A THEOREM ON PROXIMATE ORDER B OF AN ENTIRE FUNCTION

JODH PAL SING

A result concerning a function of proximate order **B** (in the sense of BOUTROUX [1]) is proved. This is connected with a result due to KAMTHAN [2].

1. BOUTROUX [1] in 1903 proved the existence of a proximate order for an entire function $f(z)$ of non-integral order, known as proximate order B, satisfying the following conditions :

(i) $\varrho(r)$ is real, continuous and piecewise differentiable for $r \geq r_0$.

(ii) $\liminf\limits_{r\to\infty} \varrho(r) > p$ where p is the genus of function.

(iii) There is a number α such that $r^{\varrho(r)-p-\alpha}$ and $r^{+1-\varrho(r)-\alpha}$ are inreasing function of r.

(iv) $r\varrho'(r)\log r \to 0$ as $r \to \infty$ where $\varrho'(r)$ is either the right hand or the left hand derivatives at the points where they are different.

(v) $n(r,a) < kr^{\varrho(r)}$, $r \geq r_0$, k any constant > 0.

(vi) $n(r,a) < \dfrac{1}{k} r^{\varrho(r)}$ for a sequence of values of r proportional to the values for which $\log M(r,f) = r^{\varrho(r)}$.

Let $f(z)$ be an entire function of order ϱ and genus p. Suppose further that $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$; then the HATAMARD representation of the function is

$$f(z) = z^m e^{Q(z)} P(z), \tag{1.1}$$

where $Q(z)$ is a polynomial of degree $q \leq \varrho$ and $P(z)$ is the canonical product of genus p formed with the zeros (other than $z = 0$) of $f(z)$. KAMTHAN [2] has proved the following theorem:

Theorem. If $P(z)$ be a canonical product of genus p and order ϱ $(\varrho > p)$ defined by

$$P(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\exp\left\{\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{z_n}\right)^p\right\} \tag{1.2}$$

where $z_1, z_2, \ldots$, etc. are the zeros of $P(z)$ whose modul $r_1, r_2, \ldots$ form a non-decreasing sequence such that $r_n > 1$ for all n and where $r_n \to \infty$ as $n \to \infty$ then for z in a domain exterior to the circles r_n^{-h} $(h > \varrho)$ described about the zeros z_n as centres we have

$$\left.\frac{P'(z)}{P(z)}\right| > k \int_0^\infty \frac{r^p\, n(x)}{x^p\,(x+r)^2}\, dx \tag{1.3}$$

where k is a constant dependent of p and $P'(z)$ is the first derivative and $n(x)$ denotes the number of zeros within and on the circle $|z| = x$.

As we are considering entire functions of non-integral order, it is sufficient to prove the theorem for canonical product $P(z)$ of $f(z)$. In this paper we prove the following theorem:

2. Theorem. If $\varrho(r)$ be the proximate order B for an entire function of non-integral order, $P(z)$ be the canonical product, for some $\lambda > 0$

$$N(r) \sim \lambda\, r^{\varrho(r)} \text{ as } r \to \infty \quad \text{then}$$

$$\left| \sin \pi (\varrho - p) \frac{P'(re^{i\theta})}{P(re^{i\theta})} - \pi (\varrho - p\, \lambda\, r^{\varrho(r)-p-1}, \right| < \varepsilon\, r^{\varrho(r)-1}, \varepsilon > 0. \tag{2.1}$$

We require the following lemmas to prove the theorem.

Lemma 1. For a sequence of values of r

$$\left| \frac{P'(re^{i\theta})}{P(re^{i\theta})} \right| < k \int_0^\infty \frac{N(x)\, r^p}{x^p\,(x+r)^2}\, dx.$$

Proof. Since

$$N(x) = \int_0^x \frac{n(t)}{t}\, dt$$

$$d\,N(x) = \frac{n(x)}{x}$$

for almost all values of x. From (1.3) we have

$$\left| \frac{P'(re^{i\theta})}{P(re^{i\theta})} \right| < k \int_0^\infty \frac{r^p d\,N(x)}{x^p\,(x+r)^2}\, dx$$

$$< k \int_0^\infty \frac{N(x)\, r^p}{x^p\,(x+r)^2} \left\{ \frac{(p+1)\,x^2 + 2pxr + (p-1)\,r^2}{(x+r)^2} \right\} dx$$

$$< k \int_0^\infty \frac{r^p\, N(x)}{x^p\,(x+r)^2} \left\{ \frac{x(p+1) + r\,(p-1)}{(x+r)} \right\} dx.$$

Now the expression written within the curly bracket inside the integral sign is bounded in $(0, \infty)$ and monotonic increasing. Hence, we have

$$\left| \frac{P'(re^{i\theta})}{P(re^{i\theta})} \right| < k \int_0^{\infty} \frac{N(x)\, r^p}{x^p (x+1)^2}\, dx.$$

Lemma 2.

$$\int_0^{\infty} \frac{x^{\varrho-p}}{(x+r)^2}\, dx = \frac{\pi(\varrho - p)}{\sin \pi(\varrho - p)}\, r^{(\varrho-p-1)}.$$

This is easily established by contour integration since $0 < \varrho - p < 1$.

Lemma 3 A. If $\mu > 1$, $N(x) \sim \lambda x^{\varrho(x)}$ then

$$\int_{r_0}^{r/\mu} \frac{N(x)\, r^p}{x^p (x+r)^2}\, dx < \varepsilon\, r^{\varrho(p)-1}$$

Proof.

$$\int_{r_0}^{r/\mu} \frac{N(x)\, r^p}{x^p (x+r)^2}\, dx < \int_{r_0}^{r/\mu} \frac{2\lambda\, x^{\varrho(x)}\, r^p}{x^p (x+r)^2}\, dx$$

$$< \int_{r_0}^{r/\mu} 2\lambda\, x^{\varrho(x)-p}\, r^{p-2}\, dx$$

$$< 2\lambda\, r^{\varrho(r)-2-\alpha} \int_{r_0}^{r/\mu} x^{\alpha}\, dx$$

$$< \varepsilon\, r^{\varrho(r)-1},$$

since given ε in advance we can choose μ such that $\frac{1}{\mu^{\varrho-p-1}} < 1$ as $\mu > 1$, $r^{\varrho(r)-p-1}$ is increasing function in proximate order B.

Lemma 3 B.

$$\int_{r_0}^{r/\mu} \frac{x\, r^p}{x^p (x+r)^2}\, dx < \varepsilon\, r^{\cdot -1}.$$

This is easily established since $\varrho - p + 1 > 0$.

Lemma 4 A. If $\mu > 1$ and $N(x) \sim \lambda x^{\ (x)}$ then

$$\int_{\mu r}^{\infty} \frac{N(x)\, r^p}{x^p (x+r)^2}\, dx < \varepsilon\, r^{\varrho(r)-1}.$$

Proof.

$$\int_{\mu r}^{\infty} \frac{N(x)\, r^p}{x^p\,(x+r)^2}\, dx < \int_{\mu r}^{\infty} \frac{2\lambda\, x^{\varrho(x)}\, r}{x^{p+2}}\, dx$$

$$< 2\lambda\, r^p \int_{\mu r}^{\infty} x^{\varrho(x)-p+\alpha-1} \cdot (x^{-1-\alpha})\, dx$$

$$< 2\lambda\, r^{\varrho(r)+\alpha-1} \left[\frac{x^{-\alpha}}{\alpha}\right]_{\infty}^{\mu r}$$

$$< \varepsilon\, r^{\varrho(r)-1},$$

since given ε in advance we can choose μ large enough such that $\mu^{\varrho-1-p+\alpha} < 1$ and $\frac{2\lambda}{\alpha\,\mu^{\alpha}} < \varepsilon$ as $r^{\varrho(r)-1-p+\alpha}$ is decreasing function in proximate order B.

Lemma 4 B.

$$\int_{\mu r}^{\infty} \frac{x\, r^p}{x^p\,(x+r)^2}\, dx < \varepsilon\, r^{\varrho-1}.$$

Proof.

$$\int_{\mu r}^{\infty} \frac{x^p\, r^p}{x^p\,(x+r)^2}\, dx < r^p \int_{\mu r}^{\infty} x^{\varrho-p-2}\, dx$$

$$< r^p \left[\frac{x^{\varrho-p-1}}{\varrho-p-1}\right]_{\mu r}^{\infty}$$

$$< r^p\, \frac{(\mu r)^{\varrho-p-1}}{p+1-\varrho}$$

$$< \varepsilon\, r^{\varrho-1}, \qquad \varepsilon > 0,$$

since $\varrho - p - 1 < 0$.

Lemma 5. If $\lambda > 1$ and $N(x) \sim \lambda\, x^{\varrho(x)}$ Then

$$E \equiv \left| \int_{r/\mu}^{\mu r} \frac{N(x)\, r^p}{x^p\,(x+r)^2}\, dx - \lambda\, r^{\varrho(r)-\varrho} \int_{r/\mu}^{\mu r} \frac{x^{\varrho}\, r^p}{x^p\,(x+r)^2} \right| < e r^{\varrho(r)-1}.$$

Proof. Now

$$E \equiv \left| \int_{\mu/r}^{\mu r} \frac{\left\{N(x) - \lambda\, r^{\varrho(r)} \left(\frac{x}{r}\right)^{\varrho}\right\} r^p}{x^p\,(x+r)^2}\, dx \right|,$$

since $\qquad N(x) \sim \lambda\, x^{\varrho(x)}$ and $\lambda\, x^{\varrho(x)} \sim \left(\frac{x}{\varrho}\right)^{\varrho} r^{\varrho(r)}$

Hence

$$E < \left| \int_{r/\mu}^{\mu r} \frac{2^8 1\, r^{\varrho(r)} r^p}{x^{p+2}}\, dx \right.$$

$$< \frac{2^8 1}{p+1} r^{\varrho(r)+p} \left[\frac{1}{x^{p+1}} \right]_{r\mu}^{r/\mu}$$

$$< \frac{2^8 1}{p+1} r^{\varrho(r)-1} \left[\mu^{p+1} - \frac{1}{\mu^{p+1}} \right]$$

$$< \varepsilon\, r^{\varrho(r)-1}.$$

Proof of the theorem. For every $\varepsilon > 0$

$$E = \left| \frac{P'(re^{i\theta})}{P(re^{i\theta})} - \pi \lambda (\varrho - p)\, r^{\varrho(r)-p-1} \operatorname{cosec} \mu (\varrho - p) \right|.$$

By lemma 1 and lemma 2, it will be sufficient to show that for every $\varepsilon > 0$

$$E < \left| \int_0^\infty \frac{K\, N(x)\, r^p}{x^p (x+r)^2}\, dx - \frac{\lambda\, r^{\varrho(r)-p-1}}{r^{\varrho-p-1}} \int_0^\infty \frac{x^{\varrho-p}}{(x+r)^2}\, dx \right.$$

$$< \left| \int_0^{r_0} \frac{K\, N(x)\, r^p}{x^p (x+r)^2}\, dx \right| + \left| \int_{r_0}^{r/\mu} \frac{K\, N(x)\, r^p}{x^p (x+r)^2}\, dx \right| + \left| \int_{\mu r}^\infty \frac{K\, N(x)\, r^p}{x^p (x+r)^2}\, dx \right.$$

$$+ \left| \int_0^{r_0} \frac{\lambda\, r^{\varrho(r)-\varrho}\, x^{\varrho-p}}{(x+r)^2}\, dx \right| + \left| \int_{r_0}^{r/\mu} \frac{\lambda\, r^{\varrho(r)-\varrho}\, x^{\varrho-p}}{(x+r)^2}\, dx \right.$$

$$+ \left| \int_{\mu r}^\infty \frac{\lambda\, r^{\varrho(r)-\varrho}\, x^{\varrho-p}}{(x+r)^2}\, dx \right|$$

$$+ \left| \int_{r/\mu}^{\mu r} \frac{K\, N(x)\, r^p}{x^p (x+r)^2}\, dx - \lambda\, r^{\varrho(r)-\varrho} \int_{r/\mu}^{\mu r} \frac{x^{\varrho-p}}{(x+r)^2}\, dx \right|.$$

On applying the lemmas 3A, 4A, 3B, 4B and 5 to above integrals, we have

$$E < e\, r^{\varrho(r)-1}$$

or

$$\left| \sin \pi (\varrho - p) \frac{P'(re^{i\theta})}{P(re^{i\theta})} - \lambda \pi (\varrho - p)\, r^{\varrho(r)-p-1} \right| < \varepsilon\, r^{\varrho(r)-1}$$

for every positive ε [1].

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REFERENCES

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43, BRAHMANPURI,
ALIGARH (U.P.)
INDIA

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ÖZET

BOUTROUX [1] anlamında B yaklaşık mertebesinden bir fonksiyon ile ilgili bir sonuç elde edilmiştir. Bu netice KAMTHAN [2] tarafından ispat edilen bir teoreme bağlıdır.