ON THE BOREL TRANSFORM OF AN ENTIRE FUNCTION OF EXPONENTIAL TYPE

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Let $Lf(z) = \sum_{n=0}^{\infty} (n!)\, a_n z^n$ denote the BOREL transform of the entire function defined by the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Then let

$$f_{L_1}(z_1) = \sum_{n=0}^{\infty} (n!)\, a_n z_1^n ,$$

where $z \cdot z_1 = 1$; similarly, let

$$f_{L_k}(z_k) = \sum_{n=0}^{\infty} (n!)^k a_n z_k^n ,$$

where $z_{k-1} \cdot z_k = 1$. Then all the $f_{L_k}(z_k)$ are entire functions on the z-plane or on the $(1/z)$-plane, according to the parity of k. A necessary and sufficient condition for $f_{L_k}(z_k)$ to be of finite order has been obtained, as well as some relations between the orders, lower orders, types and λ - types of $f(z)$ and $f_{L_k}(z_k)$ (*).

1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order ϱ and lower order λ. It is said to be of exponential type if it is of growth $(1, T)$ *i.e.*, of order $\varrho \leq 1$ and if $\varrho = 1$, then its type is at the most equal to $T (T < \infty)$. BOREL first showed ([1], p. 73) that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

is an entire function of order 1 and type σ if

$$L\, f(z) = \sum_{n=0}^{\infty} (n!)\, a_n z^n \tag{1.2}$$

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is convergent for $|z| \geq \sigma$. $L f(z)$, as usual, denotes the BOREL transform of $f(z)$. We apply the transformation $z = \frac{1}{z_1}$ to (1.2) and denote

$$\frac{1}{z_1} L f\left(\frac{1}{z_1}\right) \text{ by } f_{L_1}(z_1).$$

Thus, we get

$$f_{L_1}(z_1) = \sum_{n=0}^{\infty} a_n (n!)\, z_1^n \tag{1.3}$$

where $f(z)$ and $L f(z)$ are in the same plane while $f_{L_1}(z_1)$ is in the new plane which we denote by Z -plane. If the order ϱ of $f(z)$ is less than 1, then we show in this paper that $f_{L_1}(z_1)$ is also an entire function in the Z-plane. This is generalised by appliying the BOREL transform and inversion transform repeatedly. Thus, if $f_{L_1}(z_1)$, be of order ϱ_1 ($\varrho_1 < 1$) the application of the BOREL transform and the inversion transform $z_1 = \frac{1}{z_2}$ to (1.3) yields

$$\frac{1}{z_2} f_{L_1}\left(\frac{1}{z_2}\right) = \sum_{r=0}^{\infty} a_n\, (n!)^2\, z_2^n = f_{L_2}(z_2) \tag{1.4}$$

which will be an entire function in the z-plane. Repeating the argument k times, we can write

$$f_{L_k}(z_k) = \sum_{n=0}^{\infty} a_n\, (n!)^k\, z_k^n . \tag{1.5}$$

Evidently, if k is an odd integer the function $f_{L_k}(z_k)$ will be in the Z -plane, while if k is even, it will be in the z-plane. It will follow that if $f_{L_{k-1}}(z_{k-1})$ is an entire function of order $\varrho_{k-1} < 1$ in one of these planes then $f_{L_k}(z_k)$ is an entire function in the other plane.

In this paper I have investigated a necessary and sufficient condition under which $f_{L_k}(z_k)$ is an entire function of finite order in one of these planes. A number of relations between orders, lower orders, types, and λ -types of $f(z)$ and $f_{L_k}(z_k)$ have also been obtained. The results are given in the form of theorems with remarks.

2. **Theorem 1.** *If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order ϱ, *then*

$$f_{L_k}(z_k), \ (k = 1, 2, \ldots, m)$$

is an entire function of finite order in one of the planes, if and only if, $k\varrho < 1$.

Proof. We have

$$\frac{\log |a_n (n!)^k|^{-1}}{n \log n} \sim \frac{\log |a_n|^{-1}}{n \log n} - \frac{k \log (n!)}{n \log n}.$$

But from STIRLING's formula

$$n! \sim n^{n+\frac{1}{2}} . e^{-n} . \sqrt{2\pi} .$$

Therefore,

$$\frac{\log |a_n (n!)^k|^{-1}}{n \log n} \sim \frac{\log |a_n|^{-1}}{n \log n} - \frac{k (n+\frac{1}{2}) \log n}{n \log n} + O\left(\frac{1}{\log n}\right)$$

$$= \frac{\log |o_n|^{-1}}{n \log n} - k + 0(1).$$

Since ([1], p. 9)

$$\text{(2.1)} \qquad \liminf_{n\to\infty} \frac{\log |an|^{-1}}{n \log n} = \frac{1}{\varrho},$$

by making use of (2.1) in the above expression we get

$$\text{(2.1)}' \qquad \text{hm}\inf_{n\to\infty} \frac{\log |a_n (n!)^k|^{-1}}{n \log n} = \frac{1}{\varrho} - k .$$

This leads us to the conclusion that $f_{L_k}(z_k)$ is of finite order if $k\varrho < 1$.

Conversely, let $f(z)$ be of order ϱ (where $k\varrho < 1$); then, from (2.1)

$$\limsup_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}} = \varrho .$$

Hence for any $\varepsilon > 0$, we can find a number $N(\varepsilon)$ such that

$$\frac{n \log n}{\log |a_n|^{-1}} < (\varrho + \varepsilon) \quad \text{for all} \quad n > N_0(\varepsilon)$$

or

$$|a_n| < n^{-\frac{n}{\varrho+\varepsilon}}$$

or

$$|(n!)^k a_n|^{1/n} < n^{k-\frac{1}{\varrho+\varepsilon}} .$$

Therefore

$$\lim_{n\to\infty} |(n!)^k a_n|^{1/n} = 0$$

and hence the theorem.

Theorem 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be of order ϱ ($m\varrho < 1$) and lower order λ and let ϱ_k and λ_k denote respeetively the order and lower order of

$$f_{L_k}(z_k), \quad (k = 1, 2, \ldots, m),$$

then

(2.2) $$\varrho_k = \frac{\varrho}{1 - k\varrho}.$$

If further,

$$\left| \frac{a_n}{(n+1)^k a_{n+1}} \right|, \quad (k = 1, 2, \ldots, m)$$

forms a non-decreasing function of n for $n > n_0$ then

(2.3) $$\lambda_k = \frac{\lambda}{1 - k\lambda}.$$

Proof. (2.2) follows from (2.1)'. Further, since

$$\left| \frac{a_n}{(n+1)^k a_{n+1}} \right|, \quad (k = 1, 2, \ldots, m)$$

forms a non-decreasing function of n for $n > n_0$, in view of the fact ([2], p. 1047), that

(2.4) $$\limsup_{n\to\infty} \frac{\log |a_n|^{-1}}{n \log n} = \frac{1}{\lambda},$$

we get

$$\limsup_{n\to\infty} \frac{\log |a_n (n!)^k|^{-1}}{n \log n} = \frac{1}{\lambda_k}.$$

Using STIRLING's formula for $n!$, we easily get

$$\lambda_k = \frac{\lambda}{1 - k\lambda}.$$

Applications. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be of order ϱ, $(m\varrho < 1)$ lower order λ and

$$f_{L_k}(z_k) = \sum_{n=0}^{\infty} a_n (n!)^k z_k^n$$

be of order ϱ_k and lower order λ_k respectively and satisfy the hypothesis of theorem 2: then the direct consequences of (2.2) and (2.3) are

(2.5) $$\varrho < \varrho_1 < \varrho_2 < \ldots < \varrho_{m-1} < \varrho_m \quad \text{if and only if} \quad \varrho \neq 0,$$

(2.6) $$\lambda < \lambda_1 < \lambda_2 < \ldots < \lambda_{m-1} < \lambda_m \quad \text{if and only if} \quad \lambda \neq 0$$

(2.7) $\varrho = 0$ if and only if $\varrho_k = 0$ for any k, $1 \leq k \leq m$,

(2.8) $\lambda = 0$ if and only if $\lambda_k = 0$ for any k, $1 \leq k \leq m$,

(2.9) $\varrho = \lambda$ if and only if $\varrho_k = \lambda_k$ for any k, $1 \leq k \leq m$,

i.e. if $f(z)$ is of regular growth then so is $f_{L_k}(z_k)$ and vice versa.

Theorem 3. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order ϱ, $(0 < m\varrho < 1)$ *and type* T. *Then*

$$T_k = \frac{1}{\varrho_k} (\varrho T)^{\frac{\varrho_k}{\varrho}} \quad \text{for } k = 1, 2, \dots, m \tag{2.10}$$

where ϱ_k *and* T_k *are the order and type, respectively, of* $f_{L_k}(z_k)$.

Proof. Let

$$\psi(n) = \frac{n}{e\,\varrho_k} \,|\, a_n (n!)^k \,|^{\frac{\varrho_k}{n}} .$$

Then

$$\log \psi(n) = \log \frac{1}{e\,\varrho_k} + \log n + \frac{k\,\varrho_k}{n} \log (n!) + \frac{\varrho_k}{n} \log \,|\, a_n \,| .$$

Since, from (2.2),

$$\varrho_k = \frac{\varrho}{1 - k\varrho} \quad \text{and} \quad n! \sim n^{n+\frac{1}{2}} \,.\, e^{-n} \sqrt{2\pi},$$

by making use of these in the above expression, we get

$$\log \psi(n) \sim \log \frac{1}{e\,\varrho_k \,.\, e^{k\,\varrho_k}} + \frac{\varrho_k}{\varrho} \log n \,|\, a_n \,|^{\varrho/n} .$$

Now, proceeding to limit, we get

$$T_k = \frac{1}{\varrho_k} (\varrho T)^{\frac{\varrho_k}{\varrho}}$$

in view of the fact ([1], p. 11) that

(2.11)
$$\limsup_{n\to\infty} \frac{n}{e\varrho} |a_n|^{\varrho/n} = T$$

and hence the theorem is proved.

Let $f(z)$ be of order ϱ and lower order $\lambda, (0 \leq \lambda < \varrho < \infty)$, then [3],

(2.12)
$$\liminf_{\gamma\to\infty} \frac{\log M(\gamma)}{\gamma^{\varrho}} = 0 .$$

Further, let

(2.13)
$$\liminf_{\gamma\to\infty} \frac{\log M(\gamma)}{\gamma^{\lambda}} = t_{\lambda} .$$

We call t_{λ} the λ -type of $f(z)$. It has also been shown [3] that, if $(0 < \lambda < \infty)$ and $\left|\frac{a_n}{a_{n+1}}\right|$ forms a non-decreasing function of n for $n > n_0$, then

(2.14)
$$\liminf_{n\to\infty} \frac{n}{e\lambda} |a_n|^{\lambda/\lambda_n} = t_{\lambda} .$$

The following theorem can be proved on the lines of the proof of theorem 3.

Theorem 4. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order ϱ*, lower order* $\lambda\ (0 < \lambda < \infty)$ *and* λ*-type* t_{λ}*. If*

$$\left| \frac{a_n}{(n+1)^k a_{n+1}} \right| , \quad (k = 1, 2, \dots, m)$$

forms a non-decreasing function of n for $n > n_0$*, then*

(2.15)
$$t_{\lambda^k} = \frac{1}{\lambda_k} (\lambda t_{\lambda})^{\frac{\lambda_k}{k}}$$

for

$$k = 1, 2, \dots, m.$$

where λ_k *and* t_{λ_k} *are lower order and* λ_k*- type of* $f_{L_k}(z_k)$.

Remark. If $f(z)$ is of regular growth *i.e.* $\varrho = \lambda$ then (2.15) is the relation involving the lower types of $f_{L_k}(z_k)$ and $f(z)$. Otherwise, if $\varrho \neq \lambda$ it follows from (2.12) that lower type of $f(z)$ is zero.

Here we give a few applications of theorems 2,3 and 4.

(*i*) The relation, (2.2), (2.3), (2.10) and (2.15) are recurrence relations. Hence knowing the order and type of any one function out of the $m+1$ functions, one can find out the order and type of any of the other m functions. The same is true for the lower order and λ-type.

(*ii*) $$(\varrho_k T_k)^{1/\varrho_k} \quad \text{and} \quad (\lambda_k t_{\lambda_k})^{1/\lambda_k}$$

are invariant quantities for

$$k = 1, 2, \ldots, m.$$

(*iii*) If

$$\left| \frac{a_n}{(n+1)^k a_{n+1}} \right|, \quad (k = 1, 2, \ldots m)$$

forms a non-decreasing function of n for $n > n_0$, then $f_{L_k}(z_k)$ is of perfectly regular growth if and only if $f(z)$ is of perfectly regular growth.

(*iv*) If $\varrho \neq \lambda$ then $t_k = 0$ for $k = 1, 2, \ldots, m$, t_k is the lower type of $f_{L_k}(z_k)$.

(*v*) If $m\varrho < 1$, then $f(z)$ and $f_{L_k}(z_k)$ each have an infinity of zeroes in their respective planes.

(*vi*) If one considers $(0, \varrho), (1, \varrho_1), \ldots, (m, \varrho_m)$ as points in the cartesian plane then all lie on the curve

$$y = \frac{\varrho}{1 - x\varrho}.$$

Theorem 5. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order, ϱ *lower order* λ $(0 < \lambda,\ m\varrho < 1)$, *type* $T\,(T >)$ *and* λ-*type* t_λ $(t_\lambda > 0)$, *then*

$$\frac{\varrho_m T_m}{\varrho T} = \prod_{k=1}^{m} (\varrho_{k-1} T_{k-1})^{\varrho_k} \tag{2.16}$$

and

$$\varrho_m - \varrho = \sum_{k=1}^{m} \varrho_{k-1} \varrho_k \,. \tag{2.17}$$

If further, $\left| \frac{a_n}{a_{n+1}} \right|$ *forms a non-decreasing function of n for* $n > n_0$, *then*

$$\frac{\lambda_m t_{\lambda m}}{\lambda t_\lambda} = \prod_{k=1}^{m} (\lambda_{k-1} t_{\lambda_{k-1}})^{\lambda_k} \tag{2.18}$$

and

(2.19)
$$\lambda_m - \lambda = \sum_{k=1}^{m} \lambda_{k-1}\, \gamma_k$$

where for $k=1$, $\varrho_0 = \varrho$, $\lambda_0 = \lambda$, $T_0 = T$, $t_{\lambda_0} = t_\lambda$ *and* ϱ_k, λ_k, T_k *and* t_{λ_k} *are the same as in theorems* 2, 3 *and* 4.

Proof. Let us consider the entire functions $f_{L_{k-1}}(z_{k-1})$ and $f_{L_k}(z_k)$ then, on the basis of theorem 3, we can see that

(2.20)
$$T_k = \frac{1}{\varrho_k}\,(\varrho_{k-1}\, T_{k-1})^{\frac{\varrho_k}{\varrho_{k-1}}}.$$

Putting $k = 1, 2, \ldots, m$ in (2.20) and then multiplying the m equations thus obtained, we get

(2.21)
$$T_1\, T_2 \ldots T_m = \frac{1}{\varrho_1\, \varrho_2 \ldots \varrho_m}\,(\varrho\, T)^{\frac{\rho_1}{\rho}} \ldots (\varrho_{m-1}\, T_{m-1})^{\frac{\rho_m}{\rho_{m-1}}}.$$

Again, considering $f_{L_{k-1}}(z_{k-1})$ in place of $f(z)$ in theorem 2, then (2.2) reduces to

(2.22)
$$\varrho_k = \frac{\varrho_{k-1}}{1-\varrho_{k-1}} \quad i.e. \quad \varrho_k - \varrho_{k-1} = \varrho_k\, \varrho_{k-1}.$$

Making use of (2.22) in (2.21), we can easily see that

$$\frac{\varrho_m\, T_m}{\varrho\, T} = \prod_{k=1}^{m} (\varrho_{k-1}\, T_{k-1})^{\varrho_k}$$

which is (2.16).

In (2.22), putting $k = 1, 2, \ldots, m$ and then adding all the equations thus obtained, we get

$$\sum_{k=1}^{m} (\varrho_k - \varrho_{k-1}) = \sum_{k=1}^{m} (\varrho_k\, \varrho_{k-1})$$

or

$$\varrho_m - \varrho = \sum_{k=1}^{m} \varrho_k\, \varrho_{k-1}$$

which is (2.17). Similarly, we can prove (2.18) and (2.19).

3. Now, we obtain relations between the maximum moduli of $f(z)$ and $f_{L_k}(z_k)$ and also between their maximum terms and their ranks. We denote by $M(\gamma)$ the maximum modulus of $f(z)$ for $|z| = \gamma$, and by $M(\gamma, f_{L_k})$ the maximum modulus of $f_{L_k}(z_k)$, taking $|z_k| = |z| = \gamma$. When k is even we have $z = z_k$ but when k is odd we have $z = 1/z_k$.

In the latter case, having chosen a value γ for $|z|$, we look for the point in the Z-plane such that $|z_k| = \gamma$. Corresponding to this point z_k the point in the z-plane will have the modulus $1/\gamma$. Similar remarks apply also for the maximum terms and for their ranks in the following theorems.

Theorem 6. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order $\varrho\,(0 \leq m\varrho < 1)$ *and lower order* λ. *If*

$$\left| \frac{a_n}{(n+1)^m a_{n+1}} \right|$$

forms a non-decreasing function of for $n > n_0$, *then for any* $\varepsilon > 0$.

$$\gamma^{\frac{\lambda-\rho-\varepsilon}{1-k\lambda}} \left\{ \log M(\gamma) \right\}^{\frac{1}{1-k\lambda}} < \log M(\gamma, f_{L_k}) < \gamma^{\frac{\rho-\lambda+\varepsilon}{1-k\rho}} \cdot \left\{ \log M(\gamma) \right\}^{\frac{1}{1-k\rho}}$$

for $\gamma < \gamma_0(\varepsilon)$ *and* $k = 1, 2, \ldots, m$.

Proof. It is known ([1], p. 9) that

$$\lim_{\gamma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\gamma)}{\log \gamma} = \begin{matrix} \varrho \\ \lambda \end{matrix} .$$

Therefore, for any $\varepsilon' > 0$, we can find a positive number $\gamma_0(\varepsilon')$ such that

$$\gamma^{\lambda-\varepsilon'} < \log M(\gamma) < \gamma^{\rho+\varepsilon'}. \tag{3.1}$$

Similarly, for the integral function $f_{L_k}(z_k)$, we have

$$\gamma^{\frac{\lambda}{1-k\lambda}+\varepsilon_k} < \log M\,(\gamma, f_{L_k}) > \gamma^{\frac{\rho}{1-k\rho}+\varepsilon_k} \quad \text{for} \quad \gamma > \gamma_k(\varepsilon_k)$$

or

$$\gamma^{\frac{\lambda-\rho-\varepsilon}{1-k\lambda}} \left\{ \log M(\gamma) \right\}^{\frac{1}{1-k\lambda}} < \log M(\gamma, f_{L_k}) < \gamma^{\frac{\rho-\lambda+\varepsilon}{1-k\rho}} \left\{ \log M(\gamma) \right\}^{\frac{1}{1-k\rho}}$$

where

$$\varepsilon \geq \left(\varepsilon_k + \frac{1-k\varrho}{\varepsilon'} \right).$$

Making use of (3.1) in the above inequality, we get

$$\gamma^{\frac{\lambda-\rho-\varepsilon}{1-k\lambda}} \left\{ \log M(\gamma) \right\}^{\frac{1}{1-k\lambda}} < \log M(\gamma, f_{L_k}) < \gamma^{\frac{\rho-\lambda+\varepsilon}{1-k\rho}} \left\{ \log M(\gamma) \right\}^{\frac{1}{1-k\rho}}$$

where

$$\gamma > \gamma_0(\varepsilon) = \max_{1 \leq k \leq m} \{ \gamma_0(\varepsilon'), \quad \gamma_k(\varepsilon_k) \}$$

and hence the theorem.

Theorem 7. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order ϱ $(0 \leq \varrho < 1)$, *lower order* λ *and let*

$$\nu(\gamma, f),\ \nu(\gamma, f_{L_k}),\quad \nu(\gamma, f^{(s)}) \text{ and } \nu(\gamma, f^{(s)}_{L_k})$$

denote respectively the ranks of the maximum terms of

$$f(z),\ f_{L_k}(z)$$

and their s-th derivatives

$$f^{(s)}(z) \quad \text{and} \quad f^{(s)}_{L_k}(z_k).$$

If

$$\left| \frac{a_n}{(n+1)^m\, a_{n+1}} \right|$$

forms a non-decreasing function of n for $n > n_0$, *then for any* $\varepsilon > 0$, *we have*

$$\frac{\lambda - \varrho - \varepsilon}{1 - k\lambda} + \frac{1}{(1 - k\lambda)\, s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi(x, s)}{x}\, dx < \frac{1}{s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi_k(x, s)}{x}\, dx$$

$$< \frac{\varrho - \lambda + \varepsilon}{1 - k\varrho} + \frac{1}{(1 - k\varrho)\, s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi(x, s)}{x}\, dx$$

for $\gamma > \gamma_0'$ *and* $k = 1, 2, \ldots, m$, *where*

$$\varphi(\gamma, s) = \nu(\gamma, f^{(s)}) - \nu(\gamma, f),$$

$$\varphi_k(\gamma, s) = \nu(\gamma, f^{(s)}_{L_k}) - \nu(\gamma, f_{L_k}).$$

Proof. It is known ([4], p. 276) that

$$\lim_{\gamma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{1}{s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi(x, s)}{x}\, dx = \frac{\varrho}{\lambda}.$$

Therefore, proceeding on the same lines as in theorem 6 we easily obtain the result.

Theorem 8. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order $\varrho\ (0 \leq m\varrho < 1)$, *lower order* λ *and let*

$$\mu(\gamma, f)\ ,\ \mu(\gamma, f_{L_k}),\ \mu(\gamma, f^{(s)}) \text{ and } \mu(\gamma, f^{(s)}_{L_k})$$

denote respectively the maximum terms of

$$f(z),\ f_{L_k}(z_k)\ ,\ f^{(s)}(z) \text{ and } f^{(s)}_{L_k}(z_k).$$

If $\left| \dfrac{a_n}{(n+1)^m a_{n+1}} \right|$ *is a non-decreasing function of* n *for* $n > n_0$, *then for any* $\varepsilon > 0$,

$$\gamma^{\frac{1+\lambda-\rho-\varepsilon}{1-k\lambda}} \left\{ \frac{\mu(\gamma, f^{(s)})}{\mu(\gamma, f)} \right\}^{\frac{1}{s(1-k\lambda)}} < \gamma \left\{ \frac{\mu(\gamma, f^{(s)}_{L_k})}{\mu(\gamma, f_{L_k})} \right\} < \gamma^{\frac{1+\rho-\lambda+\varepsilon}{1-k\rho}} \left\{ \frac{\mu(\gamma, f^{(s)})}{\mu(\gamma, f)} \right\}^{\frac{1}{s(1-k\rho)}}$$

for

$$\gamma > \gamma_0(\varepsilon) \text{ and } k = 1, 2, \ldots, m.$$

Proof. It is known ([5], p. 107) that

$$\lim_{\gamma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \gamma \left\{ \dfrac{\mu(\gamma, f^{(s)})}{\mu(\gamma, f)} \right\}^{1/s}}{\log \gamma} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Again, proceeding on the same lines as in theorem 6 we get the result (1).

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ÖZET

$Lf(z) = \sum_{n=0}^{\infty} (n!)\, a_n z^n$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ seri açılımı ile verilen tam fonksiyonun BOREL dönüştürülmüşünü göstersin. Bu takdirde, $z \,.\, z_1 = 1$ olmak üzere,

$$f_{L_1}(z_1) = \sum_{n=0}^{\infty} (n!)\, a_n z_1^n$$

vaz edilsin ve buna benzer tarzda, $z_{k-1} \cdot z_k = 1$ olmak üzere,

$$f_{L_k}(z_k) = \sum_{n=0}^{\infty} (n!)^k a_n z_k^n$$

tanımlansın. $f_{L_k}(z_k)$ fonksiyonları, k sayısının çift veya tek olmasına göre, z veya $1/z$ düzleminde tam fonksiyonlardır. Bu yazıda bu fonksiyonların sonlu mertebeden olmaları için bir gerek ve yeter şart elde edilmiş, ayrıca $f_{L_k}(z_k)$ ile $f(z)$ fonksiyonlarının mertebeleri, alt mertebeleri, tipleri ve λ - tipleri arasında bazı bağıntılar ispat edilmiştir.