ON THE MEANS OF AN ENTIRE FUNCTION AND ITS DERIVATIVES [1])

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Some properties of the lower and upper limits of particular means of an entire function and its derivatives are obtained. These properties are extensions of some previously proved, under less general conditions, by R. P. SRIVASTAVA, P.K. KAMTHAN and O.P. JUNEJA.

1. Let $f(z)$ be an entire function of order ϱ and lower order λ. For $0 < \delta < \infty$ and $z = re^{i\vartheta}$, let

$$\{M_\delta(r)\}^\delta = \mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^\delta \, d\vartheta \tag{1.1}$$

and

$$\{M_\delta(r, f^{(m)})\}^\delta = \mu_\delta(r, f^{(m)}) = \frac{1}{2\pi}\int_0^{2\pi} |f^{(m)}(re^{i\vartheta})|^\delta \, d\vartheta, \tag{1.2}$$

where $f^{(m)}(z)$ denotes the m-th derivative of $f(z)$. Then the following results are known :

Theorem [6]**.** For every entire function $f(z)$, other than a polynomial,

$$\lim_{r\to\infty}\sup \frac{\log r\left\{\dfrac{M_1(r, f^{(s)})}{M_1(r, f)}\right\}^{\frac{1}{s}}}{\log r} = \varrho. \tag{1.3}$$

Theorem [2]. For $1 \leqq \delta < \infty$

$$\text{hm}\sup_{r\to\infty} \frac{\log r\left\{\dfrac{M_\delta(r, f^{(s)})}{M_\delta(r, f)}\right\}^{\frac{1}{s}}}{\log r} = \varrho \tag{1.4}$$

(here $r \to \infty$ excluding a set of values of r having measure zero).

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JUNEJA [1] proved the following result as a corollary:

If $f(z)$ is an entire function of lower order $\lambda > 1 + \frac{1}{m}$ and order ϱ ($< \infty$), then

$$(1.5)\qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \left\{ \frac{\mu_\delta(r, f^{(m)})}{\mu_\delta(r, f)} \right\}^{\frac{1}{m\delta}}}{\log r} = \frac{\varrho}{\lambda}, \quad (\delta \geq 1)$$

Theorem [1]. For every entire function $f(z)$, other than a polynomial,

$$(1.6)\qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}' \frac{\log r \left\{ \frac{\mu_\delta(r, f^{(m)})}{\mu_\delta(r, f)} \right\}^{\frac{1}{m\delta}}}{\log r} = \frac{\varrho}{\lambda} \quad (0 < \delta < 1)$$

where r tends to infinity through values excluding an exceptional set of at most finite measure.

For $\delta = 1$, R. P. SRIVASTAVA [6] has given a proof of (1.3), when the upper limit is only considered. For $1 \leq \delta < \infty$, P. K. KAMTHAN [2] has given a proof of (1.4) for the upper limit only and under the condition that $r \to \infty$ excluding a set of values of r having measure zero. Their methods, infact, fail to give the lower limit. The result (1.5) has been obtained by O. P. JUNEJA, imposing the condition on the lower order. In the case $0 < \delta < 1$ O. P. JUNEJA has given a proof of the result (1.6) (see [1]), when the upper limit is only considered. His method, in fact, fails to give the lower limit of the left-hand expression in (1.6).

Our aim in this paper is to prove some results on the means of an entire function and its derivatives. We prove the following:

2. Theorem 1. *If $f(z)$ is an entire function of lower order λ and order ϱ ($< \infty$), then*

$$(2.1)\qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \frac{M_\delta(r, f^{(m)})^{\frac{1}{m}}}{M_\delta(r, f)}}{\log r} = \frac{\varrho}{\lambda}.$$

where $\delta \geq 1$ and $m = 1, 2, \ldots, m$.

Remark. This result is stronger than that of JUNEJA in the sense that it does not impose the additional condition $\lambda > 1 + \frac{1}{m}$.

To prove this theorem we require the following lemmas:

Lemma 1. For an entire function $f(z)$

$$(2.2)\qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M_\delta(r, f)}{\log r} = \frac{\varrho}{\lambda}.$$

This lemma has been proved by RAHMAN [5].

Lemma 2. If $f(z)$ is an entire function of order ϱ and lower order λ and $\delta \geq 1$, then

(2.3) $$\frac{M_\delta(r, f^{(m)})}{M_\delta(r,f)} > \left[\frac{\log M_\delta(r,f)}{r \log r}\right]^m (1-\varepsilon)^{m-1} \cdot \alpha \ , \ o < \alpha < 1$$

Proof. It is known that [3]

$$\frac{M_\delta(r,f')}{M_\delta(r,f)} > \frac{\log M_\delta(r,f) - \log M_\delta(r_o,f)}{r \log r} .$$

Replacing $f(z)$ by $f^{(k-1)}(z)$, we have

$$\frac{M_\delta(r,f^{(k)})}{M_\delta(r,f^{(k-1)})} > \frac{\log M_\delta(r,f^{(k-1)}) - \log M_\delta(r_o, f^{(k-1)})}{r \log r}$$

$$> \frac{\log M_\delta(r,f^{(k-1)})}{r \log r} \alpha_{k-1}, \quad 0 < \alpha_{k-1} < 1.$$

Putting $k = 1, 2, \ldots, m$ and multiplying the m inequalities thus obtained, we get

$$\frac{M_\delta(r,f^{(m)})}{M_\delta(r,f)} > \frac{\prod_{k=1}^{m} \log M_\delta(r,f^{(k-1)})}{(r \log r)^m} \cdot \alpha \ , \ \alpha = \prod_{k=1}^{m} \alpha_{k-1}.$$

Now by Lemma 1, we have

$$\log M_\delta(r,f^{(k)}) \geq (1-\varepsilon) \log M_\delta(r,f).$$

Hence for all $r > r_o > 1$, we get

$$\frac{M_\delta(r,f^{(m)})}{M_\delta(r,f)} > \left\{\frac{\log M_\delta(r,f)}{r \log r}\right\}^m (1-\varepsilon)^{m-1} \cdot \alpha$$

which is (2.3).

Lemma 3. If $f(z)$ is of finite order ϱ and lower order λ, then for $r > r_o$

(2.4) $$M_\delta(r,f^{(m)}) < r^{(\rho-1+\varepsilon)\cdot m} M_\delta(r,f)$$

and for an infinite sequence of values of r tending to infinity

(2.5) $$M_\delta(r,f^{(m)}) < r^{(\lambda-1+\varepsilon)\cdot m} M_\delta(r,f)$$

where $\delta \geq 1$ and ε is positive.

Proof. It is enough to prove (2.4) since the proof of (2.5) is similar.
We know [8] that for every $\varepsilon > 0$ and large r

$$M_\delta(r,f^{(k)}) < r^{(\rho-1+\varepsilon)} M_\delta(r,f^{(k-1)}).$$

Giving k the values $1, 2, 3, \ldots, m$ and multiplying the m inequalities thus obtained, we get for large r

$$M_\delta(r, f^{(m)}) < r^{(\rho-1+\varepsilon)\,m}\, M_\delta(r, f)$$

which is (2.4).

Proof of Theorem 1. Lemma 2 leads to

$$\frac{\log r \left\{ \dfrac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}}}{\log r} > \frac{\log \log M_\delta(r)}{\log r} - \frac{\log \log r}{\log r} + o(1).$$

Proceeding to limits and using Lemma 1, we get

$$\text{(2.6)} \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \left\{ \dfrac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}}}{\log r} \geqq \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

From Lemma 3, we have

$$\text{(2.7)} \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \left\{ \dfrac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}}}{\log r} \leqq \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Comparing (2.6) and (2.7), we get (2.1).

3. Theorem 2. *For every entire function $f(z)$, other than a polynomial,*

$$\text{(3.1)} \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \left\{ \dfrac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}}}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix}, \quad 0 < \delta < 1.$$

where r tends to infinity trough values excluding an exceptional set of at most finite measure.

The proof of this theorem requires the following lemmas :

Lemma 4. For every entire function $f(z)$, other than a polynomial, outside an exceptional set of at most finite measure,

$$\text{(3.2)} \qquad M_\delta(r, f^{(m)}) \geqq M_\delta(r, f) \left(\frac{\nu(r)}{r}\right)^m \left\{ 1 - k\nu(r)^{-\frac{1}{16}} \right\}$$

where $\nu(r)$ denotes the rank of maximum term in $f(z)$ for $|z| = r$, k is positive constant and $0 < \delta < 1$,[1].

Lemma 5. If $f(z)$ is of finite order ϱ and finite lower order λ, then for $r > r_0$

$$\text{(3.3)} \qquad M_\delta(r, f^{(m)}) \leqq r^{(\rho-1+\varepsilon)\,m}\, M_\delta(r, f), \quad 0 < \delta < 1, \quad r \geqq r_0(\varepsilon)$$

and for an infinite sequence of values of r tending to infinity

$$\text{(3.4)} \qquad M_\delta(r, f^{(m)}) \leqq r^{(\lambda-1+\varepsilon)\,m}\, M_\delta(r, f), \quad 0 < \delta < 1.$$

Proof. It is known [1] that

$$M_\delta(r, f') \leq r^{(\rho-1+\varepsilon)} M_\delta(r, f) \quad , \quad 0 < \delta < 1.$$

Then the proof of the Lemma follows on the lines of that of Lemma 3.

Proof of Theorem 2. Lemma 4 leads to

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \left\{ \dfrac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}}}{\log r} \geq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \nu(r)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix} \tag{3.5}$$

while from (3.3) and (3.4) we get

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r \left\{ \dfrac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}}}{\log r} \leq \begin{matrix}\varrho\\ \lambda\end{matrix}. \tag{3.6}$$

Comparing (3.5) and (3.6), we get (3.1).

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ÖZET

Bir tam fonksiyon ve bu tam fonksiyonun türevleri ile tanımlanan bâzı özel ortalamaların alt ve üst limitlerinin bâzı özellikleri ispat edilmektedir. Bu sonuçlar, daha dar bir çerçeve içinde R. P. SRIVASTAVA, P. K. KAMTHAN ve O. P. JUNEJA tarafından elde edilen bâzı bağıntıların genelleştirilmesidir.