ON "BOREL - LAPLACE" TRANSFORMS AND INTEGRAL FUNCTIONS OF TWO COMPLEX VARIABLES

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A result due to M.M. DZRBASYAN [1] and a method used by MACINTYRE [2] in the case of a single complex variable, are used to obtain some interpolation formulae for integral functions in two complex variables of finite order and type.

1. Let

$$F(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} \frac{a_{n_1 n_2}}{n_1! \, n_2!} z_1^{n_1} z_2^{n_2}, \tag{1.1}$$

be an integral function of two complex variables z_1 and z_2. The maximum modulus of the integral function $F(z_1, z_2)$ is denoted by

$$M(\gamma_1, \gamma_2; F) = \max_{|z_1|=\gamma_1, \, |z_2|=\gamma_2} |F(z_1, z_2)|.$$

The integral function $F(z_1, z_2)$ has finite order ϱ_1 and ϱ_2 with respect to the variables z_1 and z_2 respectively, if

$$\limsup_{\gamma_2\to\infty} \left[\limsup_{\gamma_1\to\infty} \frac{\log\log M(\gamma_1, \gamma_2; F)}{\log \gamma_1} \right] = \varrho_1 \tag{1.2}$$

and

$$\limsup_{\gamma_1\to\infty} \left[\limsup_{\gamma_2\to\infty} \frac{\log\log M(\gamma_1, \gamma_2; F)}{\log \gamma_2} \right] = \varrho_2. \tag{1.3}$$

The integral function $F(z_1, z_2)$ has finite type σ_1 and σ_2 with respect to the variables z_1 and z_2, respectively, if

$$\limsup_{\gamma_2\to\infty} \left[\limsup_{\gamma_1\to\infty} \frac{\log M(\gamma_1, \gamma_2; F)}{\gamma_1^{\rho_1}} \right] = \sigma_1 \tag{1.4}$$

and

$$\limsup_{\gamma_1\to\infty} \left[\limsup_{\gamma_2\to\infty} \frac{\log M(\gamma_1, \gamma_2; F)}{\gamma_2^{\rho_2}} \right] = \sigma_2. \tag{1.5}$$

M. M. DZRBASYAN ([1], p. 21) has proved the following theorem :

Theorem. *Every integral function*

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$$F(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} \frac{a_{n_1 n_2}}{\Gamma\left(\mu_1 + \frac{n_1}{\rho_1}\right) \Gamma\left(\mu_2 + \frac{n_2}{\rho_2}\right)} z_1^{n_1} z_2^{n_2}, \quad (\mu_1 > 0, \mu_2 > 0)$$

of order $(\varrho_1, \varrho_2) > 0$ *and type* $(\sigma_1, \sigma_2) > 0$ *can be represented in the form*

$$(1.6) \qquad F(z_1, z_2) = -\frac{1}{4\pi^2} \int_{l_1} \int_{l_2} E_{\varrho_1}(z_1 \xi_1; \mu_1) E_{\varrho_2}(z_2 \xi_2; \mu_2) f(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2$$

where

$$E_\varrho(z; \mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\mu + \frac{n}{\rho}\right)}, \quad (\mu > 0, \varrho > 0)$$

is an integral function of MITTAG-LEFLER *type and*

$$f(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} \frac{a_{n_1 n_2}}{z_1^{n_1+1} z_2^{n_2+1}}$$

holomorphic in the domain $|z_1| > \sigma_1^{1/\varrho_1}$ *and* $|z_2| > \sigma_2^{1/\varrho_2}$, *is the function* $B_{\varrho_1, \varrho_2}(\mu_1, \mu)$ *associated with* $F(z_1, z_2)$ *and* l_k $(k = 1,2)$ *are arbitrary closed contours lying in the region* $|\xi_1| > \sigma_1^{1/\varrho_1}$ *and* $|\xi_2| > \sigma_2^{1/\varrho_2}$.

If we take $\mu_1 = \mu_2 = 1$ and $\varrho_1 = \varrho_2 = 1$, then (1.6) reduces to

$$(1.7) \qquad F(z_1, z_2) = -\frac{1}{4\pi^2} \int_{l_1} \int_{l_2} e^{z_1 \xi_1 + z_2 \xi_2} f(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2 .$$

In this paper we have derived a number of interpolation formulae from the BOREL-LAPLACE transformation of two variables for the integral function of order ϱ_1^1 and ϱ_2^1 and type k_1 and k_2 to be one with respect to the variables z_1 and z_2.

2. Let

$$G(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} a_{n_1 n_2} z_1^{n_1} z_2^{n_2},$$

be an integral function of exponential type k_1 and k_2, satisfying

$$|G(z_1, z_2)| < M e^{k_1 |z_1| + k_2 |z_2|} \quad (k_j < \pi, \; j = 1, 2).$$

With the integral function $G(z_1, z_2)$ we associate the function

$$g(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} \frac{n_1! \, n_2! \, a_{n_1 n_2}}{z_1^{n_1+1} z_2^{n_2+1}} .$$

Using (1.7), the relation between the integral function $G(z_1, z_2)$ and the associate function $g(z_1, z_2)$ may be expressed by means of the integral

(2.1) $$G(z_1, z_2) = -\frac{1}{4\pi^2} \int_{c_1} \int_{c_2} e^{z_1 u_1 + z_2 u_2} g(u_1, u_2)\, du_1\, du_2 .$$

The function $g(z_1, z_2)$ is regular outside the polydisc $|z_j| = k_j$, $j = 1,2$; so that the closed contours c_1 and c_2 in (2.1) can be taken as the circles

$$|z_j| = k_j + \varepsilon_j \;,\quad \varepsilon_j > 0 \;,\quad j = 1,2 .$$

We consider another associate function

(2.2) $$\psi(z_1, z_2) = \tfrac{1}{4} G(0,0) + \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0)\, e^{-n_1 z_1}$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, n_2)\, e^{-n_2 z_2} + \sum_{n_1, n_2=1}^{\infty} G(n_1, n_2)\, e^{-n_1 z_1 - n_2 z_2} ,$$

as MACINTYRE ([2], p. 3) considered for the case of one complex variable.

Raplacing $G(n_1, n_2)$ in (2.2) by means of (2.1), we have

(2.3) $$\psi(z_1, z_2) = -\frac{1}{16\pi^2} \int_{c_1} \int_{c_2} g(u_1, u_2)\, du_1\, du_2$$

$$-\frac{1}{8\pi^2} \sum_{n_1=1}^{\infty} \int_{c_1} \int_{c_2} e^{n_1(u_1 - z_1)} g(u_1, u_2)\, du_1\, du_2 - \frac{1}{8\pi^2} \sum_{n_2=1}^{\infty} \int_{c_1} \int_{c_2} e^{n_2(u_2 - z_2)} g(u_1, u_2)\, du_1\, du_2$$

$$-\frac{1}{4\pi^2} \sum_{n_1, n_2=1}^{\infty} \int_{c_1} \int_{c_2} e^{n_1(u_1 - z_1) + n_2(u_2 - z_2)} g(u_1, u_2)\, du_1\, du_2$$

$$= -\frac{1}{16\pi^2} \int_{c_1} \int_{c_2} g(u_1, u_2)\, du_1\, du_2 - \frac{1}{8\pi^2} \int_{c_1} \int_{c_2} \frac{e^{u_1 - z_1}}{1 - e^{u_1 - z_1}} g(u_1, u_2)\, du_1\, du_2$$

$$-\frac{1}{8\pi^2} \int_{c_1} \int_{c_2} \frac{e^{u_2 - z_2}}{1 - e^{u_2 - z_2}} g(u_1, u_2)\, du_1\, du_2 - \frac{1}{4\pi^2} \int_{c_1} \int_{c_2} \frac{e^{(u_1 - z_1) + (u_2 - z_2)}}{(1 - e^{u_1 - z_1})(1 - e^{u_2 - z_2})} g(u_1, u_2)\, du_1\, du_2$$

$$= -\frac{1}{16\pi^2} \int_{c_1} \int_{c_2} \frac{(1 + e^{u_1 - z_1})(1 + e^{u_2 - z_2})}{(i - e^{u_1 - z_1})(1 - e^{u_2 - z_2})} g(u_1, u_2)\, du_1\, du_2 ,$$

provided the change of order of integration and summation in (2.3) is justified.

If we take z_1 and z_2 to be inside the closed contours c_1 and c_2 respectively and no other value of $z_1 + 2 n_1 \pi i$ and $z_2 + 2 n_2 \pi i$ $(n_1, n_2 = 0, \pm 1, \pm 2, \ldots)$ is allowed to enter the contours, then we must modify the above formula in the form

$$\psi(z_1, z_2) = g(z_1, z_2) - \frac{1}{16\pi^2} \int_{c_1} \int_{c_2} \frac{(1 + e^{u_1 - z_1})(1 + e^{u_2 - z_2})}{(1 - e^{u_1 - z_1})(1 - e^{u_2 - z_2})} g(u_1, u_2)\, du_1\, du_2 .$$

For $j = 1, 2$, if we take $k_j < \pi$, then $\psi(z_1, z_2) - g(z_1, z_2)$ is regular for $|z_j| < 2\pi - k_j$ and $\psi(z_1, z_2)$ is regular for $k_j < |z_j| < 2\pi - k_j$ and therefore, we can replace $g(u_1, u_2)$ by $\psi(n_1, u_2)$, if the contours of (2.1) are inside $|z_j| < 2\pi - k_j, j = 1, 2,$ to obtain

$$G(z_1, z_2) = -\frac{1}{4\pi^2} \int_{c_1} \int_{c_2} e^{z_1 u_1 + z_2 u_2}\, \psi(u_1, u_2)\, du_1\, du_2 . \tag{2.4}$$

If we take $R(z_j) < k_j \quad , \quad j = 1, 2$; then the expression (2.3) reduces to (2.2). Similarly,

(*i*) For $R(z_j) < -k_j \quad , \quad j = 1,2$

$$\psi(z_1, z_2) = -\frac{1}{16\pi^2} \int_{c_1} \int_{c_2} \frac{(1 + e^{z_1 - u_1})(1 + e^{z_2 - u_2})}{(1 - e^{z_1 - u_1})(1 - e^{z_2 - u_2})}\, g(u_1, u_2)\, du_1\, du_2 \tag{2.5}$$

$$= \tfrac{1}{4} G(0, 0) + \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(-n_1, 0)\, e^{n_1 z_1} + \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, -n_2)\, e^{n_2 z_2}$$

$$+ \sum_{n_1, n_2=1}^{\infty} G(-n_1, -n_2)\, e^{n_1 z_1 + n_2 z_2} .$$

(*ii*) For $R(z_1) < k_1$ and $R(z_2) < -k_2$

$$\psi(z_1, z_2) = \frac{1}{16\pi^2} \int_{c_1} \int_{c_2} \frac{(1 + e^{u_1 - z_1})(1 + e^{z_2 - u_2})}{(1 - e^{u_1 - z_1})(1 - e^{z_2 - u_2})}\, g(u_1, u_2)\, du_1\, du_2 \tag{2.6}$$

$$= -\tfrac{1}{4} G(0, 0) - \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0)\, e^{-n_1 z_1} - \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, -n_2)\, e^{n_2 z_2}$$

$$- \sum_{n_1, n_2=1}^{\infty} G(n_1, -n_2)\, e^{-n_1 z_1 + n_2 z_2} .$$

(*iii*) For $R(z_1) < -k_1$ and $R(z_2) < k_2$

$$\psi(z_1, z_2) = \frac{1}{16\pi^2} \int_{c_1} \int_{c_2} \frac{(1 + e^{z_1 - u_1})(1 + e^{u_2 - z_2})}{(1 - e^{z_1 - u_1})(1 - e^{u_2 - z_2})}\, g(u_1, u_2)\, du_1\, du_2 \tag{2.7}$$

$$= -\tfrac{1}{4} G(0, 0) - \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(-u_1, 0)\, e^{n_1 z_1} - \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, u_2)\, e^{-n_2 z_2}$$

$$- \sum_{n_1, n_2=1}^{\infty} G(-n_1, n_2)\, e^{n_1 z_1 - n_2 z_2} .$$

The formula (2.4) may be regarded as a species of interpolation formula, since $\psi(z_1, z_2)$ depends on $G(n_1, n_2)$, $n_1, n_2 = 1, 2, \ldots$.

Interpolation formulae of more familiar types may be obtained under different sets of conditions. For example, we prove the following theorem :

3. **Theorem.**

Let

$$G(z_1, z_2) = \sum_{n_1, n=0}^{\infty} a^{n_1 n_2} z_1^{n_1} z_2^{n_2},$$

be an integral function of two complex variables z_1 *and* z_2, *satisfying the condition that there exist positive constants* M, k_1 *and* k_2 *such that*

$$|G(z_1, z_2)| < M e^{k_1 |z_1| + k_2 |z_2|}, \ (k_j < \pi, \ j = 1,2)$$

for all z_1 *and* z_2, *then*

$$(3.1) \quad G(z_1, z_2) = \frac{1}{\pi^2} \lim_{\delta_1, \delta_2 \to 0} \sum_{n_1, n_2 = -\infty} \frac{\sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 - n_2)}{(z_1 - n_1)(z_2 - n_2)} G(n_1, n_2) e^{-\delta_1 |n_1| - \delta_2 |n_2|},$$

provided the change of order of integration and summation involved are justified.

Consider that the series

$$(3.2) \quad \psi(z_1, z_2) = \tfrac{1}{4} G(0, 0) + \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0) e^{-n_1 z_1} + \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, n_2) e^{-n_2 z_2} + \sum_{n_1, n_2=1}^{\infty} G(n_1, n_2) e^{-n_1 z_1 - n_2 z_2}$$

converges for $R(z_1) > 0$ and $R(z_2) > 0$;

$$(3.3) \quad \psi(z_1, z_2) = \tfrac{1}{4} G(0, 0) + \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(-n_1, 0) c^{n_1 z_1} + \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, -n_2) e^{z_2 n_2} + \sum_{n_1, n_2=1}^{\infty} G(-n_1, -n_2) e^{n_1 z_1 + n_2 z_2}$$

converges for $R(z_1) < 0$ and $R(z_2) < 0$;

$$(3.4) \quad \psi(z_1, z_2) = -\tfrac{1}{4} G(0, 0) - \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0) e^{-n_1 z_1} - \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, -n_2) e^{n_2 z_2} - \sum_{n_1, n_2=1}^{\infty} G(n_1, -n_2) e^{-n_1 z_1 + n_2 z_2}$$

converges for $R(z_1) > 0$ and $R(z_2) < 0$; and

$$(3.5) \quad \psi(z_1, z_2) = -\tfrac{1}{4} G(0, 0) - \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(-n_1, 0) e^{n_1 z_1} - \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, n_2) e^{-n_2 z_2} - \sum_{n_1, n_2=1}^{\infty} G(-n_1, n_2) e^{n_1 z_1 - n_2 z_2}$$

converges for $R(z_1) < 0$ and $R(z_2) > 0$.

We now deform in (2.4), the closed contours e_1 and e_2 into the rectangles

$$-\delta_j-\omega_j i \ , \ \delta_j-\omega_j i \ , \ \delta_j+\omega_j i \ , \ -\delta_j+\omega_j i$$

respectively, where $k_j < \omega_j < 2\pi - k_j$ and $\delta_j > 0$, $j = 1,2$. Thus

$$G(z_1, z_2) = -\frac{1}{4\pi^2} \int_{c_1}\int_{c_2} e^{z_1 u_1 + z_2 u_2}\ \psi(u_1, u_2)\, du_1\, du_2 \tag{3.6}$$

$$= -\frac{1}{4\pi^2} \lim_{\delta_1, \delta_2 \to 0} \Big[\int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2}\ \psi(u_1, u_2)\, du_1\, du_2$$

$$-\int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{-\delta_2-\omega_2 i}^{-\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2}\ \psi(u_1, u_2)\, du_1\, du_2$$

$$-\int_{-\delta_1-\omega_1 i}^{-\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2}\ \psi(u_1, u_2)\, du_1\, du_2$$

$$+\int_{-\delta_1-\omega_1 i}^{-\delta_1+\omega_1 i} \int_{-\delta_2-\omega_2 i}^{-\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2}\ \psi(u_1, u_2)\, du_1\, du_2 \Big],$$

where the limit exists uniformly for z_1 and z_2 in any bounded two dimensional plane.

Using (3.2), we have

$$J_1 = \int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2}\ \psi(u_1, u_2)\, du_1\, du_2 \tag{3.7}$$

$$= \int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2} \Big[\tfrac{1}{4}\, G(0\,;0) + \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0)\, e^{-n_1 u_1}$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, n_2)\, e^{-n_2 u_2} + \sum_{n_1, n_2=1}^{\infty} G(n_1, n_2)\, e^{-n_1 u_1 - n_2 u_2} \Big] du_1\, du_2$$

$$= \tfrac{1}{4}\, G(0, 0) \int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1 + z_2 u_2}\ du_1\, du_2$$

$$+ \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0) \int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{(z_1-n_1) u_1 + z_2 u_2}\ du_1\, du_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, n_2) \int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1 + (z_2-n_2) u_2}\ du_1\, du_2$$

$$+ \sum_{n_1, n_2=1}^{\infty} G(n_1, n_2) \int_{\delta_1-\omega_1 i}^{\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{(z_1-n_1) u_1 + (z_2-n_2) u_2}\ du_1\, du_2$$

$$= -4 \Big[\tfrac{1}{4} \frac{G(0,0)}{z_1 z_2} e^{z_1 \delta_1 + z_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \frac{G(n_1, 0)}{z_2 (z_1 - n_1)} e^{(z_1 - n_1)\delta_1 + z_2 \delta_2} \sin \omega_1 (z_1 - n_1) \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} \frac{G(0, n_2)}{z_1 (z_2 - n_2)} e^{z_1 \delta_1 + (z_2 - n_2)\delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 - n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(n_1, n_2)}{(z_1 - n_1)(z_2 - n_2)} e^{(z_1 - n_1)\delta_1 + (z_2 - n_2)\delta_2} \sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 - n_2) \Big],$$

provided the change of order of integration and summation in (3.7) is justified.

Similarly, using (3.3), (3.4) and (3.5), we have

$$J_4 = \int_{-\delta_1 - \omega_1 i}^{-\delta_1 + \omega_1 i} \int_{-\delta_2 - \omega_2 i}^{-\delta_2 + \omega_2 i} e^{z_1 u_1 + z_2 u_2} \psi(u_1, u_2)\, du_1\, du_2 \tag{3.8}$$

$$= \int_{-\delta_1 - \omega_1 i}^{-\delta_1 + \omega_1 i} \int_{-\delta_2 - \omega_2 i}^{-\delta_2 + \omega_2 i} e^{z_1 u_1 + z_2 u_2} \Big[\tfrac{1}{4} G(0,0) + \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(-n_1, 0) e^{n_1 u_1}$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} G(0, -n_2) e^{n_2 u_2} + \sum_{n_1, n_2=1}^{\infty} G(n_1, -n_2) e^{n_1 u_1 + n_2 u_2} \Big] du_1 du_2$$

$$= -4 \Big[\tfrac{1}{4} \frac{G(0,0)}{z_1 z_2} e^{-z_1 \delta_1 - z_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \frac{G(-n_1, 0)}{z_2 (z_1 + n_1)} e^{-(z_1 + n_1)\delta_1 - z_2 \delta_2} \sin \omega_1 (z_1 + n_1) \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} \frac{G(0, -n_2)}{z_1 (z_2 + n_2)} e^{-z_1 \delta_1 - (z_2 + n_2)\delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 + n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(-n_1, -n_2)}{(z_1 + n_1)(z_2 + n_2)} e^{-(z_1 + n_1)\delta_1 - (z_2 + n_2)\delta_2} \sin \omega_1 (z_1 + n_1) \sin \omega_2 (z_2 + n_2) \Big],$$

$$J_2 = \int_{\delta_1 - \omega_1 i}^{\delta_1 + \omega_1 i} \int_{-\delta_2 - \omega_2 i}^{-\delta_2 + \omega_2 i} e^{z_1 u_1 + z_2 u_2} \psi(u_1, u_2)\, du_1\, du_2 \tag{3.9}$$

$$= \int_{\delta_1 - \omega_1 i}^{\delta_1 + \omega_1 i} \int_{-\delta_2 - \omega_2 i}^{-\delta_2 + \omega_2 i} e^{z_1 u_1 + z_2 u_2} \Big[-\tfrac{1}{4} G(0,0) - \tfrac{1}{2} \sum_{n_1=1}^{\infty} G(n_1, 0) e^{-n_1 u_1}$$

$$-\tfrac{1}{2}\sum_{n_2=1}^{\infty} G(0,-n_2)\, e^{n_2 u_2} - \sum_{n_1,\, n_2=1}^{\infty} G(n_1,-n_2)\, e^{-n_1 u_1+n_2 u_2}\Big]\, du_1\, du_2$$

$$= 4\Big[\tfrac{1}{4}\,\frac{G(0,0)}{z_1 z_2}\, e^{z_1 \delta_1 - z_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 z_2$$

$$+\tfrac{1}{2}\sum_{n_1=1}^{\infty} \frac{G(n_1,0)}{z_2(z_1-n_1)}\, e^{(z_1-n_1)\delta_1 - z_2\delta_2} \sin \omega_1 (z_1-n_1) \sin \omega_2 z_2$$

$$+\tfrac{1}{2}\sum_{n_2=1}^{\infty} \frac{G(0,-n_2)}{z_1(z_2+n_2)}\, e^{z_1\delta_1-(z_2+n_2)\delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2+n_2)$$

$$+\sum_{n_1,\, n_2=1}^{\infty} \frac{G(n_1,-n_2)}{(z_1-n_1)(z_2+n_2)}\, e^{(z_1-n_1)\delta_1-(z_2+n_2)} \sin \omega_1 (z_1-n_1) \sin \omega_2 (z_2+n_2)\Big],$$

(3.10)
$$J_3 = \int_{-\delta_1-\omega_1 i}^{-\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1+z_2 u_2}\, \psi(n_1, n_2)\, du_1\, du_2$$

$$= \int_{-\delta_1-\omega_1 i}^{-\delta_1+\omega_1 i} \int_{\delta_2-\omega_2 i}^{\delta_2+\omega_2 i} e^{z_1 u_1+z_2 u_2} \Big[-\tfrac{1}{4}\, G(0,0) - \tfrac{1}{2}\sum_{n_1=1}^{\infty} G(-n_1,0)\, e^{n_1 u_1}$$

$$-\tfrac{1}{2}\sum_{n_2=1}^{\infty} G(0,n_2)\, e^{-n_2 u_2} - \sum_{n_1,\, n_2=1}^{\infty} G(-n_1,n_2)\, e^{n_1 u_1-n_2 u_2}\Big]\, du_1 dn_2$$

$$= 4\Big[\tfrac{1}{4}\,\frac{G(0,0)}{z_1 z_2}\, e^{-z_1\delta_1+z_2\delta_2} \sin \omega_1 z_1 \sin \omega_2 z_2$$

$$+\tfrac{1}{2}\sum_{n_1=1}^{\infty} \frac{G(-n_1,0)}{z_2(z_1+n_1)}\, e^{-(z_1+n_1)\delta_1+z_2\delta_2} \sin \omega_1 (z_1+n_1) \sin \omega_2 z_2$$

$$+\tfrac{1}{2}\sum_{n_2=1}^{\infty} \frac{G(0,n_2)}{z_1(z_2-n_2)}\, e^{-z_1\delta_1+(z_2-n_2)\delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2-n_2)$$

$$+\sum_{n_1,\, n_2=1}^{\infty} \frac{G(-n_1,n_2)}{(z_1+n_1)(z_2-n_2)}\, e^{-(z_1+n_1)\delta_1+(z_2-n_2)\delta_2} \sin \omega_1 (z_1+n_1) \sin \omega_2 (z_2-n_2)\Big]$$

respectively, provided the change of order of integration and summation in (3.8) and (3.10) are justified. Using (3.7), (3.8), (3.9) and (3.10) we have, from (3.6), that

$$G(z_1, z_2) = \frac{1}{\pi^2} \lim_{\delta_1, \delta_2 \to 0} \Big[\frac{G(0,0)}{4 z_1 z_2} \sin \omega_1 z_1 \sin \omega_2 z_2 \cdot \big(e^{z_1 \delta_1 + z_2 \delta_2} + e^{z_1 \delta_1 - z_2 \delta_2}$$

$$+ e^{-z_1 \delta_1 + z_2 \delta_2} + e^{-z_1 \delta_1 - z_2 \delta_2} \big)$$

$$+ e^{z_1 \delta_1 + z_2 \delta_2} \Big\{ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \frac{G(n_1, 0)}{z_2 (z_1 - n_1)} e^{-n_1 \delta_1} \omega_1 (z_1 - n_1) \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} \frac{G(0, n_2)}{z_1 (z_2 - n_2)} e^{-n_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 - n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(n_1, n_2)}{(z_1 - n_1)(z_2 - n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 - n_2) \Big\}$$

$$+ e^{z_1 \delta_1 - z_2 \delta_2} \Big\{ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \frac{G(n_1, 0)}{z_2 (z_1 - n_1)} e^{-n_1 \delta_1} \sin \omega_1 (z_1 - n_1) \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} \frac{G(0, -n_2)}{z_1 (z_2 + n_2)} e^{-n_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 + n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(n_1, -n_2)}{(z_1 - n_1)(z_2 + n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 + n_2) \Big\}$$

$$+ e^{-z_1 \delta_1 + z_2 \delta_2} \Big\{ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \frac{G(-n_1, 0)}{z_2 (z_1 + n_1)} e^{-n_1 \delta_1} \sin \omega_1 (z_1 + n_1) \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} \frac{G(0, n_2)}{z_1 (z_2 - n_2)} e^{-n_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 - n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(-n_1, n_2)}{(z_1 + n_1)(z_2 - n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 + n_1) \sin \omega_2 (z_2 - n_2) \Big\}$$

$$+ e^{-z_1 \delta_1 - z_2 \delta_2} \Big\{ \tfrac{1}{2} \sum_{n_1=1}^{\infty} \frac{G(n_1, 0}{z_2 (z_1 + n_1)} e^{-n_1 \delta_1} \sin \omega_1 (z_1 + n_1) \sin \omega_2 z_2$$

$$+ \tfrac{1}{2} \sum_{n_2=1}^{\infty} \frac{G(0, -n_2)}{z_1 (z_2 + n_2)} e^{-n_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 + n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(-n_1, -n_2)}{(z_1 + n_1)(z_2 + n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 + n_1) \sin \omega_2 (z_2 + n_2) \Big\} \Big]$$

$$= \frac{1}{\pi^2} \lim_{\delta_1, \delta_2 \to 0} \Big[\frac{G(0,0)}{z_1 z_2} \sin \omega_1 z_1 \sin \omega_2 z_2 + \sum_{n_1=1}^{\infty} \frac{G(n_1, 0)}{z_2(z_1 - n_1)} e^{-n_1 \delta_1} \sin \omega_1 (z_1 - n_1) \sin \omega_2 z_2$$

$$+ \sum_{n_2=1}^{\infty} \frac{G(0, n_2)}{z_1(z_2 - n_2)} e^{-n_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 - n_2)$$

$$+ \sum_{n_1=1}^{\infty} \frac{G(-n_1, 0)}{z_2(z_1 + n_1)} e^{-n_1 \delta_1} \sin \omega_1 (z_1 + n_1) \sin \omega_2 z_2$$

$$+ \sum_{n_2=1}^{\infty} \frac{G(0, -n_2)}{z_1(z_2 + n_2)} e^{-n_2 \delta_2} \sin \omega_1 z_1 \sin \omega_2 (z_2 + n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(n_1, n_2)}{(z_1 - n_1)(z_2 - n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 - n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(n_1, -n_2)}{(z_1 - n_1)(z_2 + n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 + n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(-n_1, n_2)}{(z_1 + n_1)(z_2 - n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 + n_1) \sin \omega_2 (z_2 - n_2)$$

$$+ \sum_{n_1, n_2=1}^{\infty} \frac{G(-n_1, -n_2)}{(z_1 + n_1)(z_2 + n_2)} e^{-n_1 \delta_1 - n_2 \delta_2} \sin \omega_1 (z_1 + n_1) \sin \omega_2 (z_2 + n_2) \Big].$$

Thus

$$G(z_1, z_2) = \frac{1}{\pi^2} \lim_{\delta_1, \delta_2 \to 0} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{\sin \omega_1 (z_1 - n_1) \sin \omega_2 (z_2 - n_2)}{(z_1 - n_1)(z_2 - n_2)} G(n_1, n_2) e^{-\delta_1 |n_1| - \delta_2 |n_2|}.$$

Particular Cases.

(*i*) If we put $\omega_1 = \pi$, then the theorem reduces to :

$$G(z_1, z_2) = \frac{\sin \pi z_1}{\pi^2} \lim_{\delta_1, \delta_2 \to 0} \sum_{n_1, n_2 = -\infty}^{\infty} (-1)^{n_1} \frac{\sin \omega_2 (z_2 - n_2)}{(z_1 - n_1)(z_2 - n_2)} G(n_1, n_2) e^{-\delta_1 |n_1| - \delta_2 |n_2|}. \tag{3.11}$$

(*ii*) If we put $\omega_2 = \pi$, then the theorem reduces to :

$$G(z_1, z_2) = \frac{\sin \pi z_2}{\pi^2} \lim_{\delta_1, \delta_2 \to 0} \sum_{n_1, n_2 = -\infty}^{\infty} (-1)^{n_2} \frac{\sin \omega_1 (z_1 - n_1)}{(z_1 - n_1)(z_2 - n_2)} G(n_1, n_2) e^{-\delta_1 |n_1| - \delta_2 |n_2|}. \tag{3.12}$$

(*iii*) In the special case, if we take $\omega_1 = \pi$, $\omega_2 = \pi$, then the theorem reduces to :

(3.13) $$G(z_1, z_2) = \frac{\sin \pi z_1 \sin \pi z_2}{\pi^2} \sum_{n_1, n_2=-\infty}^{\infty} (-1)^{n_1+n_2} \frac{G(n_1, n_2)}{(z_1 - n_1)(z_2 - n_2)},$$

that is the series is summable to $G(z_1, z_2)$.

4. Set of interpolation formulae.

From (3.11), we have

$$G(z_1, z_2) = \frac{\sin \pi z_1}{\pi^2} \sum_{n_1, n_2=-\infty}^{\infty} (-1)^{n_1} \frac{\sin \omega_2 (z_2 - n_2)}{(z_1 - n_1)(z_2 - n_2)} G(n_1, n_2)$$

$$= \frac{\sin \pi z_1}{\pi^2} \sum_{n_2=-\infty}^{\infty} \frac{\sin \omega_2 (z_2 - n_2)}{(z_2 - n_2)} \left[\sum_{n_1=-\infty}^{\infty}{}' (-1)^{n_1} \frac{G(n_1, n_2)}{(z_1 - n_1)} + \frac{G(0, n_2)}{z_1} \right],$$

so long as the right hand side is uniformly convergent.

Now

$$\frac{\partial G(z_1, z_2)}{\partial z_1} = \frac{1}{\pi^2} \sum_{n_2=-\infty}^{\infty} \frac{\sin \omega_2 (z_2 - n_2)}{(z_2 - n_2)} \left[\sum_{n_1=-\infty}^{\infty}{}' (-1)^{n_1} \left\{ \frac{\pi \cos \pi z_1}{(z_1 - n_1)} - \frac{\sin \pi z_1}{(z_1 - n_1)^2} \right\} G(n_1, n_2) \right.$$

$$\left. + \frac{\pi z_1 \cos \pi z_1 - \sin \pi z_1}{z_1^2} G(0, n_2) \right].$$

Therefore

$$\frac{\partial G(0, z_2)}{\partial z_1} = -\frac{1}{\pi} \sum_{n_2=-\infty}^{\infty} \frac{\sin \omega_2 (z_2 - n_2)}{(z_2 - n_2)} \left[\sum_{n_1=-\infty}^{\infty}{}' \frac{(-1)^{n_1}}{n_1} G(n_1, n_2) \right],$$

so that

(4.1) $$G(z_1, z_2) - \frac{\sin \pi z_1}{\pi} \frac{\partial G(0, z_2)}{\partial z_1}$$

$$= \frac{\sin \pi z_1}{z_1 \pi^2} \sum_{n_2=-\infty}^{\infty} \frac{\sin \omega_2 (z_2 - n_2)}{(z_2 - n_2)} \left[G(0, n_2) + z_1^2 \sum_{n_1=-\infty}^{\infty}{}' (-1)^{n_1} \frac{G(n_1, n_2)}{n_1 (z_1 - n_1)} \right].$$

Similarly, from (3.12), we have

$$G(z_1, z_2) = \frac{\sin \pi z_2}{\pi^2} \sum_{n_1, n_2=-\infty}^{\infty} (-1)^{n_2} \frac{\sin \omega_1 (z_1 - n_1)}{(z_1 - n_1)(z_2 - n_2)} G(n_1, n_2),$$

so long as the right hand side is uniformly convergent and therefore

(4.2) $$G(z_1, z_2) - \frac{\sin \pi z_2}{\pi} \frac{\partial G(z_1, 0)}{\partial z_2}$$

$$= \frac{\sin \pi z_2}{z_2 \pi^2} \sum_{n_1=-\infty}^{\infty} \frac{\sin \omega_1 (z_1 - n_1)}{(z_1 - n_1)} \left[G(n_1, 0) + z_2^2 \sum_{n_2=-\infty}^{\infty}{}' (-1)^{n_2} \frac{G(n_1, n_2)}{n_2 (z_2 - n_2)} \right].$$

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ÖZET

İki değişkenli, sonlu mertebe ve tipten tam fonksiyonlara, M. M. DZRBASYAN [1] 'ın bir sonucu ve tek bir kompleks değişken için MACINTYRE [2] tarafından kullanılan bir usulü uygulamak suretiyle bunlar için bazı interpolasyon formülleri elde edilmiştir.