$ of a geometric object $\Omega(x, \dot{x})$ is the difference of its value at the point $\bar{x}^i$ and its component obtained from the coordinate transformation (2.1) in the $\bar{x}^i$-system, *i.e.*

(2.4) $$\Delta\Omega(x, \dot{x}) \doteq \Omega(\bar{x}, \dot{\bar{x}}) - {}'\Omega(\bar{x}, \dot{\bar{x}}),$$

where ${}'\Omega(\bar{x}, \dot{\bar{x}})$ is the component of the geometric object obtained from its value at x^i when (2.1) is regarded as a coordinate transformation.

DEFINITION: *The* LIE-*derivative of a geometric object is the limit of the* LIE-*differential divided by $d\tau$ when $d\tau$ tends to zero, i.e.*

(2.5) $$\underset{L}{D}\, \Omega(x, \dot{x}) = \lim_{d\tau \to 0} \frac{\Delta \Omega(x, \dot{x})}{d\tau},$$

where $\underset{L}{D}\, \Omega(x, \dot{x})$ *denotes the* LIE-*derivative of the geometric object* $\Omega(x, \dot{x})$.

3. **LIE-derivatives of various entities.** Let $X^i(x, \dot{x})$ be a vector whose value at the point $\bar{x}^i$ is given by

$$X^i(\bar{x}, \dot{\bar{x}}) = X^i(x, \dot{x}) + \{ v^j \partial_j X^i + (\dot{\partial}_j X^i)(\dot{x}^k \partial_k v^j + \ddot{x}^k \dot{\partial}_k v^j) \} \, d\tau .$$

The value of $X^i(x, \dot{x})$ at $\bar{x}^i$, considered as having undergone the transformation (2.1), is

$${}'X^i(\bar{x}, \dot{\bar{x}}) = X^j \partial_j \bar{x}^i = \{ \delta^i_j + (\partial_j v^i)\, d\tau \} X^j = X^i(x, \dot{x}) + X^j (\partial_j v^i)\, d\tau .$$

Therefore the LIE-derivative of $X^i(x, \dot{x})$ is

(3.1) $$\underset{L}{D}\, X^i = v^j \partial_j X^i - X^j \partial_j v^i + (\dot{\partial}_j X^i)(\dot{x}^k \partial_k v^j + \ddot{x}^k \dot{\partial}_k v^j).$$

Using (1.3) and (1.4) the equation (3.1) may be reduced to the form

(3.2) $$\underset{L}{D}\, X^i = X^i_{\,|j}\, v^j - X^j (v^i_{\,|j} + G^k_j \dot{\partial}_k v^i) + (\dot{\partial}_j X^i)\{ v^j_{\,|k}\, \dot{x}^k + (\dot{\partial}_k v^j)(\ddot{x}^k + 2 G^k) \} .$$

The derivation formula (3.2) can be extended to an arbitrary tensor $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x, \dot{x})$ in the following manner :

(3.3) $$\left\{ \begin{aligned} \underset{L}{D}\, T^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x, \dot{x}) &= v^k\, T^{i_1 \ldots i_p}_{j_1 \ldots j_q | k}(x, \dot{x}) \\ &- \sum_{\alpha=1}^{p} T^{i_1 \ldots i_{\alpha-1}\, l\, i_{\alpha+1} \ldots i_p}_{j_1 \ldots \ldots \ldots \ldots j_q} (v^{i_\alpha}_{\;\,|l} + G^r_l \dot{\partial}_r v^{i_\alpha}) \\ &+ \sum_{\beta=1}^{q} T^{i_1 \ldots \ldots \ldots \ldots i_p}_{j_1 \ldots j_{\beta-1}\, l\, j_{\beta+1} \ldots j_q} (v^l_{\,|j_\beta} + G^r_{j_\beta} \dot{\partial}_r v^l) \\ &+ \left(\dot{\partial}_l T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \right) \{ v^l_{\,|\gamma}\, \dot{x}^\gamma + (\dot{\partial}_\gamma v^l)(\ddot{x}^\gamma + 2 G^\gamma) \} . \end{aligned} \right.$$

The LIE-derivative of a scalar function $S(x, \dot{x})$ may be similarly calculated : it has the form

(3.4a) $$\underset{L}{D}\, S(x, \dot{x}) = S_{|k}\, v^k + (\dot{\partial}_k S)\{ v^k_{\,|h}\, \dot{x}^h + (\dot{\partial}_h v^k)(\ddot{x}^h + 2 G^h) \} .$$

In case the scalar $S(x, \dot{x})$ is replaced by the metric function in (3.4a), in consequence of (1.5) we have

(3.4b) $$\underset{L}{D} F(x, \dot{x}) = (\dot{\partial}_k F) \{ v^k_{|h} \dot{x}^h + (\dot{\partial}_h v^k)(\ddot{x}^h + 2 G^h) \}.$$

The LIE-derivative of the element of support $\dot{x}^i$ follows from (3.1):

(3.5) $$\underset{L}{D} \dot{x}^i = \dot{x}^k \partial_k v^i.$$

In our applications of the LIE-derivatives we shall require, in particular, the LIE-derivatives of the metric tensor and of the connection coefficients of the space. The former is easily evaluated from (3.3). By virtue of (1.5) we have

(3.6) $$\underset{L}{D} g_{ij} = 2 g_{m(i} \{ v^m_{|j)} + G^r_{j)} \dot{\partial}_r v^m \} + (\dot{\partial}_m g_{ij}) \{ v^m_{|r} \dot{x}^r + (\dot{\partial}_r v^m)(\ddot{x}^r + 2 G^r) \}.$$

In order to find the LIE-derivatives of the connection coefficients $\Gamma^{*i}_{jk}(x, \dot{x})$ we can not apply (3.3) directly, because the Γ^{*i}_{jk} do not form the components of a tensor, so that we have to revert to the definition given by (2.5). Firstly, we note that

$$\Gamma^{*i}_{jk}(\bar{x}, \dot{\bar{x}}) = \Gamma^{*i}_{jk}(x, \dot{x}) + \{ v^h \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_h \Gamma^{*i}_{jk})(\dot{x}^m \partial_m v^h + \ddot{x}^m \dot{\partial}_m v^h) \} d\tau.$$

Application of (1.3) and (1.4) reduces the above identity to the form

(3.7) $$\Gamma^{*i}_{jk}(\bar{x}, \dot{\bar{x}}) = \Gamma^{*i}_{jk}(x, \dot{x}) + [(\partial_h \Gamma^{*i}_{jk} - G^m_h \dot{\partial}_m \Gamma^{*i}_{jk}) v^h + (\dot{\partial}_h \Gamma^{*i}_{jk}) \{ v^h_{|m} \dot{x}^m + (\dot{\partial}_m v^h)(\ddot{x}^m + 2 G^m) \}] d\tau.$$

Secondly, we remark that the law of transformation for the Γ^{*i}_{jk} may be written as

(3.8) $$'\Gamma^{*i}_{jk}(\bar{x}, \dot{\bar{x}}) = (\partial_r \bar{x}^i) \left\{ \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} + \Gamma^{*r}_{st} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial x^t}{\partial \bar{x}^k} \right\}.$$

Using (2.3) in (3.8) and simplifying we get

(3.9) $$'\Gamma^{*i}_{jk}(\bar{x}, \dot{\bar{x}}) = \Gamma^{*i}_{jk}(x, \dot{x}) - \{ \partial_j \partial_k v^i - \Gamma^{*r}_{jk} \partial_r v^i + 2 \Gamma^{*i}_{r(j} \partial_{k)} v^r \} d\tau.$$

Therefore, the LIE-derivative of $\Gamma^{*i}_{jk}(x, \dot{x})$ is given by

(3.10) $$\left\{ \begin{aligned} \underset{L}{D} \Gamma^{*i}_{jk} &= (\partial_h \Gamma^{*i}_{jh} - G^m_h \dot{\partial}_m \Gamma^{*i}_{jk}) v^h \\ &+ \partial_j \partial_k v^i - \Gamma^{*r}_{jk} \partial_r v^i + 2 \Gamma^{*i}_{r(j} \partial_{k)} v^r \\ &+ (\dot{\partial}_h \Gamma^{*i}_{jk}) \{ v^h_{|m} \dot{x}^m + (\dot{\partial}_m v^h)(\ddot{x}^m + 2 G^m) \}. \end{aligned} \right.$$

Considering the expansion

$$
\begin{aligned}
v^i_{|jk} = \partial_j \partial_k v^i - \Gamma^{*r}_{jk} \partial_r v^i + 2\, \Gamma^{*i}_{r(j} \partial_{k)} v^r \\
+ (\partial_k \Gamma^{*i}_{jh} - G^m_k \dot{\partial}_m \Gamma^{*i}_{jh} + 2\, \Gamma^{*i}_{m[k} \Gamma^{*m}_{h]j})\, v^h \\
- (\partial_j \dot{\partial}_h v^i + \Gamma^{*i}_{rj} \dot{\partial}_h v^r)\, G^h_k - (G^h_j \dot{\partial}_h v^i)_{|k},
\end{aligned}
$$

and the expression of the curvature tensor given by [3]

$$
K^i_{jkh} = 2\, \{ \partial_{[h} \Gamma^{*i}_{k]j} - (\dot{\partial}_m \Gamma^{*i}_{j[k})\, G^m_{h]} + \Gamma^{*i}_{m[h} \Gamma^{*m}_{k]j} \} \tag{3.11}
$$

the equation (3.10) reduces to

$$
\left\{
\begin{aligned}
& \underset{L}{D}\, \Gamma^{*i}_{jk} = v^i_{|jk} + v^h K^i_{jkh} \\
& + (\dot{\partial}_h \Gamma^{*i}_{jk})\, \{ v^h_{|m}\, \dot{x}^m + (\dot{\partial}_m v^h)\, (\ddot{x} + 2\, G^m) \} \\
& + (\partial_j \dot{\delta}_h v^i + \Gamma^{*i}_{rj} \dot{\delta}_h v^r)\, G^h_k + (G^h_j \dot{\delta}_h v^i)_{|k}\,.
\end{aligned}
\right. \tag{3.12}
$$

Particularly, if the points x^i are chosen to be on the geodesies of F_n and the element of support is taken along the unit tangent to the geodesies,

$$
x''^i + 2\, G^i\, (x, x') = 0
$$

being the equation of the geodesies, the Lie-derivatives of various geometric objects discussed above reduce to their simpler forms.

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ÖZET

LIE-türevi kavramı, bir FINSLER uzayının sonsuz küçük şekil değişimleri teorisine ilk defa DAVIES tarafından sokulmuştur. Ancak, bu yazarın verdiği tanımda, şekil değişimlerinin belirtilmesine yarayan vektörün doğrultulara bağlı olmadığı kabul edilmiştir. Bu araştırmada ise, sonsuz küçük dönüşümler en genel ifadeleri ile ele alınmış ve yukarda söz konusu edilen vektörün hem noktanın, hem de noktaya bağlı doğrultu elemanının bir fonksiyonu olduğu kabul edilmiştir.