ON CONVEX FUNCTIONS OF ORDER ZERO

PAWAN KUMAR JAIN

Let $M(r)$ be a non - decreasing non negative convex function with respect to an absolutely continuous function $\varphi(r)$ in $(0,\infty)$ [1,2] The order [3] ϱ of $M(r)$ is the quantity

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{\varphi(r)} = \varrho \qquad (0 \leq \varrho \leq \infty).$$

The object of this paper is to prove certain rusults concerning convex functions of order zero. These results contain some theorems proved for entire functions represented by power or DIRICHLET expansions [5,6].

1. **Introduction.** Let $M(r)$ be a non-decreasing convex function with respect to an absolutely continuous function $\varphi(r)$ in $(0, \infty)$ and non-negative (for details refer to the papers [1] and [2]). Let $M(\delta) = 0$, for some $\delta \geq 0$ (depending on M), δ being a constant. We may, then, represent $M(r)$ in terms of an integral of a non-decreasing function $n(r)$. $n(r) \to \infty$ as $r \to \infty$, given by [4]

$$M(r) = \int_{\delta}^{r} n(t)\, d\varphi(t), \tag{1.1}$$

the integral being considered as a LEBESGUE-STIELTJES integral.

By the order ϱ of $M(r)$ we mean the quantity [3]:

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{\varphi(r)} = \varrho \quad , \quad 0 \leq \varrho \leq \infty . \tag{1.2}$$

In this paper we wish to prove certain results exclusively devoted to the convex functions of order zero. The importance of these results is that they envelop many of the results already known for entire functions represented by power and DIRICHLET series, for instance see [5], [6]. The results are given in the form of theorems.

2. **Theorem 1.** Let $M(r)$ be a convex function with respect to $\varphi(r)$ of order zero, then

$$M(r) = O\left((\varphi(r))^{\alpha-\beta}\right) \tag{2.1}$$

if and only if

$$Q(r) = O\left((\varphi(r))^{\alpha-\beta-1}\right), \tag{2.2}$$

where

$$Q(r) = e^{\varphi(r)} \int_{r}^{\infty} n(x)\, e^{-\varphi(x)}\, d\varphi(x),$$

and $2 \leq \alpha < \infty$; $\alpha - 2 \leq \beta < \alpha - 1$.

Proof. Under the hypothesis of the theorem, it follows from (1.1) and (1.2) that

$$\overline{\lim_{r\to\infty}} \frac{\log n(r)}{\varphi(r)} = 0,$$

and hence for $x \geq x_0$ and $\varepsilon > 0$ arbitrarily small,

$$\frac{\exp.\,(\eta^{-1}\,\varphi(x))}{n(x)} > \frac{\exp.\,(\eta^{-1}\,\varphi(x))}{\exp.\,(\varepsilon\,\varphi(x))}\,,\quad 0 < \eta^{-1} < 1,$$

and so $\{\exp.\,(\eta^{-1}\,\varphi(x))/n(x)\}$ is an increasing function for $x \geq x_0$ and $\to \infty$ with x. Now for $R > r$.

$$M(R) > \int_r^R n(x)\; d\,\varphi(x) > n(r)\,\{\varphi(R) - \varphi(r)\}. \tag{2.3}$$

Let us choose $R = R(r)$ such that $\varphi(R) - \varphi(r) \to \varphi(r)$, which is always possible for a proper choice of $\varphi(r)$ and then of R (for example, $\varphi(r) = r$, $R = 2r$ and $\varphi(r) = \log r$, $R = r^2$ etc.). Hence if (2.1) holds, then (2.3) gives

$$n(r) = O\left((\varphi(r))^{\alpha-\beta-1}\right),$$

so that

$$Q(r) = O\left(e^{\varphi(r)} \int_r^\infty (\varphi(x))^{\alpha-\beta-1}\, e^{-\varphi(x)}\, d\,\varphi(x)\right) = O\left((\varphi(r))^{\alpha-\beta-1}\right).$$

Conversly, next suppose that (2.2) holds. Now

$$Q(r) = e^{\varphi(r)} \int_r^\infty n(x)\; e^{-\varphi(x)}\; d\;\varphi(x) \geq n(r),$$

and therefore $n(r)$ is atmost of $O\,((\varphi(r))^{\alpha-\beta-1})$.

Again, let $R > r$ where $R = R(r, k)$ such that $\varphi(R) - \varphi(r) \to \varphi(k) \geq 0$, k being a constant, then

$$Q(r) = e^{\varphi(r)} \int_r^R n(x)\; e^{-\varphi(x)}\; d\,\varphi(x) + e^{\varphi(r)} \int_R^\infty n(x)\; e^{-\varphi(x)}\; d\,\varphi(x)$$

$$\leq n(R)\;(1 - e^{-\varphi(k)}) + \frac{e^{\varphi(r)}\; n(R)}{e^{\eta^{-1}\,\varphi(R)}} \int_R^\infty e^{-\eta_1^{-1}\,\varphi(x)}\; d\,\varphi(x)$$

$$= n(R)\,\{1 - e^{-\varphi(k)} + \eta_1\; e^{-\varphi(k)}\},\quad 0 < \eta_1^{-1} < 1\;,\; \eta^{-1} + \eta_1^{-1} = 1\,;$$

and therefore $n(r)$ is at least of $O\left((\varphi(r))^{\alpha-\beta-1}\right)$. Hence we conclude that

$$n(r) = O\left((\varphi(r))^{\alpha-\beta-1}\right),$$

which when substituted in (1.1), gives (2.1). Hence the result is proved.

Theorem 2. If $L(r)$ be any function satisfying the following condition :

(i) $L(r)$ is positive, continuous for $r > \alpha \geq 0$:

$$\lim_{r\to\infty} \frac{\log L(r)}{\varphi(r)} = 0,$$

where $\{\log L(r) / \varphi(r)\}$ is assumed to tend to zero steadily as $r \to \infty$.

(ii) $L(r) \to +\infty$ as $r \to \infty$;

(iii) $L(R) \sim L(r)$. $R > r$ with $R = R(r, k)$ such that $\varphi(R) - \varphi(r) \to \varphi(k) \geq 0$, k being a constant; then, if either $M(r) \sim L(r)$ or $n(r) \sim L(r)$.

(2.4) $$\lim_{r\to\infty} \frac{M(r)}{n(r)} = \infty.$$

For proof of this theorem we need the following lemma:

Lemma. If $L(r)$ satisfies the above conditions, then

$$\lim_{r\to\infty} \frac{\int_{\delta}^{r} L(x)\, d\,\varphi(x)}{L(r)} = \infty, \quad (\delta = a \text{ positive constant}).$$

Proof of the lemma. For we have

$$\underline{\lim}_{n\to\infty} \frac{\log L(r_n)}{\varphi(r_n)} = 0.$$

There will, therefore, exists a sequence $r_{n_i} = r_N$ (say) of $r \to \infty$, such that ([7], p. 18, Prob. 107)

$$\frac{\log L(r_N)}{\varphi(r_N)} < \frac{\log L(r_n)}{\varphi(r_n)}, \quad n = 1, 2, \ldots, N-1;$$

$$\frac{\log L(x)}{\varphi(x)} > \frac{\log L(r_N)}{\varphi(r_N)}, \quad \delta \leq x < r_N.$$

Let $r_N = r$. Then

$$(L(r))^{-1} \int_{\delta}^{r} L(x)\, d\,\varphi(x) > (L(r))^{-1} \int_{\delta}^{r} (L(r))^{\frac{\varphi(x)}{\varphi(r)}}\, d\,\varphi(x)$$

$$= \frac{\varphi(r)}{\log L(r)} \left\{ 1 - (L(r))^{\frac{\varphi(\delta)}{\varphi(r)} - 1} \right\}$$

$$\sim \frac{\varphi(r)}{\log L(r)} \, (1 + 0(1)).$$

Hence the lemma follows.

Proof of the theorem. (i) Since $M(r) \sim L(r)$, therefore, for all $r \geq r_0$ and $\varepsilon > 0$

$$(1-\varepsilon)^{-2} \; L(r) > (1-\varepsilon)^{-1} \; L(R) > M(R) = A + \int_{a}^{R} n(x)\, d\,\varphi(x),$$

$$(1+\varepsilon)^{-2}\, L(r) < M(r) = A + \int_a^r n(x)\, d\,\varphi(x);$$

so that

$$\{(1-\varepsilon)^{-2} - (1+\varepsilon)^{-2}\}\, L(r) > \int_r^R n(x)\, d\,\varphi(x)$$

$$> n(r)\,\{\varphi(R) - \varphi(r)\}$$

$$\sim n(r)\,\varphi(k).$$

i. e.

$$\lim_{r\to\infty} \frac{n(r)}{L(r)} = 0.$$

Hence, as $L(r) \sim M(r)$, (2.4) follows.

(ii) Here $n(r) \sim L(r)$, therefore, for $r \geq r_0$, $\delta > r_0$, $\varepsilon > 0$

$$M(r) > (1-\varepsilon) \int_\delta^r L(x)\, d\,\varphi(x).$$

Thus

$$\lim_{r\to\infty} \frac{M(r)}{n(r)} \geq \lim_{r\to\infty} \frac{(1-\varepsilon)\int_\delta^r L(x)\, d\,\varphi(x)}{L(r)} = \infty,$$

by the lemma and (2.4) follows again [1].

[1]) Finally, I must thank Dr. P. K. KAMPTHAN, Professor of Pure Mathamtics, WATERLOO UNIVERSITY (Presently at I. I. T., KANPUR) for his valuable suggestions and guidance in the preprations of this paper. I am also grateful to Professor R. S. VARMA, Dean, Faculty of Mathematics, DELHI UNIVERSITY for his kind encouragement and inspiration to me.

REFERENCES

[1] JAIN, P. K. : *Convex functions and their applications*, Istanbul Univ. Fen Fak. Mec. Ser. A, 30, (1965).

[2] — : *Applications of convex functions*, Bull. PGS, Delhi Uni., 3, 18—23, (1966).

[3] KAMPTHAN, P. K. AND JAIN, P. K. : *On a certain class of convex functions*, Rend. Circ. Matematico di Palermo, (To appear).

[4] KAMPTHAN, P. K. : *Convex functions and their applications*, Istanbul Univ. Fen Fak. Mec. Ser. A, 28, 71—78, (1936).

[5] — : *On entire functions of order (R) zero represented by Dirichlet series*, Mathematica Japonicae, 8, 1—2, 17—22, (1963).

[6] — : *On the maximum term and its rank of an entire function represented by Dirichlet series IV*, Mathematics Student, 33, 2—3, 63—71, (1965).

[7] POLYA, G. AND SZEGÖ, G. : Aufgaben und Lehrsatzens Analysis, SPRINGER-VERLAG, Berlin, (1954).

HANS RAJ COLLEGE
DELHI UNIVERSITY
DELHI 7 — INDIA

(Manuscript, received August, 5, 1968)

ÖZET

$(0, \infty)$ aralığında mutlak olarak sürekli bir $\varphi(r)$ fonksiyonuna nazaran konveks olan, azalmayan ve negatif olmayan bir fonksiyon $M(r)$ ile gösterilsin [1,2]. $M(r)$ fonksiyonunun «derecesi» [3]

$$\overline{\lim_{r \to \infty}} \frac{\log M(r)}{\varphi(r)} = \varrho \qquad (0 \leq \varrho \leq \infty)$$

bağıntısı ile tanımlanır. Bu yazının gâyesi sıfırıncı dereeden konveks fonksiyonlar hakkında bâzı sonuclar ispatetmektir. Bu sonuclar kuvvet veya DIRICHLET serileri ile verilen tam fonksiyonlar icin bâzı teoremleri ihtiva etmektedir [5,6].