ON THE ORDER AND GROWTH NUMBER OF INTEGRAL FUNCTIONS DEFINED BY DIRICHLET SERIES*

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The object of this paper is to establish under what conditions two integral functions, defined by their DIRICHLET series, will have the same order, type and growth number. Necessary and sufficient conditions, in terms of the exponents of the integral functions, have been obtained for the order and the growth number, while the problem of finding such a condition for the type is still open.

Consider the DIRICHLET series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n},$$

where [1])

$$\lambda_{n+1} > \lambda_n, \qquad \lambda_1 \geq 0, \lim_{n\to\infty} \lambda_n = \infty, \quad s = \sigma + it,$$

and [2])

(1) $$\limsup_{n\to\infty} \frac{\log n}{\log \lambda_n} = 0.$$

Let σ_c, σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. Also let $\sigma_c = \infty$, then σ_a will also be infinite, since, according to a known result ([1], p. 4) a DIRICHLET series which satisfies (1) has its abscissa of convergence equal to its abscissa of absolute convergence and therefore $f(s)$ represents an integral function. In the sequel all integral functions represented by DIRICHLET series will be chosen so as to satisfy (1).

Let

$$\mu(\sigma, f) = |a_{N(\sigma,f)}| \, e^{\lambda_{N(\sigma,f)}}$$

1) $f(s)$ is not an exponential polynomial.

2) $$\limsup_{n\to\infty} \frac{\log n}{\log \lambda_n} = 0 \text{ implies } \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0.$$

be the maximum term of rank $N(\sigma, f)$ in $f(s)$. It is known ([2], p. 6) that

$$\log \mu(\sigma, f) = \log \mu(\sigma_0, f) + \int_{\sigma_0}^{\sigma} \lambda_{N(t,f)}\, dt, \quad 0 \leqq \sigma_0 < \sigma. \tag{2}$$

Let ϱ be the linear RITT order ([8], p. 78) of $f(s)$, and $M(\sigma, f)$ be the *l. u. b.* of

$$| f(\sigma + it) |, \quad -\infty < t < \infty$$

where

$$\sigma < \sigma_a;$$

it is also known ([2], p. 73) that for functions of finite order,

$$\log M(\sigma, f) \sim \log \mu(\sigma, f). \tag{3}$$

The order ϱ satisfies the following equalities in view of (2) and (3):

$$\limsup_{\sigma \to \infty} \frac{\log_2 M(\sigma, f)}{\sigma} = \varrho = \limsup_{n \to \infty} \frac{\log_2 \mu(\sigma, f)}{\sigma} = \limsup_{\sigma \to \infty} \frac{\log \lambda_{N(\sigma, f)}}{\sigma} \tag{4}$$

where

$$\log_2 M(\sigma, f) = \log \log M(\sigma, f)$$

and similarly for

$$\log_2 \mu(\sigma, f).$$

$\lambda_{N(\sigma, f)}$ is the exponent corresponding to the maximum term $\mu(\sigma, f)$ whose rank is $N(\sigma, f)$.

The growth number γ ([6], p. 94) of an integral function of order

$$\varrho\,(0 < \varrho < \infty)$$

is defined as

$$\limsup_{\sigma \to \infty} \frac{\lambda_{N(\sigma, f)}}{e^{\varrho\sigma}} = \gamma. \tag{5}$$

Various researchers (*e. g.* [4], [5], [6], [7]) have given many relations between the orders, types and growth numbers of two or more integral functions. A question naturally arises: Under what conditions will two integral functions be of the same order, type and growth number? One such sufficient condition trivially is that one of the functions is the derivative of the other.

In this paper, we have found non-trivial necessary and sufficient conditons, in terms of the exponents of integral functions, for order and growth number. Since the author is not aware of any relation connecting the type of integral function with its exponents only, the question of finding such a condition for the type is still open.

Theorem 1. *Let*

$$f_1(s) = \sum_{n=1}^{\infty} a_{1,n} \, e^{s\lambda_{1,n}}$$

and

$$f_2(s) = \sum_{n=1}^{\infty} a_{2,n} \, e^{s\lambda_{2,n}}$$

be integral functions of linear RITT *orders* ϱ_1 *and* ϱ_2 *respectively. If* $\lambda_{N(\sigma, f_1)}$ *and* $\lambda_{N(\sigma, f_2)}$ *are the exponents corresponding to the maximum terms in* $f_1(s)$ *and* $f_2(s)$ *respectively, then a sufficient condition for* $\varrho_1 = \varrho_2$ *is that*

$$\limsup_{\sigma \to \infty} \left[\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}\right]$$

exists and is equal to a finite number a. The condition is also necessary if ϱ_1 *and* ϱ_2 *are both finite.*

Proof. First we shall show that the condition is sufficient. For, if

$$\limsup_{\sigma \to \infty} \left[\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}\right] = a,$$

we have, for any $\varepsilon > 0$ and sufficiently large σ,

$$\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)} < a + \varepsilon,$$

or

$$\frac{\lambda_{N(\sigma, f_2)}}{\lambda_{N(\sigma, f_1)}} - 1 < \frac{a + \varepsilon}{\lambda_{N(\sigma, f_1)}}.$$

Hence

$$\lim_{\sigma \to \infty} \left[\frac{\lambda_{N(\sigma, f_2)}}{\lambda_{N(\sigma, f_1)}} - 1\right] = 0.$$

Therefore

$$\lambda_{N(\sigma, f_2)} \sim \lambda_{N(\sigma, f_1)}.$$

Hence

$$\varrho_2 = \limsup_{\sigma \to \infty} \frac{\log \lambda_{N(\sigma, f_2)}}{\sigma} = \limsup_{\sigma \to \infty} \frac{\log \lambda_{N(\sigma, f_1)}}{\sigma} = \varrho_1$$

in view of (4).

We shall establish the necessity of the condition by showing that if $\varrho_1 \neq \varrho_2$ then

$$\limsup_{\sigma \to \infty} \left[\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}\right]$$

is not finite. We suppose that $\varrho_2 > \varrho_1$: then

$$\limsup_{\sigma\to\infty} \frac{\log \lambda_{N(\sigma, f_2)}}{\sigma} > \limsup_{\sigma\to\infty} \frac{\log \lambda_{N(\sigma, f_1)}}{\sigma}$$

or

$$\limsup_{\sigma\to\infty} \frac{\log \lambda_{N(\sigma, f_2)}}{\sigma} - \limsup_{\sigma\to\infty} \frac{\log \lambda_{N(\sigma, f_1)}}{\sigma} = c\,,$$

say, where c is some positive number, or

$$\limsup_{\sigma\to\infty} \log \frac{\lambda_{N(\sigma, f_2)}}{\lambda_{N(\sigma, f_1)}} \geq c\,\sigma\,.$$

Therefore

$$\limsup_{\sigma\to\infty} [\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}] \geq (e^{c\sigma} - 1)\,\lambda_{N(\sigma, f_1)}\,. \tag{6}$$

The right hand side of (6) tends to infinity with σ and hence

$$\limsup_{\sigma\to\infty} [\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}]$$

is not finite.

Theorem 2. *Let*

$$f_1(s) = \sum_{n=1}^{\infty} a_{1,n}\; e^{s\lambda_{1,n}}$$

and

$$f_2(s) = \sum_{n=1}^{\infty} a_{2,n}\; e^{s\lambda_{2,n}}$$

be integral functions of orders

$$\varrho_1\,(0 < \varrho_1 < \infty)\,, \qquad \varrho_2\,(0 < \varrho_2 < \infty),$$

and growth numbers

$$\gamma_1\,(0 < \gamma_1 < \infty)\,, \qquad \gamma_2\,(0 < \gamma_2 < \infty),$$

respectively. If $\lambda_{N(\sigma, f_1)}$ *and* $\lambda_{N(\sigma, f_2)}$ *are the exponents corresponding to the maximum terms in* $f_1(s)$ *and* $f_2(s)$, *respectively, then a sufficient condition for* $\gamma_1 = \gamma_2$ *is that*

$$\limsup_{\sigma\to\infty} [\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}]$$

exists and is equal to a finite number a. *The condition is also necessary provided* $\varrho_1 = \varrho_2$ *and* γ_1, γ_2 *are finite.*

Suppose

$$\limsup_{\sigma \to \infty} [\lambda_{N(\sigma, f_2)} - \lambda_{N(\sigma, f_1)}]$$

exists and is equal to a finite number a. Then $\varrho_1 = \varrho_2$ in view of the theorem 1. The rest of the proof follows on the same lines as that of Theorem I with trivial modifications in view of (5). [1]

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(*Manuscript received November 11, 1966*)

ÖZET

Bu çalışmanın gayesi, DIRICHLET serileri ile tanımlanmış iki tam fonksiyonun hangi şartlar altında ayni mertebe, tip ve artma sayısını haiz olacaklarını tesbit etmektir. Mertebe ve artma sayısı için gerek ve yeter şartlar elde edilmiş, buna karşılık tip için böyle bir şart elde etmek problemi balâ cevapsız kalmış bulunmaktadır.

1) I am grateful to Dr. J. S. GUPTA for his help and guidance in this paper.