# ON THE MEAN VALUES OF INTEGRAL FUNCTIONS AND THEIR DERIVATIVES DEFINED BY DIRICHLET SERIES

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The function $f(s)$ defined by means of the Dirichlet series

$$f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

is considered: the $\lambda_n$ which appear in the definition form a monotonic increasing sequence and satisfy some particular limit conditions. The object of this paper is to prove some properties of the lower and upper limits of the mean value of this function and of its derivatives and to obtain some inequalites related to these quantities.

1. Consider the Dirichlet Series

$$f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \quad ,$$

where

$$\lambda_{n+1}>\lambda_n \, , \, \lambda_1 \geq 0, \lim_{n\to\infty} \lambda_n=\infty \, , \, s=\sigma+it$$

and

$$\lim_{n\to\infty} \sup \frac{\log n}{\log \lambda_n}=E<\infty \, . \tag{1.1}$$

Let $\sigma_c$ and $\sigma_a$ be the abscissa of convergence and abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c=\infty$ then $\sigma_a$ will also be infinite, since according to a known result ([1], p. 4) a Dirichlet series which satisfies (1.1) has its abscissa of convergence, equal to its abscissa of absolute convergence, and so $f(s)$ is an integral function.

The Mean Value of $f(s)$ is

$$I_2(\sigma)=I_2(\sigma, f)=\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2 \, dt \, , \tag{1.2}$$

and extending this definition to $f^{(p)}(s)$, the $p^{th}$ —derivative of $f(s)$,

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(1.3) $$I_2(\sigma, f^{(p)}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f^{(p)}(\sigma+it)|^2\, dt.$$

Let

$$\mu(\sigma) = \max_{n\geq 1}.\{ |a_n| e^{\sigma\lambda_n}\} \quad ; M(\sigma) = \underset{-\infty<t<\infty}{l.\,u.\,b.} \; |f(\sigma+it)|$$

be respectively the maximum term and the maximum modulus of an integral function.

It is known ( [2], p. 67) that

(1.4) $$\log\mu(\sigma) = \int_{\sigma_0}^{\sigma_{\iota}} \lambda_{\nu(t)}\, dt + O(1),$$

where $\nu(\sigma)$ is the rank of the maximum term.

Further, we know ( [3], p. 265, Theorem 5) that

(1.5) $$\log M(\sigma) \sim \log\mu(\sigma),$$

provided (1.1) holds and $f(s)$ is of finite order.

It is also known* ( [4] , p. 523) that for functions of finite non-zero linear order $\varrho$ and lower order $\lambda$,

(1.6) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log I_2(\sigma)}{\sigma} \right\} = \varrho_{\lambda}$$

and

(1.7) $$\log \{I_2(\sigma)\}^{1/2} \sim \log M(\sigma).$$

Throughout this paper we shall assume that the function $f(s)$ is of finite non-zero linear order and satisfies (1.1). In this paper we have obtained a few properties of $I_2(\sigma)$ and its derivative, and also of $I_2(\sigma, f^{(p)})$.

2. **Theorem 1.** *Let $f(s)$ be an integral function of linear order $\sigma$ and lower order $\lambda$, then*

(2.1) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log(I_2'(\sigma)/I_2(\sigma))}{\sigma} \right\} = \varrho_{\lambda}.$$

**Proof.** We know ( [4], p. 521) that $\log I_2(\sigma)$ is an increasing convex function of $\sigma$. Therefore, $\log I_2(\sigma)$ is differentiable almost everywhere with an increasing derivative, the set of points where the left hand derivative is less than the right hand derivative is of measure zero. This enables us to express $\log I_2(\sigma)$ in the following form :

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* Results (1.6) and (1.7) have been proved under the condition $\limsup_{n\to\infty} \frac{\log n}{\lambda n} = D = 0$ though these results also hold for $E < \infty$.

$$\log I_2(\sigma) = \log I_2(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{I_2'(x)}{I_2(x)}\,dx\,,$$

for an arbitrary $\sigma_0$.

We have

$$\log I_2(\sigma) \leqq \log I_2(\sigma_0) + (\sigma - \sigma_0)\frac{I_2'(\sigma)}{I_2(\sigma)}$$

or

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log I_2(\sigma)}{\sigma} \right\} \leqq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\left(\frac{I_2'(x)}{I_2(x)}\right)}{\sigma} \right\}.$$

Again, for an arbitrary fixed $k < 0$

$$\log I_2(\sigma + k) = \log I_2(\sigma) + \int_{\sigma}^{\sigma+k} \frac{I_2'(x)}{I_2(x)}\,dx \;\geqq\; k\,\frac{I_2'(\sigma)}{I_2(\sigma)}\,,$$

and therefore

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log I_2(\sigma + k)}{\sigma} \right\} \geqq \lim_{\sigma\to\infty} \left\{ \frac{\log\left(\frac{I_2'(\sigma)}{I_2(\sigma)}\right)}{\sigma} \right\}.$$

Thus

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log I_2(\sigma)}{\sigma} \right\} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\left(\frac{I_2'(\sigma)}{I_2(\sigma)}\right)}{\sigma} \right\}.$$

Further, from (1.7) we have

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log I_2(\sigma)}{\sigma} \right\} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log M(\sigma)}{\sigma} \right\}.$$

Hence

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\left(\frac{I_2'(\sigma)}{I_2(\sigma)}\right)}{\sigma} \right\} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log\log M(\sigma)}{\sigma} \right\} = \varrho_{\lambda}.$$

**Theorem 2.** *Let $f(s)$ be an integral function of linear order $\varrho$ and lower order $\lambda$, then*

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\sigma\, \lambda\nu(\sigma)} \right\} \leqq 2\,(1 - \lambda/\varrho) \tag{2.2}$$

*and*

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\lambda_{\nu}(\sigma) \log \lambda_{\nu}(\sigma)} \right\} \leqq 2\left(\frac{1}{\lambda} - \frac{1}{\varrho}\right) \tag{2.3}$$

**Proof.** It is known* ( [5], p. 84) that, for $0 < \varrho < \infty$

$$\lim_{\sigma\to\infty} \operatorname{snp} \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu}(\sigma)} \right\} \leq 1 - \frac{\lambda}{\varrho} \tag{2.4}$$

and

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)} \log \lambda_{\nu}(\sigma)} \right\} \leq 1/\lambda - 1/\varrho. \tag{2.5}$$

Using (1.5) and (1.7), we have

$$\begin{aligned}\lim_{\sigma\to\infty} \sup \left\{ \frac{\log \{I_2\}(\sigma)^{\nu_2}}{\sigma \lambda_{\nu(\sigma)}} \right\} &= \lim_{\sigma\to\infty} \sup \left\{ \frac{\log M(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \right\} \\ &= \lim_{\sigma\to\infty} \sup \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \right\} \\ &\leq 1 - \lambda/\varrho .\end{aligned}$$

Hence,

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_{\nu}(\sigma)} \right\} \leq 2\,(1 - \lambda/\varrho).$$

Proceeding as above and using (2.5), we have

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\lambda_{\nu(\sigma)} \log \lambda_{\nu(\sigma)}} \right\} \leq 2\,({}^1/_\lambda - {}^1/_\varrho).$$

**Theorem 3.** *Let $f(s)$ be an integral function of finite linear order $\varrho$ and lower order $\lambda$, then*

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \right\} \leq 2 . \tag{2.6}$$

**Proof.** Using (1.5) and (1.7), we have

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \right\} = 2 \lim_{\sigma\to\infty} \sup \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \right\}.$$

From (1.4), we get

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \right\} \leq 1.$$

Hence

$$\lim_{\sigma\to\infty} \sup \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_{\nu}(\sigma)} \right\} \leq 2 .$$

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* Results (2.4) and (2.5) have been proved under the condition

$$\lim_{n\to\infty} \sup \left\{ \frac{\log n}{\lambda_n} \right\} = D = 0,$$

though the results also hold for $E < \infty$.

**Theorem 4.** *Let $f(s)$ be an integral function of finite linear order $\varrho$ and lower order $\lambda$, then for*

$$\sigma < \sigma_0 \, , \; \varepsilon > 0$$

$$I_2'(\sigma) > \frac{I_2(\sigma) \log I_2(\sigma)}{(1+\varepsilon)\sigma}, \tag{2.7}$$

*where*

$$\varepsilon = \varepsilon(\sigma) \to 0 \;\; as \;\; \sigma \to \infty \; .$$

**Proof.** For the left hand derivative of $\log I_2(\sigma)$, we have

$$\frac{I_2'(\sigma)}{I_2(\sigma)} \geq \frac{\log I_2(\sigma) - \log I_2(\sigma_1)}{\sigma - \sigma_1}$$

$$> \frac{\log I_2(\sigma)}{(1+\varepsilon)\sigma},$$

where $\sigma_1 < \varrho$ and $\varepsilon = \varepsilon(\sigma) \to 0$ as $\sigma \to \infty$.

Hence

$$I_2'(\sigma) < \frac{I_2(\sigma) \log I_2(\sigma)}{(1+\varepsilon)\sigma},$$

where

$$\varepsilon \equiv \varepsilon(\sigma) \quad \text{as} \quad \sigma \to \infty \, .$$

**Corollary.** *Let $f(s)$ be an integral function of finite order $\varrho$ and if $I_2'(\sigma)$ is the derivative of $I_2(\sigma)$, then for $\sigma_0 < \sigma_1 < \sigma_2$*

$$\frac{I_2'(\sigma_1)}{I_2(\sigma_1)} \leq \frac{\log I_2(\sigma) - \log I_2(\sigma_1)}{\sigma_2 - \sigma_1} \leq \frac{I_2'(\sigma_1)}{I_2(\sigma_2)} . \tag{2.8}$$

**Proof.** From (1.4), we have

$$\log I_2(\sigma_2) = \log I_2(\sigma_1) + \int_{\sigma_1}^{\sigma_2} \frac{I_2'(x)}{I_2(x)} \, dx$$

$$\leq \log I_2(\sigma_1) + (\sigma_2 - \sigma_1) \frac{I_2'(\sigma_2)}{I_2(\sigma_2)} \tag{2.9}$$

and

$$\log I_2(\sigma) = \log I_2(\sigma_1) + \int_{\sigma_1}^{\sigma_2} \frac{I_2'(x)}{I_2(x)} \, dx$$

$$\geq \log I_2(\sigma_1) + (\sigma_2 - \sigma_1) \frac{I_2'(\sigma_1)}{I_2(\sigma_1)} . \tag{2.10}$$

Combining (2.9) and (2.10) we get the result.

3. **Theorem 5.** *Let $f(s)$ be an integral function of linear order $\varrho$ and lower order $\lambda$, then for $Re(s)=\sigma$ and $\lambda \geqq \delta > 0$*

$$I_2(\sigma, f) < I_2(\sigma, f^{(1)}) < \cdots\cdots < I_2(\sigma, f^{(b)}), \tag{3.1}$$

*where $p$ is an integer.*

**Proof.** It is known ( [4], p. 522)

$$I_2(\sigma, f^{(1)}) \leqq \frac{1}{2^2}\left(\frac{\log I_2(\sigma, f)}{\sigma}\right)^2 I_2(\sigma, f). \tag{3.2}$$

Taking limits on both the sides and using (1.6), we get

$$\liminf_{\sigma\to\infty} \left[\frac{1}{\sigma}\log\left\{\frac{I_2(\sigma, f^{(1)})}{I_2(\sigma, f)}\right\}^{\frac{1}{2}}\right] \geqq \lambda .$$

Again, for $\varepsilon > 0$ and $\sigma$ sufficiently large

$$\left\{\frac{I_2(\sigma, f^{(1)})}{I_2(\sigma, f)}\right\} > e^{2\sigma(\tau-\varepsilon)},$$

If $\lambda \geqq \delta > 0$,

$$I_2(\sigma, f) < I_2(\sigma, f^{(1)}),$$

and the result follows for subsequent derivatives.

**Theorem 6.** *Let $f(s)$ be an integral function, then for*

$$\sigma > 0 \quad \lambda \geqq \delta > 0$$

*and*

$$I_2(\sigma, f^{(P)}) \geqq \frac{1}{2^{\alpha p}}\left\{\frac{\log I_2(\sigma, f)}{\sigma}\right\}^{\alpha p} I_2(\sigma, f), \tag{3.3}$$

*where $p$ is an integer.*

**Proof.** Writing (3.2) for $p$ th—derivative, we have

$$\begin{aligned} I_2(\sigma, f^{(P)}) &\geqq \frac{1}{2^2}\left\{\frac{\log I_2(\sigma, f(p-1))}{\sigma}\right\}^2 I_2(\sigma, f^{(p-1)}) \\ &\geqq \frac{1}{(2^2)^2}\left\{\frac{\log I_2(\sigma, f(p-1))}{\sigma}\right\}^2 \left\{\frac{\log I_2(\sigma, f(p-2))}{\sigma}\right\} . I_2(\sigma, f(p-2)) \\ &\geqq \frac{1}{2^{2p}}\left\{\frac{\log I_2(\sigma f^{(P-1)})}{\sigma}\right\}^2 \left\{\frac{\log I_2(\sigma, f^{(P-2)})}{\sigma}\right\}^2 \left\{\frac{\log I_2(\sigma, f)}{\sigma}\right\}^2 I_2(\sigma). \end{aligned}$$

Using (3.1), we get

$$I_2(\sigma, f^{(p)}) \geq \frac{1}{2^{2p}} \left\{ \frac{\log I_2(\sigma, f)}{\sigma} \right\}^{2p} I_2(\sigma, f).$$

**Corollary.** *Let $f(s)$ be an integral function of linear order $\varrho$ and lower order $\lambda$, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left[ \frac{1}{\sigma} \log \left\{ \frac{I_2(\sigma f^{(p)})}{I_2(\sigma, f)} \right\}^{1/2p} \right] \geq \varrho_\lambda, \tag{3.4}$$

*where $p$ is an integer.*

This follows immediately from (3.3). (*)

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## ÖZET

$\{\lambda_n\}$ dizisi monoton artan ve bazı limitleri özel şartlar sağlamak üzere,

$$f(s) = \sum a_n e^{s\lambda_n}$$

Dirichlet serisi ile tanımlanan fonksiyon göz önüne alınıyor. Bu fonksiyonun ortalama değeri ve ortalama değerinin ardışık türevleri tanımlanarak, bu fonksiyonların alt ve üst limitleri tarafından sağlanan bazı bağıntılar elde edilmekte ve bunlar için bazı eşitsizlik ispat edilmektedir.

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