# CONGRUENCES OF CURVES IN A RIEMANNIAN SPACE — III

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Consider a set of m-n congruences of curves in a Rimannian space $V_m$, such that one curve of each congruence passes through each point of a given subspace of reference $V_n$. The object of this paper is to determine the equations of the *focal subspaces* of the congruence and those of the *orthogonal normal directions* at a point of a curve of the congruence.

**1. Introduction.** Congruences of curves in a Riemannian space $V_n$ of coordinates $x^i$ ($i = 1,2 \ldots n$) [1] imbedded in a Riemannian space $V_m$ of coordinates $Y^\alpha$ ($\alpha = 1,2 \ldots m$) have been studied by UPADHYAY [1],[2][2]) and other authors. In this paper following the notations of UPADHYAY [1],[2] we shall obtain the equations determining «*Focal subspaces*» and «*Orthogonal normal directions*» at a point of a curve of the congruence.

Let us consider a set of $m - n$ congruences of curves in $V_m$ ($m > n$) such that one curve of each congruence passes through each point of $V_n$. Let $s_{\tau|}$ be the length of a curve of the congruence-$\lambda_{\tau|}$ (say) measured from a point $P$ with coordinates $Y^\alpha$ at which the curve meets $V_n$ to another point on the curve. $V_n$ is known as the subspace of reference. The fundamental tensors of $V_n$ and $V_m$ are connected by the relation

$$g_{ij} = a_{\alpha\beta}\, Y^\alpha{}_{;i}\, Y^\beta{}_{;j} \tag{1.1}$$

where semi colon (;) followed by a Latin letter denotes tensor derivative with respect to $x$'s.

**2. Focal Subspace.** The infinitesimal distance between two adjacent curves of the congruence-$\lambda_{\tau|}$ from a point $y^\alpha_{\tau|}(x^1, x^2, \ldots, x^n, s_{\tau|})$ on the former curve to another point $y^\alpha_{\tau|}$ $(x^1 + dx^1, x^2 + dx^2, \ldots, x^n + dx^n, s_{\tau|} + ds_{\tau|})$ on the latter is of the second or higher order. Therefore, neglecting quantities of the second and higher order, we obtain

$$y^\alpha_{\tau|}(x^1, x^2, \ldots, x^n, s_{\tau|}) = y^\alpha_{\tau|}(x^1 + dx^1, x^2 + dx^2, \ldots, x^n + dx^n, s_\tau + ds_{\tau|})$$

i.e.

$$\frac{\partial y^\alpha_{\tau|}}{\partial x^i}\, dx^i + \frac{\partial y^\alpha_{\tau|}}{\partial s_{\tau|}}\, ds_{\tau|} = 0. \tag{2.1}$$

1) Throughout this paper Latin letters $i, j, k, \ldots$ take values from 1 to $n$, early letters $\alpha, \beta, \gamma, \delta, \ldots$ of the Greek alphabet take values from 1 to $m$ and later letters $\mu, \nu, \sigma, \tau \ldots$ take values from n+1 to $m$.

2) The numbers in square brackets indicate references which are given at the end of the paper.

Multiplying (2.1) by $a_{\alpha\beta}\left(\frac{\partial y^{\beta}_{\tau|}}{\partial s_{\tau|}}\right)_{;j} dx^j$ we get

$$(2.2) \qquad a_{\alpha\beta}\frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\left(\frac{\partial y^{\beta}_{\tau|}}{\partial s_{\tau|}}\right)_{;j} dx^i\, dx^j = 0 \cdot$$

From (2.2.) we have

$$(2.3) \qquad C_{\tau|ij}\, dx^i\, dx^j = 0$$

which is satisfied when

$$(2.4) \qquad Det \mid c_{\tau|ij} \mid = 0$$

where

$$(2.5) \qquad Det \mid c_{\tau|ij} \mid = a_{\alpha\beta}\frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\left(\frac{\partial y^{\beta}_{\tau|}}{\partial s_{\tau|}}\right)_{;j}$$

The value of $s_{\tau|}$ in (2.2) corresponds to the focal points on the curves of the congruence - $\lambda_{\tau|}$ by analogy with the focal points of a curvilinear congruence in an Euclidean space of three dimensions [3]. Hence (2.4) is satisfied at all points of a subspace, which we call the «*Focal Subspace*». Thus (2.4) determines such a subspace.

**3. Orthogonal normal directions.** From (2.1) the infinitesimal displacement $\delta y^{\alpha}_{\tau|}$ normal to a curve of the congruence - $\lambda_{\tau|}$ at the point $M(y^{\alpha}_{\tau|})$ is given by [1]

$$(3.1) \qquad \delta y^{\alpha}_{\mu|} = y^{\alpha}_{\tau|;i}\, \delta x^i + \frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\, \delta s_{\tau|}$$

But we have [1]

$$(3.2)_a \qquad ds_{\tau|} = -p_{\tau|}\, dx^i$$

where

$$(3.2)_b \qquad p_{\tau|} = a_{\alpha\beta}\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\, y^{\beta}_{\tau|;i} \cdot$$

By virtue of $(3.2)_a$ and $(3.2)_b$ the equation (3.1) takes the form

$$(3.3) \qquad \frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\, dx^i - p_{\tau|}\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\, dx^i \overset{def}{=} u^{\alpha}_{\tau|}\,,$$

so that $u^{\alpha}_{\tau|}$ are the components of an infinitesimal vector normal to the curve of the congruence - $\lambda_{\tau|}$.

Let $u^{\alpha}_{\tau|}$ and $u^{\beta}_{\mu|}$ be the components of two such normal directions, $\tau$ and $\mu$ having fixed values. If these directions are orthogonal, we have

$$(3.4) \qquad a_{\alpha\beta}\, u^{\alpha}_{\tau|}\, u^{\beta}_{\mu|} = 0 \cdot$$

With the help of (3.3), (3.4) can be written as

$$a_{\alpha\beta}\left[\frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\frac{\partial y^{\beta}_{\mu|}}{\partial x^j} - p_{\tau|}\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\frac{\partial y^{\beta}_{\mu|}}{\partial x^j} - p_{\mu|}\frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\frac{\partial y^{\beta}_{\mu|}}{\partial s_{\mu|}}\right.$$

$$\left. + p_{\tau|}p_{\mu|}\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\frac{\partial y^{\beta}_{\mu|}}{\partial s_{\mu|}}\right] dx^i\, dx^j = 0. \tag{3.5}$$

At the point M, the normal directions which are orthogonal are determined by the ratio $dx^i : dx^j$. The determinant of (3.5) is

(3.6) *a*

$$Det\ |\ A_{\tau\mu|ij}\ | = 0$$

where

$$A_{\tau\mu|ij} \stackrel{def}{=} a_{\alpha\beta}\left[\frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\frac{\partial y^{\beta}_{\mu|}}{\partial x^j} - p_{\tau|}\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\frac{\partial y^{\beta}_{\tau|}}{\partial x^j}\right.$$

(3.6) *b*

$$\left. - p_{\mu|}\frac{\partial y^{\alpha}_{\tau|}}{\partial x^i}\frac{\partial y^{\beta}_{\tau|}}{\partial s_{\mu|}} + p_{\tau|}\,p_{\mu|}\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\frac{\partial y^{\beta}_{\tau|}}{\partial s_{\mu|}}\right].$$

In particular, for a hypersurface, $m = n + 1$, and the $m - n$ congruences will reduce to a single congruence, Therefore $s_{\tau|}$ will have only one value, say, $s$. Then the equation (3.5) reduces to

$$ds\,(\Omega - 1) = 0 \tag{3.7}$$

where we have used (1.1) and the definition

$$p_{\tau|}\frac{\partial x^i}{\partial s_{\tau|}} \stackrel{def}{=} \Omega. \tag{3.8}$$

## REFERENCES

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(*Manuscript received February 18, 1969*)

## ÖZET

Bir $V_m$ Riemann uzayında, $V_n$ ile gösterilen bir karşılaştırma altuzaynın her bir noktasından sadece birer eğrisi geçecek şekilde çizilen m-n eğrisel kongrüanslarından meydana gelen bir sistem gözönüne alınıyor. Bu araştırmanın gâyesi, bu kongrüans sisteminin *odak altuzay*'larının denklemleriyle kongrüanslara ait bir eğrinin bir noktasındaki *ortogonal normal doğrultu*'larının denklemlemlerini bulmaktadır.