ON THE ZEROS OF AN ENTIRE FUNCTION-I

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and suppose ρ denotes its order, λ its lower order, $\mu(r, f)$ its maximum term when $|z| = r$, $M(r, f)$ and $m(r, f)$ its maximum and minimum modulus on $|z| = r$, $n(r, f)$ the uumer of its zeros in $|z| \leq r$. Some inequalities concerning these quantities are obtained,

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ϱ and lower order λ and let $\mu(r, f)$ be the maximum term of $f(z)$ for $|z| = r$ while $M(r, f)$ and $m(r, f)$ denote the maximum modulus and minimum modulus of $f(z)$ on $|z| = r$ respectively. Finally, let

$n(r, f) = n(r)$ be the number of zeros of $f(z)$ in $|z| \leq r$.

We prove the following Theorems.

Theorem 1.

i)
$$\frac{\{I_\delta(R)\}^{\frac{1}{\delta}}}{m(r)} \geq \left(\frac{R}{r}\right)^{n(r)}.$$

ii)
$$\frac{m(R)}{\{I_\delta(r)\}^{\frac{1}{\delta}}} \leq \left(\frac{R}{r}\right)^{n(R)}.$$

for

$$R \geq r > 0.$$

where

$$I_\delta(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^\delta \, d\vartheta.$$

aud δ is any positive number.

We observe that if $f(z)$ has an infinity of zeros

$$r_1 e^{i\vartheta_1}, r_2 e^{i\vartheta_2}, \ldots, r_n e^{i\vartheta_n}, \ldots$$

the above inequalities are obviously true for a sequence $r_1, r_2, \ldots$ of values of r.

The main feature of the theorem is that it is true for all $R \geq r > 0$.

Theorem 1 improves the result due to S. K. SINGH [1].

Theorem 2.

Let

$$P(r) = \log \{ I_\delta (r) \}^{1/\delta} - \frac{1}{2\pi} \int_0^{2\pi} \log | f(r e^{i\vartheta}) | \, d\vartheta \geq 0.$$

Then

i) $$\left(\frac{R}{r} \right)^{\delta \, n(r)} \leq \frac{I_\delta (R)}{I_\delta (r)}, \text{ if } P(r) - P(R) \leq 0.$$

ii) $$\left(\frac{R}{r} \right)^{\delta \, n(R)} \geq \frac{I_\delta (R)}{I_\delta (r)}, \text{ if } P(r) - P(R) \geq 0.$$

iii) $$\left(\frac{R}{r} \right)^{\delta \, n(r)} \leq \frac{I_\delta (R)}{I_\delta (r)} \leq \left(\frac{R}{r} \right)^{\delta \, n(R)} \text{ if } P(r) - P(R) = 0.$$

for $R \geq r > 0$ *and* δ *is any positive number.*

We give an example for which all the above three cases will hold.

Theorem 3.

If $f(z)$ *is an entire function having no zeros in the unit circle, then,*

$$\frac{\{ I_\delta (R) \}^{1/\delta}}{m(r)} \geq \left(\frac{R}{r} \right)^{N(R)/\log R} \quad \text{for } R \geq r > 0,$$

where

$$N(r) = \int_0^r \frac{n(t)}{t} \, dt.$$

Theorem 4.

$$\frac{M(R)}{M(r)} \geq \left(\frac{R}{r} \right)^{n(r) \log \left(\frac{R}{r} \right) / \log R}, \quad R \geq r > 0.$$

Corollary.

$$\frac{M(R)}{M(r)} \geq n(r) \frac{\log (R/r)}{\log R} \left(\frac{R}{r} \right)$$

$$\text{if } R \geq e . r, \text{ and } n(r) \frac{\log (R/r)}{\log R} > 1.$$

Remark.

In Theorem 1, (*i*) and Theorem 3 we can easily replace $m(r)$ by $M(r)$, if we choose r such that

$$|f(r\,e^{i\vartheta})| = M(r).$$

Similarly we can replace $m(R)$ by $M(R)$ in Theorem 1 (*ii*).

Theorem 3 improves the result of S. K. SINGH [2],

Theorem 5.

If $f(z)$ is an arbitrary entire function of lower order λ $(1 \leq \lambda < \infty)$, $x = H(y)$ denotes the inverse function of $y = \log M(x)$. Further, if $n_k(f(z), 1)$ denotes the number of zeros of $f^k(z)$ in the unit circle, where $f^k(z)$ is the k^{th} derivative of $f(z)$, we have

$$\liminf_{k\to\infty} n_k(f(z), 1)\; H(k) \leq e^{2-1/\lambda}.$$

The above result inmproves the result ot P. ERDÖS & A. RENYI [3].

Theorem 6.

Let $0 < a_1 < a_2 < \ldots\ldots < a_n \ldots.$ be a sequence of numbers tending to infinity and let $n_1(r)$ denote the number of these $[o_n]$ not exceeding r. Similarly let $0 < b_1 < b_2 \ldots. < b_n \ldots.$ be a sequence of numbers tending to intinity and let $n_2(r)$ denote the number of these $[b_n]$ not exceeding r. We set,

$$Q_1(z) = \prod_{n=1}^{\infty} (1 - z^2/a_n^2) \text{ and } Q_2(z) = \prod_{n=1}^{\infty} (1 - z^2/b_n^2).$$

Let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(r, Q_2) - n(r, Q_1)}{r} = \begin{matrix}B\\ A\end{matrix}, \quad 0 < A \leq B.$$

Then

$$0 < \pi A \leq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left\{ \dfrac{M(r, Q_2)}{M(r, Q_1)} \right\}}{r} \leq \pi B.$$

Proof of Theorem 1.

By JENSEN's formula [2]

(1) $$\int_0^r \frac{n(x)}{x}\, dx = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r\,.\,e^{i\vartheta})|\, d\vartheta - \log |f(0)|\,.$$

We know that [5]

(2) $$\log\{I_\delta(r)\} \geq \frac{\delta}{2\pi}\int_0^{2\pi} \log|f(r\,.\,e^{i\vartheta})|\,d\vartheta.$$

From (1) and (2)

$$\log\{I_\delta(r)\} \geq \frac{\delta}{2\pi}\int_0^{2\pi} \log|f(r\,.\,e^{i\vartheta})|\,d\vartheta = \delta\int_0^{r}\frac{n(t)}{t}\,dt + \delta\log|f(e)|\,.$$

But

$$\int_r^R \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\log|f(R\,.\,e^{i\vartheta})|\,d\vartheta - \frac{1}{2\pi}\int_0^{2\pi}\log|f(r\,.\,e^{i\vartheta})|\,d\vartheta\,.$$

So

$$n(r)\log R/r \leq \log\{I_\delta(R)\}^{1/\delta} - \log m(r)$$

and

$$n(R)\log R/r \geq \log M(R) - \log\{I_\delta(r)\}^{1/\delta}.$$

So

$$\frac{\{I_\delta(R)\}^{1/\delta}}{m(r)} \geq \left(\frac{R}{r}\right)^{n(r)}$$

and

$$\frac{m(R)}{\{I_\delta(r)\}^{1/\delta}} \leq \left(\frac{R}{r}\right)^{n(R)}.$$

Proof of Theorem 2.

We know that [2]

$$\log\{I_\delta(r)\}^{1/\delta} \geq \frac{1}{2\pi}\int_0^{2\pi}\log|f(r\,.\,e^{i\vartheta})|\,d\vartheta\,.$$

$$\log\{I_\delta(r)\}^{1/\delta} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(r\,.\,e^{i\vartheta})|\,d\vartheta + P(r),$$

where $P(r) \geq 0$.

So

$$\int_r^R \frac{n(x)}{x}\,dx = \log\left\{\frac{I_\delta(R)}{I_\delta(r)}\right\}^{1/\delta} + P(r) - P(R).$$

So

$$\left(\frac{R}{r}\right)^{n(r)\,\delta} \leq \left\{\frac{I_\delta(R)}{I_\delta(r)}\right\} \text{ if } P(r) - P(R) \leq 0 .$$

which proves (*i*).

Similarly (*ii*) and (*iii*) follow.

Example.

Let

$$f(z) = e^z \quad , \quad \delta = 1 .$$

Then

$$\log\ I(r) = r - \frac{1}{2} \log\ r + 0\,(1).$$

So

$$P(r) = r - \frac{\log\ r}{2} + 0\,(1) \quad \text{and} \quad P(R) = R - \frac{\log R}{2} + 0\,(1),$$

where $R > r > 0$.

$$P(r) - P(R) = r - \frac{\log r}{2} - R + \frac{\log\ R}{2} + 0\,(1).$$

Let

$$R = kr \quad , \quad k > 1 .$$

Then

$$P(r) - P(R) = v - \frac{\log r}{2} - kr + \frac{\log r}{2} + \frac{\log\ k}{2} + 0\,(1) .$$

$$P(r) - P(R) = \frac{c \log k}{2} - r\,(k-1), \quad c \text{ is a constant.}$$

So

$$P(r) - P(kr) < 0 \quad \text{if} \quad c \log\ k/2\,(k-1) < r.$$

$$P(r) - P(kr) > 0 \quad \text{if} \quad c \log\ k/2\,(k-1) > r.$$

$$P(r) - P(kr) = 0 \quad \text{if} \quad c \log\ k/2\,(k-1) = r.$$

Proof of Theorem 3.

$$N(r) = \int_{r_0}^{r} \frac{n(t)}{t}\, dt$$

So

$$(1) \qquad N(R) - N(r) = \int_r^R \frac{n(t)}{t}\, dt \leq \log \left\{ \frac{\{ I_\delta(R)\}^{1/\delta}}{m(r)} \right\}$$

from the proof of Theorem 1.

$N(x)$ is an increasing convex function of $\log x$. If we draw the graph of $N(x)$, it will pass through the origin.

Let O be the origin and $A(\log R, N(R))$, and $B(\log r, N(r))$ be two points on the graph.

Then

$$\text{Slope of } OA \geq \text{Slope of } OB.$$

So

$$\frac{N(R)}{\log R} \geq \frac{N(r)}{\log r}$$

and it follows that

$$\frac{N(R) - N(r)}{\log R - \log r} \geq \frac{N(R)}{\log R}.$$

Thus from (1) we have

$$\frac{N(R)}{\log R} \leq \frac{\log \left\{ \frac{(I_\delta(R))^{1/\delta}}{m(r)} \right\}}{\log (R/r)}.$$

So

$$\frac{\{ I_\delta(R)\}^{1/\delta}}{m(r)} \geq \left(\frac{R}{r} \right).$$

Proof of Theorem 4.

We know that [2] $\log M(r)$ is an increasing convex function of $\log r$.

So

$$\frac{\log M(R)}{\log R} \geq \frac{\log M(r)}{\log r} \text{ for } R \geq r > 0.$$

$$\frac{\log M(R) - \log M(r)}{\log R - \log r} \geq \frac{\log M(R)}{\log R}.$$

So

$$\frac{\log M(R)}{\log R}$$

$$\frac{M(R)}{M(r)} \geq \left(\frac{R}{r} \right).$$

But

$$\log M(R) \geq \int_r^R \frac{n(t)}{t}\, dt \geq n(r) \log (R/r).$$

So

$$n(r) \frac{\log (R/r)}{\log (R)}$$

$$\frac{M(R)}{M(r)} \geq \left(\frac{R}{r} \right).$$

The Corollary follows from the following Lemma.

Lemma.

$$a^x \geqq ax \quad if \quad a \geqq e \quad and \quad x > 1.$$

Proof of the Lemma.

$$e^x \geqq e\,x.$$

$$x \geqq 1 + \log x.$$

$$\frac{\log x}{x-1} \leqq 1.$$

But

$$a \geqq e$$

$$\log a \geqq 1.$$

So

$$\log a \geqq \frac{\log x}{x-1}$$

i.e.,

$$x \log a \geqq \log a + \log x.$$

So,

$$a^x \geqq ax.$$

Proof of Corollary.

From the Lemma we have

$$\frac{M(R)}{M(r)} \geqq \frac{n(r)\log(R/r)}{\log R}\left(\frac{R}{r}\right)$$

And, if we put $R = r^k$, $k \geqq 1$, in theorem 4, then, we can easily show that,

$$\left\{\frac{M(kr)}{M(r)}\right\}^k \geqq (r)^{n(r)\,(k-1)^2}.$$

Proof of Theorem 5.

It is known that [7] if $\nu(r)$ denotes the central index of the power series of $f(z)$ for $|z| = r$.

Then,

(1) $$n_{\nu(r)}\ (f(z), 1) \leqq (\nu(r) + 1)\log\left(\frac{1}{1-\frac{e}{r}}\right).$$

It follows from (1) that

(2) $$\limsup_{r\to\infty} \frac{n_{\nu(r)}\ (f(z), 1)\, r}{\nu(r)} \leqq e$$

Now we may suppose without loss of generality $|f(0)| = 1$. In that ease,

(3) $$\log \mu(r) = \int_0^r \frac{\nu(t)}{t}\, dt.$$

It follows from (3) that, if $c > 1$, taking into account that $\nu(r)$ is a non decreasing function of r [8], [9], we have,

(4) $$\log \mu(rc) - \log \mu(r) = \int_r^{rc} \frac{\nu(t)}{t}\, dt \geq \nu(r) \log c.$$

It is known that

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leq \lambda.$$

Thus to any $\varepsilon > 0$, there can be found a sequence r_n $(n = 1, 2, \dots)$ for which $r_n \to \infty$ and $\nu(r_n) < (\lambda + \varepsilon) \log \mu(r_n)$.

From (4)

$$\nu(r_n)\,(\log c + 1/\lambda + \varepsilon) < \log \mu(r_n c).$$

Choosing

$$c = e^{1-1/(\lambda+\varepsilon)},$$

it follows that

(5) $$\nu(r_n) < \log \mu(r_n\, e^{1-1/(\lambda+\varepsilon)}).$$

As

$$\mu(r) \leq M(r),$$

(6) $$(5) \Rightarrow, \nu(r_n) < \log M(r_n\, e^{1-1/(\lambda+\varepsilon)}).$$

and thus

(7) $$H(\nu(r_n)) < (r_n\, e^{1-1/(\lambda+\varepsilon)}).$$

As by (2),

(8) $$\limsup_{n\to\infty} \frac{n_{\nu(r_n)}(f(z), 1)\, r_n}{\nu(r_n)} \leq e$$

and with respect to (7) we obtain

(9) $$\limsup_{n\to\infty} \frac{n_{\nu(r_n)}(f(z), 1)\, H(\nu(r_n))}{\nu(r_n)} \leq e^{2-1/(\lambda+\varepsilon)}.$$

But, (9) clearly implies

(10) $$\liminf_{k\to\infty} \frac{n_k(f(z), 1)\, H(k)}{(k)} \leq e^{2-1/(\lambda+\varepsilon)}.$$

As (10) is valid for any $\varepsilon > 0$, the assertion of Theorem 5 is proved.

Proof of Theorem 6.

$$\log M(r, Q_1) = \sum_{i=1}^{\infty} \log (1 + r^2/(a_i^2))$$

$$\log M(r, Q_1) = \int_1^{\infty} \log (1 + r^2/t^2)\, dn_1(t)$$

$$\log M(r, Q_1) = 2r^2 \int_1^{\infty} \frac{n_1(t)}{t(t^2+r^2)}\, dt - n_1(1) \log (r^2+1).$$

Similarly,

$$\log M(r, Q_2) = 2r^2 \int_1^{\infty} \frac{n_2(t)}{t(t^2+r^2)}\, dt - n_2(1) \log (r^2+1).$$

since

$$\frac{n(u)}{u^2} \to 0$$

by [10].

So

$$(11') \quad \log \left\{ \frac{M(r, Q_2)}{M(r, Q_1)} \right\} = 2r^2 \int_1^{\infty} \frac{n_2(t) - n_1(t)}{t(t^2+r^2)}\, dt - \log (r^2+1)\,(n_2(1) - n_1(1)).$$

Again by [10] we get

$$2(A-\varepsilon) r^2 \int_1^{\infty} \frac{dt}{t^2+r^2} + 0(\log r) < \log \left\{ \frac{M(r, Q_2)}{M(r, Q_1)} \right\} < 2(B+\varepsilon) r^2 \int_1^{\infty} \frac{dt}{t^2+r^2} + 0(\log r).$$

Taking limits we get

$$0 < \pi A \leq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left\{ \frac{M(r, Q_2)}{M(r, Q_1)} \right\}}{r} \leq \pi B \text{ [1]}.$$

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ÖZET

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ şeklinde tanımlanmış bir tam fonksiyonun derecesi ρ, alt derecesi λ, $|z| = r$ için maksimum terimi $\mu(r, f)$, $|z| = r$ üzerinde maksimum ve minimum modülü $M(r, f)$ ve $m(r, f)$, $|z| \leq r$ dairesel bölgesindeki sıfırlarının sayısı $n(r, f) = n(r)$ olsun. Bu yazıda bu büyüklüklerin sağladıkları birkaç eşitsizlik elde edilmiştir.