ON THE RECURRENT TENSOR FIELDS OF FINSLER SPACES

by

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The spaces with recurrent curvature were studied by H. S. RUSE [2] 1), A. G. WALKER [4], Y. C. WONG [5], B. B. SINHA and S. P. SINGH [3] and many others. The purpose of present paper is to obtain certain theorems on the recurrence vector field, the relative curvature tensor field and the curvature tensor field in the sense of CARTAN. We are concerned here with the local properties and shall use as notations those of H. RUND [1].

1. Introduction. We consider an n-dimensional FINSLER space in which the arc-length of a curve $x^i = x^i(t)$ 2)$(t_1 \leqq t \leqq t_2)$ is defined by

$$(1.1) \qquad s = \int_{t_1}^{t_2} F(x^i, \dot{x}^i)\, dt, \quad \dot{x}^i = \frac{dx^i}{dt}$$

where $F(x^i, \dot{x}^i)$ is a positively homogeneous function of degree one in $\dot{x}^i$.

Let us take a vector field $X^i(x^k, \xi^k)$, depending on position as well as on a direction field : $\xi^k = \xi^k(x^k)$. By definiton, its covariant δ-derivative at the point x^k in the direction ξ^k is given by

$$(1.2) \qquad X^i_{;k} = \frac{\partial X^i}{\partial x^k} + \frac{\partial X^i}{\partial \dot{x}^l}\frac{\partial \xi^l}{\partial x^k} + \Gamma^{*i}_{rk}(x, \xi)\, X^r$$

while

$$(1.3) \qquad X^i_{;kh} = \frac{\partial X^i_{;k}}{\partial x^h} + \frac{\partial X^i_{;k}}{\partial \dot{x}^l}\frac{\partial \xi^l}{\partial x^h} + \Gamma^{*i}_{jh} X^j_{;k} - X^i_{;j}\Gamma^{*j}_{kh}.$$

Interchanging the indices k and h in (1.3) and subtracting the result from it, we find

$$(1.4) \qquad X^i_{;kh} - X^i_{;hk} = \widetilde{K}^i_{jkh}(x, \xi)\, X^j,$$

where $\widetilde{K}^i_{jkh}$ is the relative curvature tensor [1].

The commutation formulae involving the relative curvature tensor $\widetilde{K}^i_{jkh}$ are as follows :

$$(1.5) \qquad T_{ij;kh} - T_{ij;hk} = - T_{ir}\widetilde{K}^r_{jkh} - T_{rj}\widetilde{K}^r_{ikh}$$

and

$$(1.6) \qquad T^i_{j;kh} - T^i_{j;hk} = T^r_j \widetilde{K}^i_{rkh} - T^i_r \widetilde{K}^r_{jkh}.$$

1) Numbers in brackets refer to the references at the end of this paper.

2) The Roman small letters $a, b, c, \dots, i, j, k, \dots$ take values $1,2,3,\dots,n$.

The relative curvature tensor field $\widetilde{K}^i_{jkh}$ satisfies the identities

$$\widetilde{K}^i_{jkh} + \widetilde{K}^i_{khj} + \widetilde{K}^i_{hjk} = 0 \tag{1.7}$$

and

$$\widetilde{K}^i_{jkh;l} + \widetilde{K}^i_{jhl;k} + \widetilde{K}^i_{jlk;h} = 0. \tag{1.8}$$

Suppose that the vector field ξ^l is "stationary" at the point under consideration i.e. it satisfies the relation $\xi^l_{;k}(x, \xi) = 0.$

We then have at this point

$$\frac{\partial \xi^l}{\partial x^k} = -\Gamma^{*l}_{rk}(x, \xi)\, \xi^r = -\frac{\partial G^l(x, \xi)}{\partial \dot{x}^k}. \tag{1.9}$$

If we substitute (1.9) in (1.2) we obtain CARTAN's covariant derivative $X^i_{/k}$, provided we regard the line-element ξ^i as the element of support $\dot{x}^i$. Then the relative curvature tensor $\widetilde{K}^i_{jkh}$ becomes CARTAN's curvature tensor K^i_{jkh}.

The covariant derivative of a tensor field, say T^i_j in the sense of CARTAN is given by

$$T^i_{j/k} = \frac{\partial T^i_j}{\partial x^k} - \frac{\partial T^i_j}{\partial \dot{x}^l}\, \Gamma^{*l}_{rk}\, \dot{x}^r + T^l_j\, \Gamma^{*i}_{lk} - T^i_l\, \Gamma^{*l}_{jk}. \tag{1.10}$$

The covariant derivative of $\dot{x}^i$ is identically zero.

The curvature tensor field K^i_{jkh} satisfies the identities

$$K^i_{jkh} + K^i_{khj} + K^i_{hjk} = 0 \tag{1.11}$$

and

$$(K^i_{jkh/l} + K^i_{jhl/k} + K^i_{jlk/h})\, l^j + (A^i_{km/s}\, K^m_{jhl} + A^i_{hm/s}\, K^m_{jlk} + A^i_{lm/s}\, K^m_{jkh})\, l^j\, l^s = 0 \tag{1.12}$$

where $A^i_{jk} = FC^i_{jk}$ is a symmetric tensor and l^j is a unit vector.

The commutation formulae involving the curvature tensor field K^i_{jkh} are as follows:

$$T_{/kh} - T_{/hk} = -\frac{\partial T}{\partial \dot{x}^l}\, K^l_{rkh}\, \dot{x}^r, \tag{1.13}$$

$$T^i_{j/kh} - T^i_{j/hk} = -\frac{\partial T^i_j}{\partial \dot{x}^l}\, K^l_{rkh}\, \dot{x}^r + T^r_j\, K^i_{rkh} - T^i_r\, K^r_{jkh} \tag{1.14}$$

and

$$\left(\frac{\partial T^i_j}{\partial \dot{x}^k}\right)_{/h} - \frac{\partial T^i_{j/h}}{\partial \dot{x}^k} = \frac{\partial T^i_j}{\partial \dot{x}^l}\, C^l_{kh/r}\, \dot{x}^r - T^r_j\, \frac{\partial \Gamma^{*}_{rh}}{\partial \dot{x}^k} + T^i_r\, \frac{\partial \Gamma^{*}_{jh}}{\partial \dot{x}^k}. \tag{1.15}$$

The «affinely connected spaces» are characterised by the condition $C_{ijk/h} = 0$. Conversely, the equations $\dfrac{\partial G^i_{hk}}{\partial \dot{x}^l} = 0$ imply $C_{ijk/h} = 0$ and hence $\dfrac{\partial \Gamma^{*i}_{hk}}{\partial \dot{x}^l} = 0$ because $G^i_{hk} = \Gamma^{*i}_{hk}$.

2. Recurrent relative curvature tensor field. A FINSLER space in which there exists a non-zero vector field V_m such that the relative curvature tensor field $\widetilde{K}^i_{jkh}$ satisfies

$$\widetilde{K}^i_{jkh;m} = V_m \widetilde{K}^i_{jkh} \tag{2.1}$$

is said to be a FINSLER space with the recurrent relative curvature tensor field and this V_m the recurrence vector field.

We shall prove certain theorems on the recurrence vector field V_m. We shall denote hereafter by $(\widetilde{K})$ the following property : the value of the tensor field $\widetilde{K}^i_{jkh}$ is not the zero tensor at each element (x, ξ) of the space.

Theorem 2.1. *In* FINSLER *space with the recurrent relative curvature tensor field having the property* $(\widetilde{K})$, *the recurrence vector field* V_m *satisfies the relation*

$$(V_{m;n} - V_{n;m})_{;l} = V_l (V_{m;n} - V_{n;m}). \tag{2.2}$$

Proof. Differentiating (2.1) covariandy, we have

$$\widetilde{K}^i_{jkh;mn} = V_{m;n} \widetilde{K}^i_{jkh} + V_m V_n \widetilde{K}^i_{jkh}, \tag{2.3}$$

which yields

$$\widetilde{K}^i_{jkh;mn} - \widetilde{K}^i_{jkh;nm} = (V_{m;n} - V_{n;m}) \widetilde{K}^i_{jkh}. \tag{2.4}$$

By virtue of (1.6),it becomes

$$(V_{m;n} - V_{n;m}) \widetilde{K}^i_{jkh} = \widetilde{K}^r_{jkh} \widetilde{K}^i_{rmn} - \widetilde{K}^i_{rkh} \widetilde{K}^r_{jmn} - \widetilde{K}^i_{jrh} \widetilde{K}^r_{kmn} - \widetilde{K}^i_{jkr} \widetilde{K}^r_{hmn}. \tag{2.5}$$

Differentiating it covariantly and using (2.1), we get

$$(V_{m;n} - V_{n;m})_{;l} \widetilde{K}^i_{jkh} = V_l (V_{m;n} - V_{n;m}) \widetilde{K}^i_{jkh}. \tag{2.6}$$

From the assumption $(\widetilde{K})$, we have the result.

Theorem 2.2 *In a* FINSLER *space with the recurrent relative curvature tensor field having the property* $(\widetilde{K})$, *the relation*

$$V_l (V_{m;n} - V_{n;m}) + V_m (V_{n;l} - V_{l;n}) + V_n (V_{l;m} - V_{m;l}) = 0 \tag{2.7}$$

is true.

Proof. Adding the expressions obtained by a cyclic change of the indices l, m and n in (2.2), we obtain

$$\begin{aligned} V_l (V_{m;n} - V_{n;m}) + V_m (V_{n;l} - V_{l;n}) + V_n (V_{l;m} - V_{m;l}) \\ = (V_{m;n} - V_{n;m})_{;l} + (V_{n;l} - V_{l;n})_{;m} + (V_{l;m} - V_{m;l})_{;n}. \end{aligned} \tag{2.8}$$

By virtue of (1.5) and (1.7),we have the result.

3. Recurrent curvature tensor field in the sense of CARTAN. A FINSLER space in which there exists a non-zero vector V_m, such that the curvature tensor field K^i_{jkh} satisfies

$$K^i_{jkh/m} = V_m K^i_{jkh} \tag{3.1}$$

is called to be a FINSLER space with the recurrent curvature tensor field and V_m the recurrence vector field.

Contracting the indices i and h in (3.1), we have

$$K_{jk/m} = V_m K_{jk} \tag{3.2}$$

where

$$K_{jk} = K^i_{jki}\,. \tag{3.3}$$

Here we shall prove certain theorems on the recurrence vector field V_m. We shall denote hereafter by (K) the following property: the value of the tensor field K^i_{jkh} is not the zero tensor at each element $(x, \dot{x})$ of the space.

Theorem 3.1. *In a* FINSLER *space with the recurrent curvature tensor field the relations*

$$(\dot{x} \quad K^i_{rkh})\left(\frac{\partial \Gamma^{*r}_{jm}}{\partial \dot{x}^n} - \frac{\partial \Gamma^{*r}_{jn}}{\partial \dot{x}_m}\right) = 0 \tag{3.4}$$

and

$$(\dot{x}^j K^i_{rk})\left(\frac{\partial \Gamma^{*r}_{jm}}{\partial \dot{x}^n} - \frac{\partial \Gamma^{*r}_{jn}}{\partial \dot{x}^m}\right) = 0 \tag{3.5}$$

hold good.

Proof. Multiplying (3.1) by $\dot{x}^j$ and differentiating it with respect to $\dot{x}^n$, we have

$$(\dot{x}^j K^i_{jkh})\frac{\partial V_m}{\partial \dot{x}^n} = \left[\frac{\partial}{\partial \dot{x}^n}\{(\dot{x}^j K^i_{jkh})_{/m}\} - V_m \frac{\partial}{\partial \dot{x}^n}(\dot{x}^j K^i_{jkh})\right]. \tag{3.6}$$

By virtue of (3.1), it yields

$$(\dot{x}^j K^i_{jkh})\frac{\partial V_m}{\partial \dot{x}^n} = \left[\frac{\partial}{\partial \dot{x}^n}\{(\dot{x}^j K^i_{jkh})_{/m}\} - \left\{\frac{\partial}{\partial \dot{x}^n}(\dot{x}^j K^i_{jkh})\right\}_{/m}\right] + \dot{x}^j\left(\frac{\partial}{\partial \dot{x}^n} K^i_{jkh}\right)_{/m} - \dot{x}^j V_m \frac{\partial}{\partial \dot{x}^n} K^i_{jkh}\,. \tag{3.7}$$

Using (1.15) in (3.7), we get

$$-\frac{\partial}{\partial \dot{x}^l}(\dot{x}^j K^i_{jkh})\, C^l_{mn/r}\, \dot{x}^r + \dot{x}^j \frac{\partial}{\partial \dot{x}^l} K^i_{jkh}\, C^l_{mn/r}\, \dot{x}^r + \dot{x}^j K^i_{rkh} \frac{\partial \Gamma^{*r}_{jm}}{\partial \dot{x}^n} = 0 \tag{3.8}$$

Interchanging the indices m and n in (3.8) and subtracting the result from it, we obtain (3.4). Contracting (3.4) with respect to the indices i and h, we get (3.5) Thus Theorem 3.1 is proved.

Theorem 3.2. *In a* FINSLER *space with the recurrent curvature tensor field having the property* (K), *if* K^i_{jkh} *is independent of* $\dot{x}^i$, *the relation*

$$(V_{m/n} - V_{n/m})_{/l} = V_l (V_{m/n} - V_{n/m}) \tag{3.9}$$

is true.

Proof. Differentiating (3.1) covariantly with respect to the index n, we have

$$K^i_{jkh/mn} = V_{m/n} K^i_{jkh} + V_m V_n K^i_{jkh}, \tag{3.10}$$

which yields

$$K^i_{jkh/mn} - K^i_{jkh/nm} = (V_{m/n} - V_{n/m}) K^i_{jkh}. \tag{3.11}$$

From (1.14) and (3.11), we get

$$(V_{m/n} - V_{n/m}) K^i_{jkh} = K^r_{jkh} K^i_{rmn} - K^i_{rkh} K^r_{jmn} - K^i_{jrh} K^r_{kmn} - K^i_{jkr} K^r_{hmn}. \tag{3.12}$$

Differentiating (3.12) covariantly and using (3.1), we obtain,

$$(V_{m/n} - V_{n/m})_{/l} K^i_{jkh} = V_l (V_{m/n} - V_{n/m}) K^i_{jkh}. \tag{3.13}$$

From the assumption (K), we have the result.

Theorem 3.3 *In a* FINSLER *space with the recurrent curvature tensor field having the property* (K), *if* K^i_{jkh} *and* V_m *are independent of* $\dot{x}^i$, *then* V_m *satisfies the relation*

$$V_l (V_{m/n} - V_{n/m}) + V_m (V_{n/l} - V_{l/m}) + V_n (V_{l/m} - V_{m/l}) = 0. \tag{3.14}$$

Proof. Adding the expressions obtained by a cyclic change of the indices l, m and n in (3.9), we get

$$\begin{aligned} V_l (V_{m/n} - V_{n/m}) + V_m (V_{n/l} - V_{l/n}) + V_n (V_{l/m} - V_{m/l}) \\ = (V_{m/n} - V_{n/m})_{/l} + (V_{n/l} - V_{l/n})_{/m} + (V_{l/m} - V_{m/l})_{/n}. \end{aligned} \tag{3.15}$$

By virtue of (1.14) and (1.11), (3.15) yields the result.

Theorem 3.4. *In a* FINSLER *space with the recurrent curvature tensor field, if the tensor* K_{jk} *is independent of* $\dot{x}^i$ *and is non-null,* V_m *satisfies the relation*

$$\frac{\partial V_m}{\partial \dot{x}^l} = \frac{\partial V_l}{\partial \dot{x}^m}. \tag{3.16}$$

Proof. Differentiating (3.2), multiplied by $\dot{x}^j$, with respect to $\dot{x}^l$, we obtain

$$\frac{\partial}{\partial \dot{x}^l} \{(\dot{x}^j K_{jk})_{/m}\} = \frac{\partial V_m}{\partial \dot{x}^l} (\dot{x}^j K_{jk}) + V_m \frac{\partial}{\partial \dot{x}^l} (\dot{x}^j K_{jk}). \tag{3.17}$$

Using the commutation formula (1.15) in (3.17), we get

$$(3.18)\qquad \left\{\frac{\partial}{\partial \dot{x}^l}(\dot{x}^j K)\right\}_{/m} - \frac{\partial}{\partial \dot{x}^p}(\dot{x}^j K_{jk})\, C^p_{lm/r}\, \dot{x}^r - (\dot{x}^j K_{jr})\frac{\partial \Gamma^{*r}_{km}}{\partial \dot{x}^l}$$
$$= \frac{\partial V_m}{\partial \dot{x}^l}(\dot{x}^j K_{jk}) + V_m \frac{\partial}{\partial \dot{x}^l}(\dot{x}^j K_{jk}),$$

which yields

$$(3.19)\qquad -\frac{\partial}{\partial \dot{x}^p}(\dot{x}^j K_{jk})\, C^p_{lm/r}\, \dot{x}^r (\dot{x}^j K_{jr})\frac{\partial \Gamma^{*r}_{km}}{\partial \dot{x}^l} = \frac{\partial V_m}{\partial \dot{x}^l}(\dot{x}^j K_{jk}).$$

Subtracting the result, obtained by interchanging the indices l an m in (3.19), from it and using (3.5), we have

$$(3.20)\qquad (\dot{x}^j K_{jk})\left(\frac{\partial V_m}{\partial \dot{x}^l} - \frac{\partial V_l}{\partial \dot{x}^m}\right) = 0$$

On the assumption, $K_{jk} \neq 0$, (3.20) yields the result.

Theorem 3.5. *In an affinely connected* FINSLER *space with the recurrent curvature tensor field, if K_{jk} is independent of $\dot{x}^i$ and is non-null, then the recurrence vector field V_m is independent of $\dot{x}^i$.*

Prof. It is obvious from (3.19).

We shall prove some theorems on the recurrent curvature tensor fieid. We shall denote here by (V) the following property : the value of the recurrence vector field V_m is not the zero vector at each element $(x,\dot{x})$ of the space.

Theorem 3.6. *In a affinely connected* FINSLER *space with the recurrent curvature tensor field having the properties (K) and (V), there exists a tensor field whose components S^i_j are positively homogeneous functions of degree one in $\dot{x}^i$ such that the tensor field K^i_{jkh} can be expressed as*

$$(3.21)\qquad K^i_{jkh} = V_h\, S^i_{jk} - V_k\, S^i_{jh}, \qquad S^i_{jk} = \frac{\partial S^i_k}{\partial \dot{x}_j}$$

and

$$(3.22)\qquad (\dot{x}^j K^i_{jkh}) = V_h\, S^i_k - V_k\, S^i_h.$$

Proof. From the expressions (1.12) and (3.1), wc obtain

$$(3.23)\qquad V_l\, K^i_{jkh} + V_k\, K^i_{jhl} + V_h\, K^i_{jhl} = 0.$$

By virtue of Theorem 3.5 the component V_m of the recurrence vector field do not contain the variables $\dot{x}^i$ and therefore from the assumption (V) concerning V_m, there exists a contravariant vector field whose components X^i do not contain the variables $\dot{x}^i$, such that $X_m V^m = 1$.

Transvecting (3.23) by X^l, we have

$$K^i_{jkh} = V_h\, K^i_{jkl}\, X^l - V_k\, K^i_{jhl}\, X^l,$$

that is

(3.24) $$K^i_{jkh} = V_h\, S^i_{jk} - V_k\, S^i_{jh},$$

where $$S^i_{jk} = \frac{\partial S^i_k}{\partial \dot{x}^j} = K^i_{jkl}\, X^l.$$

Multiplying (3.24) by $\dot{x}^j$, we get (3.22). Thus Theorem 3.6 is established.

Theorem 3.7. *In a* FINSLER *space with the recurrent curvature tensor field having the properties* (*K*) *and* (*V*), *in order that the tensor field* $\bar{S}^i_k$ *satisfy the expression*

$$(\dot{x}^j\, K^i_{jkh}) = \bar{S}^i_k\, V_h - \bar{S}^i_h\, V_k$$

and the component are positively homogeneous functions of degree one in $\dot{x}^i$, *it is necessary and sufficient that there exist a vector field whose components* p^i *are positively homogeneous of degree one in* $\dot{x}^i$ *and such that*

(3.25) $$\bar{S}^i_k = S^i_k + V_k\, p^i$$

holds, where S^i_k *is a tensor field satisfying* (3.22).

Proof. Les us assume that $\bar{S}^i_k$ be any given tensor field whose components are positively homogeneous functions of degree one in $\dot{x}^i$ such that

(3.26) $$(\dot{x}^j\, K^i_{jkh}) = V_h\, \bar{S}^i_k - V_k\, \bar{S}^i_h$$

holds. The expressions (3.22) and (3.26) yield

(3.27) $$V_h\, (\bar{S}^i_k - S^i_k) = V_k\, (\bar{S}^i_h - S^i_h).$$

Using the assumptions (*K*) and (*V*) in (3.27), we have a vector field whose components p^i are positively homogeneous functions of degree one in $\dot{x}^i$ and that satisfy the expression (3.25).

Conversely, if p^i is a vector field whose components p^i are positively homogeneous of degree one in $\dot{x}^i$ and satisfy (3.25), then it is evident that $\bar{S}^i_k$ satisfies our condition.

Theorem 3.8. *In a* FINSLER *space with the recurrent curvature tensor field, if* K^i_{jkh} *is a recurrent curvature tensor field, then* R^i_{jkh} *is not a recurrent curvature tensor field.*

Proof. The curvature tensor R^i_{jkh} can be expressed in terms of the curvature tensor K^i_{jkh} as

(3.28) $$R^i_{jkh} = K^i_{jkh} + C^i_{jm}\, K^m_{rkh}\, \dot{x}^r.$$

Differentiating it covariantly and using (3.1), we have

(3.29) $$R^i_{jkh/l} = V_l\, R^i_{jkh} + C^i_{jm/l}\, K^m_{rkh}\, \dot{x}^r,$$

which yields the result.

Remark. In an affinely connected FINSLER space R^i_{jkh} is also a recurrent curvature tensor field along with K^i_{jkh} and hence in this space R^i_{jkh} will satisfy all those theorems which we have proved for K^i_{jkh} in a FINSLER space with the recurrent curvature tensor field.

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ÖZET

Rekürent eğriliği hâiz uzaylar H. S. RUSE [2], A. G. WALKER [4], Y. C. WONG [5], B. B. SINHA ve S. P. SINGH [3] ve birçok başka yazar tarafından incelenmiştir. Bu araştırmanın gayesi rekürent vektör alanı, relatif egrilik tensör alanı ve CARTAN anlamında eğrilik tensör alanı ile ilgili bâzı sonuçları elde etmektir. Lokal özelikler üzerinde durulmakta ve H. RUND'un [1] notasyonları kullanılmaktadır.