ON PSEUDO-UNION CURVES IN A HYPERSURFACE OF A RIEMANNIAN SPACE

S. C. RASTOGI

The purpose of this paper is to define the pseudo-union curves on hypersurface of a Riemannian space. The differential equation of these curves and an expression for their curvature is obtained. Pseudo-union curves then studied in relation to pseudo-asymptotic and pseudo-geodesic curves.

1. **Introduction.** Pseudo-geodesic curve and pseudo-geodesic curvature have been defined by PAN [1][1]). The author [2] has defined and studied pseudo-asymptotic curves, pseudo-asymptotic curvature and totally pseudo-geodesic surfaces in a hypersurface of a Riemannian space. The purpose of the present paper is to define the pseudo-union curves in the hypersurface of a Riemannian space. The differential equation of pseudo-union curves and the expression for pseudo-union curvature is obtained. The pseudo-union curves are studied in relation to pseudo-asymptotic curves and pseudo-geodesic curves.

2. **Vector field in** V_n. Let x^i $(i = 1, \ldots, n)$ be the coordinates of a point P in the hypersurface V_n which is embedded in a Riemannian space V_{n+1}, whose coordinates are denoted by y^α $(\alpha = 1, \ldots, n+1)$[2]). For points in V_n the matrix $\| \partial y^\alpha / \partial x^i \|$ is of rank n. Let the metrics of V_n and V_{n+1}, which are supposed to be positive definite, be denoted by $g_{ij}\, dx^i\, dx^j$ and $a_{\alpha\beta}\, dy^\alpha\, dy^\beta$ respectively. The metric tensors of V_n and V_{n+1} are related as follows:

$$g_{ij} = a_{\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j}\,, \tag{2.1}$$

where $y^\alpha{}_{,i}$ are the covariant derivatives of the y^α with respect to the x^i.

Let N^α be the contravariant components of a unit vector orthogonal to $\vec{t}$ at the point P of the curve c in V_n ($\vec{t}$ being the unit tangent vector). Then

$$a_{\alpha\beta}\, N^\alpha N^\beta = 1, \tag{2.2}$$

and

$$a_{\alpha\beta}\, N^\alpha t^\beta = 0. \tag{2.3}$$

If a vector field in V_n has components U^α in the y's and components u^i in the x's, then

$$U^\alpha = y^\alpha{}_{,i}\, u_i\,. \tag{2.4}$$

1) Numbers in square brackets refer to references at the end.

2) Greek indices take the values $(1, \ldots, n+1)$ and Latin indices, $(1, \ldots, n)$.

If q^α and p^i represent the derived vectors of the unit tangent vector $\vec{t}$ of c with respect to V_{n+1} and V_n respectively, we have [1],

$$q^\alpha = y^\alpha{}_{,i}\, p^i + \left(\Omega_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}\right) \xi^\alpha, \tag{2.5}$$

where ξ^α are the contravariant components of the unit vector normal to V_n and where Ω_{ij} is the second fundamental tensor for V_n [3, 151].

Let λ^α be the contravariant components of a unit vector $\vec{\lambda}$, in V_{n+1}. The totality of these vectors $\vec{\lambda}$ associated with V_n is called a λ - congruence, which is a congruence of unit vectors, if λ^α are functions of x^i only, or a congruence of hypercones of unit vectors if λ^α are functions of both x^i and dx^i. We suppose that $\vec{\lambda}$ is in V_n if and only if $\vec{\lambda}$ and the corresponding dx^i and x^i are coincident with an asymptotic direction in V_n. Expressing λ^α as in [1] we have

$$\lambda^\alpha = y^\alpha{}_{,i}\, \omega^i + \omega\, \xi^\alpha, \tag{2.6}$$

where ω^i are the components of a contravariant vector in V_n and ω is a scalar.

Since

$$a_{\alpha\beta}\, \lambda^\alpha \lambda^\beta = 1, \tag{2.7}$$

from (2.5), (2.7) and (2.1) we have

$$g_{ij}\, \omega^i \omega^j + \omega^2 = 1. \tag{2.8}$$

With the help of (2.8) and the fact that the contravariant components of $\vec{t}$ in V_{n+1} are $y^\alpha{}_{,i}\, dx^i/ds$ we obtain

$$N^\alpha = \pm \frac{y^\alpha{}_{,i}\left\{-g_{hk}\, \omega^h \left(\frac{dx^k}{ds}\right)\left(\frac{dx^i}{ds}\right) + \omega^i\right\} + \omega\, \xi^\alpha}{\left\{1 - g_{ij} g_{hk}\, \omega^i \omega^h \frac{dx^j}{ds} \frac{dx^k}{ds}\right\}^{1/2}}. \tag{2.9}$$

The plus sign in (2.9) is to be taken when $\omega > 0$, and the minus sign when $\omega < 0$. Thus (2.9) will reduce to $N^\alpha = \xi^\alpha$, when $\vec{\lambda}$ is linearly dependent on $\vec{t}$ and ξ^α; that is, $\omega^i = k dx^i/ds$, k being any constant different from unity. Eliminating ξ^α from (2.5) and (2.9) we get

$$\begin{aligned} q^\alpha = {} & y^\alpha{}_{,i}\left(p^i - K_n \varrho^i + K_n g_{hk}\, \varrho^h \frac{dx^k}{ds} \frac{dx^i}{ds}\right) \\ & + N^\alpha K_n \left(1 - g_{ij} g_{hk}\, \omega^i \omega^h \frac{dx^j}{ds} \frac{dx^k}{ds}\right)^{1/2} \Big/ |\omega| \end{aligned} \tag{2.10}$$

where K_n is the normal curvature of c and where $\varrho^i = \omega^i/\omega$.

3. Pseudo-union curves. The totally pseudo-geodesic surface is determined by the tangent to the curve c and by the relative first curvature vector in V_{n+1} of the curve c. Let μ^α be the contravariant components in the y's of a unit vector in the direction of the curve of the congruence of curves, one curve of which passes through each point of V_n. The components μ^α, in general are not normal to V_n, and therefore may be specified by

$$\mu^\alpha = t^i y^\alpha{}_{,i} + r N^\alpha, \tag{3.1}$$

where t^i and r are parameters.

We have

$$a_{\alpha\beta}\, \mu^\alpha \mu^\beta = 1 ; \tag{3.2}$$

and

$$a_{\alpha\beta}\, y^\alpha{}_{;i} N^\beta = 0 . \tag{3.3}$$

With the help of equations (3.1), (3.2) and (3.3) it follows that

$$a_{\alpha\beta}\, \mu^\alpha \mu^\beta = a_{\alpha\beta} (t^i y^\alpha{}_{,i} + r N^\alpha)(t^j y^\beta{}_{,j} + r N^\beta)$$

$$1 = t^i t_i + r^2.$$

Hence we have

$$t^i t_i = 1 - r^2. \tag{3.4}$$

If the pseudo-geodesic in V_{n+1} in the direction of the curve of the congruence with contravariant components μ^α is to be a pseudo-geodesic of the totally pseudo-geodesic surface, then it is necessary that μ^α be a linear combination of $y^\alpha{}_{,i}\, dx^i/ds$ and q^α, therefore

$$\mu^\alpha = a\, y^\alpha{}_{,i} \frac{dx^i}{ds} + b\, q^\alpha. \tag{3.5}$$

From (3.1) and (3.5) we have

$$t^i y^\alpha{}_{;i} + r N^\alpha = a\, y^\alpha{}_{,i} \frac{dx^i}{ds} + b\, q^\alpha. \tag{3.6}$$

From (2.10) and (3.6) we obtain

$$t^i y^\alpha{}_{,i} + r N^\alpha = a\, y^\alpha{}_{i}, \frac{dx^i}{ds} + b\, (\bar{K}_n N^\alpha + y^\alpha{}_{,i}\, \bar{p}^i), \tag{3.7}$$

where

$$\bar{K}_n = K_n \left(1 - g_{ij} g_{hk}\, \omega^i \omega^h \frac{dx^j}{ds} \frac{dx^k}{ds} \right)^{1/2} \Big/ |\omega|, \tag{3.8}$$

and

$$\bar{p}^i = p^i - K_n\, \varrho^i + K_n\, g_{hk}\, \varrho^h \frac{dx^k}{ds} \frac{dx^i}{ds} . \tag{3.9}$$

Multiplying (3.7) by $a_{\alpha\beta}\, y^\beta{}_{,j}$ and summing with respect to α and using (2.1) and (3.3) we get

$$g_{ij}\, t^i = a\, g_{ij} \frac{dx^i}{ds} + b\, g_{ij}\, \bar{p}^i . \tag{3.10}$$

Multiplying (3.7) by $a_{\alpha\beta} N^\beta$, summing on α and using (2.2) and (3.3) we get

(3.11) $$r = b\,\bar{K}_n.$$

From equation (3.9) we obtain

(3.12) $$g_{ij}\,\bar{p}^i \frac{dx^j}{ds} = 0,$$

where we have used $g_{ji} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1$.

Multiplying equation (3.10) by dx^j/ds and using (3.12) we get

(3.13) $$a = g_{ij}\, t^i \frac{dx^j}{ds}.$$

Putting for a and b from (3.13) and (3.11) respectively in (3.10) we have

(3.14) $$g_{ij}\, t^i = g_{ij} \frac{dx^i}{ds}\left(g_{lm}\, t^l \frac{dx^m}{ds}\right) + \frac{r}{\bar{K}_n} g_{ij}\, \bar{p}^i.$$

Multiplication of (3.12) by g^{jk} and summation with respect to j and the replacement of t^k/r by l^k leads to

(3.15) $$\bar{p}^k - \bar{K}_n \left(l^k - g_{lm}\, l^l \frac{dx^m}{ds} \frac{dx^k}{ds}\right) = 0.$$

(3.15) represents the differential equation of the pseudo-union curves.

In the next section we shall discuss some properties of the pseudo-union curves.

4. Some properties. For a congruence specified by the parameters t^k, the solutions of the n equations (3.15) determine the pseudo-union curves in V_n relative to that congruence. The parameter r can not vanish under the assumption that the direction μ^a is not in V_n. We denote the left hand members of (3.15) by $\bar{\eta}^k$ and call it the contravariant components of the pseudo-union curvature vector.

A pseudo-union curve of V_n with respect to a congruence determined by the parameters l^k may therefore be defined as a curve along which the pseudo-union curvature vector is a null vector.

Equation (3.15) can be written in the form

(4.1) $$\bar{\eta}^k \equiv \bar{p}^k - \bar{K}_n\, \nu^k = 0,$$

where

(4.2) $$\nu^k = g_{ij} \frac{dx^i}{ds}\left(l^k \frac{dx^j}{ds} - l^j \frac{dx^k}{ds}\right).$$

For a pseudo-union curve $\bar{\eta}^k = 0$, and for a pseudo-asymptotic curve $\bar{K}_n = 0$, therefore from (4.1) if follows that $\bar{p}^k = 0$, *i.e.*, the curve is a pseudo-geodesic. Hence we have:

Theorem (4.1). *If the curve c has any two of the following properties it also has the third:*

(*i*) *it is a pseudo-union curve,*

(*ii*) *it is a pseudo-asymptotic curve,*

(*iii*) *it is a pseudo-geodesic curve,*

provided that ν^k are not the components of a null vector.

The magnitude $\bar{K}_u$ of the vector $\bar{\eta}^k$ is given by $\bar{K}_u^2 = g_{ij}\,\bar{\eta}^i\,\bar{\eta}^j$. From (3.1) it follows that angle between the vectors μ^α and N^α in V_{n+1} is given by $\cos \Phi = r$, and by virtue of the relation $t^k/r = l^k$ and the equation (3.4) we obtain $g_{ij}\, l^i\, l^j = \tan^2\Phi$. The angle α between the tangent vector to c and the vector l^k is given by $\cos \alpha = g_{ik}\, l^i \dfrac{dx^k}{ds}$. In terms of Φ and α the magnitue $\bar{K}_u$ of the pseudo-union curvature vector is given by

$$\bar{K}_u = \bar{K}_g - \bar{K}_n \tan \Phi \sin \alpha. \tag{4.3}$$

In (4.3) if $\Phi = 0$, or $\alpha = 0$, or $\bar{K}_n = 0$, we have $\bar{K}_u = \bar{K}_g$. Hence we have

Theorem (4.2). *The necessary and sufficient condition for a pseudo-union curve to be pseudo-geodesic is one of the following :*

(*i*) *it is a pseudo-asymptotic curve,*

(*ii*) *the congruence consists of the normals,*

(*iii*) *the direction of the tangent vector to c coincides with that of the vector l^k.* [1]

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DEPARTMENT OF MATHEMATICS,
LUCKNOW UNIVERSITY,
LUCKNOW, (INDIA).

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ÖZET

Bu yazının gâyesi bir RIEMANN uzayına ait bir hiperyüzeyin üzerine çizilmiş psödo-birleşik eğrileri tanımlamaktır. Bu eğrilerin diferansiyel denklemi ve eğriliklerinin bir ifadesi elde edildikten sonra, bu eğrilerin özellikleri, psödo-asemptotik ve psödo-geodezik eğrilerle bağlı olarak incelenmektedir

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