ON THE ZEROS OF ENTIRE FUNCTIONS

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The present paper takes into consideration an inequality due to BOAS [1], concerning the number of zeros of an entire function and aims to give both this inequality and some analogous relations derived here, a somewhat sharper form.

1. Let $f(z)$ be an entire function of order ϱ and lower order λ. If $f(z)$ has at least one zero in $|z| \leqq r$, the exponent of convergence $\sigma (\leqq \varrho)$ of its zeros is given by

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \sigma, \tag{1.1}$$

where $n(r)$ denotes the number of zeros of $f(z)$ in $|z| \leqq r$.

We call δ the lower exponent of convergence, if

$$\liminf_{r\to\infty} \frac{\log n(r)}{\log r} = \delta. \tag{1.2}$$

If the entire function $f(z)$ has no zero at the origin, *i.e.*, $n(0) = 0$, let

$$N(r) = \int_0^r t^{-1} n(t)\, dt. \tag{1.3}$$

It can be easily seen that

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log N(r)}{\log n} = \begin{matrix}\sigma \\ \delta\end{matrix}. \tag{1.4}$$

If

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{N(r)}{n(r)} = \begin{matrix}c \\ d\end{matrix} \tag{1.5}$$

then it is known [1] that

$$d \leq \frac{1}{\sigma} \leq \frac{1}{\delta} \leq c. \tag{1.6}$$

If $0 < \varrho < \infty$, let

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{N(r)}{r^{\varrho}} = \begin{matrix} M \\ m \end{matrix} \tag{1.7}$$

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{n(r)}{r^{\varrho}} = \begin{matrix} L \\ l \end{matrix} \tag{1.8}$$

In the present paper we sharpen (1.6) in a certain sense and obtain some relations involving the constants L, l, c, M, etc. We also derive relations between the exponents of convergence of two or more entire functions All the constants involved are assumed to be non-zero finite.

2. Theorem 1. *If the constants have the meaning as defined in section* 1, *we have*

(*i*)
$$\frac{l}{\varrho L} \leq d \leq c \leq \frac{L}{\varrho l},$$

(*ii*) *If* $0 < m \leq M < \infty$ *then* $0 < l \leq L < \infty$ *and conversely.*

(*iii*) *If* (*ii*) *holds then*

$$\frac{1}{\varrho K} < d \leq c < \frac{K}{\varrho}$$

where $x = K$ *is that root of the equation* $e M \log x = xm - e M$ *which lies in the interval* (e, ∞).

Proof. By (1.8), for any $\varepsilon > 0$ and for all $r > r_0 = r_0(\varepsilon)$,

$$(l - \varepsilon) r^{\varrho} < n(r) < (L + \varepsilon) r^{\varrho}. \tag{2.1}$$

But,

$$N(r) = N(r_0) + \int_{r_0}^{r} x^{-1} n(x)\, dx,$$

or

$$\frac{N(r)}{n(r)} = o(1) + \frac{1}{n(r)} \int_{r_0}^{r} x^{-1} n(x)\, dx$$

$$< o(1) + \frac{1}{n(r)} \int_{r_0}^{r} \frac{(L + \varepsilon) x^{\varrho}}{x}\, dx,$$

by (2.1).

$$= o(1) + \frac{L + \varepsilon}{\varrho} \cdot \frac{r^{\varrho}}{n(r)}$$

So, on proceeding to limits, we get

$$\limsup_{r\to\infty} \frac{N(r)}{n(r)} \leq \frac{L}{\varrho\, l}.$$

In a similar manner, it can be shown that

$$\liminf_{r\to\infty} \frac{N(r)}{n(r)} \geq \frac{\varrho\, L}{l}.$$

Combining these results, we get (*i*),

(*ii*) If $a \geq 1$, $L < \infty$, then

$$N(ra^{1/\varrho}) \sim o(1) + \left(\int_{r_0}^{r} + \int_{r}^{ra^{1/\varrho}} \right) n(t)\, t^{-1}\, dt$$

$$< o(1) + (L+\varepsilon) \int_{r_0}^{r} x^{\varrho-1}\, dx + \frac{n(ra^{1/\varrho}) \log a}{\varrho}$$

$$\sim (L+\varepsilon)\frac{r^{\varrho}}{\varrho} + \frac{n(ra^{1/\varrho}) \log a}{\varrho}.$$

Hence, dividing by ar^{ϱ} and proceeding to limits, we get

(2.2) $$\varrho\, a\, M \leq L + L\, a \log a,$$

(2.3) $$\varrho\, a\, m \leq L + l\, a \log a,$$

which hold also when $L = \infty$. Similarly we get

(2.4) $$\varrho\, a\, M \geq l + L \log a,$$

(2.5) $$\varrho\, a\, m \geq l\,(1 + \log a).$$

Suppose now $o < m \leq M < \infty$. From (2.4) we get $L < \infty$. Further $l > 0$. For, if $l = 0$ we get from (2.3) $m \leq \frac{L}{\varrho\, a}$ and since a is arbitrary it follows that $m = 0$. Hence we have a contradiction and so $l < 0$.

If $0 < l \leq < \infty$ then we have from (2.2) $M < \infty$ and from (2.5) $m > 0$.

(*iii*) Take $a = \exp\{(L-l)/L\}$ in (2.4). Then

$$L \leq \varrho\, M \exp\left(1 - \frac{l}{L}\right) < \varrho\, M\, e$$

and hence from (2.3)

$$\varrho\, a\, m < \varrho\, M\, e + l\, a \log a.$$

Consider now the equation

$$e\, M \log x = x\, m - e\, M.$$

It has one and only one root in the interval (e, ∞). Let it be K, then taking $K = a$, we get

$$\varrho\,(K\, m - M\, e) < l\, K \log K$$

or,

$$\varrho\, M\, e \log K < l\, K \log K,$$

i.e.,

$$l > \frac{\varrho\, e\, M}{K} > \frac{L}{K}$$

and hence, by the above relation and (*i*) it follows that

$$\frac{1}{\varrho\, K} < \frac{l}{L\, \varrho} \leq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{N(r)}{n(r)} \leq \frac{L}{l\, \varrho} < \frac{K}{\varrho}.$$

Hence the result.

The inequalities (*i*) of theorem 1 and (1.6) can be further sharpened as is evident from the following

Theorem 2. *If the constants have the meaning as defined in Sec.* 1, *we have*

$$\frac{1}{\varrho} \leq c \leq \frac{1}{\varrho}\left[1 + \log\frac{L}{l}\right] \leq \frac{L}{\varrho\, l} \tag{2.6}$$

$$\frac{l}{\varrho L} \leq \frac{e^{l/K}}{e\, \varrho} \leq d \leq \frac{1}{\varrho}. \tag{2.7}$$

Proof. We have, for $K \geq 1$,

$$N(r\, K^{1/\varrho}) = O(1) + \int_{r}^{r} x^{-1}\, n(x)\, dx + \int_{r_0}^{rK^{1/\varrho}} n(x)\, x^{-1}\, dx \tag{2.8}$$

$$> \frac{(l-\varepsilon)\, r^{\varrho}}{\varrho} + \frac{n(r) \log K}{\varrho},$$

by (2.1).

So,

$$\limsup_{r\to\infty} \frac{N(r\, K^{1/\varrho})}{n(r\, K^{1/\varrho})} \geq \frac{l}{\varrho\, K} \limsup_{r\to\infty} \frac{(r\, K^{1/\varrho})^{\varrho}}{n(r\, K^{1/\varrho})} + \liminf_{r\to\infty} \frac{n(r)}{r^{\varrho}} \liminf_{r\to\infty} \frac{(r\, K^{1/\varrho})^{\varrho}}{n(r\, K^{1/\varrho})} \cdot \frac{\log K}{\varrho\, K}$$

which gives

$$C \geq \frac{L + l \log K}{\varrho L K}.$$

Putting $K = 1$, we get $C \geq \frac{1}{\varrho}$.

Further,

(2.9) $$N(r K^{1/\varrho}) < \frac{(L+\varepsilon) r^\varrho}{\varrho} + \frac{n(r K^{1/\varrho})}{\varrho} \log K.$$

Or,

$$\limsup_{r\to\infty} \frac{N(r K^{1/\varrho})}{n(r K^{1/\varrho})} \leq \frac{L}{\varrho K} \limsup_{r\to\infty} \frac{(r K^{1/\varrho})^\varrho}{n(r K^{1/\varrho})} + \frac{\log K}{\varrho}$$

which gives

$$C \leq \frac{L + l K \log K}{\varrho l K}.$$

Taking $K = L/l$ in the right-hand side, wc get

$$C \leq \frac{1}{\varrho}\left[1 + \log\frac{L}{l}\right] \leq \frac{1}{\varrho}\cdot\frac{L}{l} \quad \text{since} \quad 1 + \log x \leq x \quad \text{for} \quad x \geq 1.$$

This proves (2.6).

Now, by (2.9), we get

$$\liminf_{r\to\infty} \frac{N(r K^{1/\varrho})}{n(r K^{1/\varrho})} \leq \frac{L}{K\varrho} \liminf_{r\to\infty} \frac{(r K^{1/\varrho})^\varrho}{n(r K^{1/\varrho})} + \frac{\log K}{\varrho}$$

i.e.,

$$d \leq \frac{1}{\varrho}\cdot\frac{1 + K\log K}{K}.$$

Putting $K = 1$, this gives $d \leq \frac{1}{\varrho}$.

Again, by (2.8),

$$\frac{N(r K^{1/\varrho})}{n(r K^{1/\varrho})}\cdot\frac{n(r K^{1/\varrho})}{(r K^{1/\varrho})^\varrho}\cdot\frac{r^\varrho}{n(r)}\cdot K > \frac{l-\varepsilon}{\varrho}\cdot\frac{r^\varrho}{n(r)} + \frac{\log K}{\varrho}.$$

So,

$$d \times \frac{1}{L} \times L \times K \geq \frac{l}{\varrho L} + \frac{\log K}{\varrho}$$

i.e.,

$$d \geq \frac{1}{\varrho L}\cdot\frac{l + L\log K}{K}.$$

Taking $K = \exp \{ (L - l) / L \}$, we get

$$d \geq \frac{1}{e \varrho} e^{l/L} \geq \frac{l}{\varrho L}$$

since for

$$x \geq 0, \qquad e^x \geq e x.$$

This proves (2.7).

Remark. Since $(e^x - e x)$ has a minimum at $x = 1$ and is always non-negative, it follows that if $L \neq l$, then (*i*) of theorem 1, becomes

$$\frac{l}{L \varrho} < d \leq C < \frac{L}{l \varrho}.$$

Next we prove

Theorem 3. *If the constants have the same meaning as before, then*

(2.10) $$L + \varrho m \leq e \varrho M$$

(2.11) $$e l + \varrho M \leq e \varrho m$$

(2.12) $$l \leq \delta M.$$

Proof. We have, if $0 < \varrho < \infty$,

$$n(r) \leq \varrho \int_r^{r e^{1/\varrho}} x^{-1} n(x) \, dx.$$

Adding $\varrho N(r)$ to both sides, we get

$$n(r) + \varrho N(r) \leq \varrho N(r) + \varrho \int_r^{r e^{1/\varrho}} x^{-1} n(x) \, dx$$

i.e.,

$$n(r) + \varrho N(r) \leq \varrho N(r e^{1/\varrho}).$$

Dividing throughout by r^ϱ and proceeding to limits, it gives

$$\lim_{r\to\infty} \sup \frac{n(r)}{r^\varrho} + \varrho \lim_{r\to\infty} \inf \frac{N(r)}{r^\varrho} \leq e \varrho \lim_{r\to\infty} \sup \frac{N(r e^{1/\varrho})}{(r e^{1/\varrho})^\varrho}$$

whence (2.10) follows.

To prove (2.11) we note that

$$n(r e^{1/\varrho}) \geq \varrho \int_{r}^{r e^{1/\varrho}} n(x) x^{-1} dx .$$

Again adding $\varrho N(r)$ to both sides, we get

$$n(r e^{1/\varrho}) + \varrho N(r) \geq \varrho N(r e^{1/\varrho}) .$$

Dividing throughout by r^{ϱ} and proceeding to limits, we get

$$e \liminf_{r\to\infty} \frac{n(r e^{1/\varrho})}{(r e^{1/\varrho})} + \varrho \limsup_{r\to\infty} \frac{N(r)}{r^{\varrho}} \geq e \varrho \liminf_{r\to\infty} \frac{N(r e^{1/\varrho})}{(r e^{1/\varrho})^{\varrho}}$$

which gives (2.11).

Further, by (1.7), for any $\varepsilon > 0$, $r > r_0 = r_0(\varepsilon)$,

$$\frac{1}{(M+\varepsilon) r^{\varrho}} < \frac{1}{N(r)} < \frac{1}{(m-\varepsilon) r^{\varrho}} .$$

Also, for $r > r'_0$,

$$(l-\varepsilon) r^{\varrho} < n(r) < (L+\varepsilon) r^{\varrho}$$

and so for $r > \max(r_0, r'_0)$

$$\frac{l-\varepsilon}{M+\varepsilon} < \frac{n(r)}{N(r)} < \frac{L+\varepsilon}{m-\varepsilon} .$$

Proceeding to limits,

$$\frac{l}{M} \leq \frac{1}{C} \leq \frac{1}{d} \leq \frac{L}{m} .$$

Hence by (1.6),

$$\frac{l}{M} \leq \delta$$

which gives (2.12).

3. In this section we derive relations between the exponents of convergence and the lower exponents of convergence of two or more entire functions.

Theorem 4. *Let*

$$n(r, f_1), n(r, f_2), n(r, f)$$

denote respectively the number of zeros of the entire functions

$$f_1(z), f_2(z), f(z)$$

each having at least one zero in $|z| \leq r$. *Further, let* $\delta_1, \delta_2, \delta$ *denote the lower exponents* *of convergence and* σ_1, σ_2, σ *the exponents of convergence of the zeros of* $f_1(z)$, $f_2(z)$, $f(z)$ *respectively. Then, if*

$$\log n(r, f) \sim \log \{ [n(r, f_1)]^{p_1} [n(r, f_2)]^{p_2} \} \tag{3.1}$$

$$(0 < p_1, p_2 < \infty)$$

for $r \to \infty$, *we have,*

$$p_1 \delta_1 + p_2 \delta_2 \leq \delta \leq \sigma \leq p_1 \sigma_1 + p_2 \sigma_2 \tag{3.2}$$

while, if

$$\log n(r, f) \sim \sqrt{\{ \log [n(r, f_1)]^{p_1} \} \{ \log [n(r, f_2)]^{p_2} \}} \tag{3.3}$$

for $r \to \infty$, *then*

$$\sqrt{p_1 p_2 \delta_1 \delta_2} \leq \delta \leq \sigma \leq \sqrt{p_1 p_2 \sigma_1 \sigma_2}\,. \tag{3.4}$$

Proof. Using (1.1) for $f_1(z)$, we have for $\varepsilon > 0$ and $r > r_0 = r_0' = r_0(\varepsilon, f_1)$

$$\log n(r, f_1) < (\sigma_1 + \varepsilon) \log r\,. \tag{3.5}$$

Similarly for the function $f_2(z)$, for $\varepsilon > 0$ and $r > r_0' = r_0'(\varepsilon, f_2)$,

$$\log n(r, f_2) < (\sigma_2 + \varepsilon) \log r\,. \tag{3.6}$$

Hence, multiplying the inequalities (3.5) and (3.6) by p_1 and p_2 respectively and adding, we have, for sufficiently large r,

$$\log \{ [n(r, f_1)]^{p_1} [n(r, f_2)]^{p_2} \} < (p_1 \sigma_1 + p_2 \sigma_2 + \varepsilon') \log r\,.$$

Using (3.1) and dividing by $\log r$, we have

$$\frac{\log n(r, f)}{\log r} < (p_1 \sigma_1 + p_2 \sigma_2 + \varepsilon').$$

Now proceeding to limits and using (1.1) for $f(x)$, we get

$$\sigma \leq p_1 \sigma_1 + p_2 \sigma_2\,.$$

Similarly, using (1.2), it may be shown that

$$p_1 \delta_1 + p_2 \delta_2 \leq \delta\,.$$

To prove (3.4), we multiply the inequalities (3.5) and (3.6) after multiplying them by p_1 and p_2 respectively and get

$$\log \{ n(r, f_1) \}^{p_1} \cdot \log \{ n(r, f_2) \}^{p_2} < (p_1 \sigma_1 + \varepsilon')(p_2 \sigma_2 + \varepsilon'')(\log r)^2\,.$$

Now using (3.3) and proceeding to limits, we get

$$\sigma \leq \sqrt{p_1 \, p_2 \, \sigma_1 \, \sigma_2}\,.$$

A similar procedure on using (1.2) and (3.3) yields

$$\sqrt{p_1 \, p_2 \, \delta_1 \, \delta_2} \leq \delta.$$

Hence the theorem.

Corollary 1 : *If*

$$n(r, f_1),\ n(r, f_2),\ \ldots,\ n(r, f_m),\ n(r, f)$$

denote respectively the number of zeros of the entire functions

$$f_1(z),\ f_2(z),\ \ldots, f_m(z), f(z)$$

each having at least one zero in

$$|\,z\,| \leq r,$$

and

$$\delta_1\,,\ \delta_2\,,\ \ldots,\ \delta_m\,,\ \delta$$

denote the lower exponents of convergence and

$$\sigma_1\,,\ \sigma_2\,,\ \ldots,\ \sigma_m\,,\ \sigma$$

the exponents of convergence of the zeros of

$$f_1(z),\ f_2(z),\ \ldots,\ f_m(z),\ f(z)$$

respectively; then, if

$$\text{(3.7)} \qquad \log n(r, f) \sim \log \{ [\, n(r, f_1) \,]^{p_1} \, [\, n(r, f_2) \,]^{p_2} \cdots [\, n(r, f_m) \,]^{p_m} \},$$

$$(0 < p_k < \infty \;;\; K = 1, 2, \ldots, m),$$

we have

$$p_1 \, \delta_1 + p_2 \, \delta_2 + \cdots + p_m \, \delta_m \leq \delta \leq \sigma \leq p_1 \, \sigma_1 + p_2 \, \sigma_2 + \cdots + p_m \, \delta_m$$

while if

$$\text{(3.8)} \qquad \log n(r, f) \sim \{ \log [\, n(r, f_1) \,]^{p_1} \cdot \log [\, n(r, f_2) \,]^{p_2} \cdots \log [n(r, f_m) \,]^{p_m} \}^{1/m},$$

then

$$(p_1 \, p_2 \cdots p_m \, \delta_1 \, \delta_2 \cdots \delta_m)^{1/m} \leq \delta \leq \sigma \leq (p_1 \, p_2 \cdots p_m \, \sigma_1 \, \sigma_2 \cdots \sigma_m)^{1/m}\,.$$

Corollary 2: *If*

$$f_1(z), f_2(z), \ldots, f_m(z), f(z)$$

be entire functions of regular growth, having non-integral orders

$$\varrho_1, \varrho_2, \ldots, \varrho_m, \varrho$$

respectively and (3.7) *holds then*

$$\varrho \leq p_1 \varrho_1 + p_2 \varrho_2 + \cdots + p_m \varrho_m$$

while if (3.8) *holds then*

$$\varrho \leq (p_1 p_2 \cdots p_m \varrho_1 \varrho_2 \cdots \varrho_m)^{1/m}.$$

Corollary 1 follows as an immediate generalization of theorem 4, while corollary **2** follows as a direct consequence of corollary 1 and the fact that for entire functions of regular growth and non-integral orders, the exponents of convergence of their zeros are equal to their orders[1].

REFERENCE

[1] BOAS, R.P. : *Some elementary theorems on entire functions*, Rend. Circ. Mat. Palermo (2), 2, pp. 323 - 331 (1952).

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ÖZET

Bu yazıda bir tam fonksiyonun sıfırları hakkında BOAS (1) tarafından elde edilen bir eşitsizlikten hareket edilerek bu eşitsizliğe ve burada elde edilen benzer bazı bağıntılara daha kesin bir şekil verilmiştir.

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