FOURIER SERIES FOR FOX'S H-FUNCTIONS [1]

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The object of the present paper is to establish two FOURIER series expansions for Fox's H-functions.

1. The object of the present paper is to establish the following two FOURIER series expansions for Fox's H-functions:

$$\sum_{r=0}^{\infty} H^{m+1,\,n+1}_{p+2,\,q+2}\left[Z \left| \begin{matrix} (1-r,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p),(2+r,1) \\ \left(\frac{3}{2},1\right),(b_1,\beta_1),\ldots,(b_q,\beta_q),(1,1) \end{matrix} \right. \right] \sin(2r+1)\theta$$

(1.1)

$$= \frac{\sqrt{\pi}}{2}\sin\theta \, . \, H^{m,n}_{p,q}\left[\frac{Z}{\sin^2\theta} \left| \begin{matrix} (a_1,\alpha_1),\ldots,(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots,(b_q,\beta_q) \end{matrix} \right. \right],$$

where $0 \leq \theta \leq \pi, \sum_1^p \alpha_j - \sum_1^q \beta_j \leq 0, \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$

and $|\arg Z| < \frac{1}{2}\lambda\pi$.

$$H^{m+1,\,n}_{p+1,\,q+1}\left[Z \left| \begin{matrix} (a_1,\alpha_1),\ldots,(a_p,\alpha_p),(1,1) \\ \left(\frac{1}{2},1\right),(b_1,\beta_1),\ldots,(b_q,\beta_q) \end{matrix} \right. \right] +$$

(1.2)

$$+ 2\sum_{r=0}^{\infty} H^{m+1,\,n+1}_{p+2,\,q+2}\left[Z \left| \begin{matrix} (1-r,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p),(1+r,1) \\ \left(\frac{1}{2},1\right),(b_1,\beta_1),\ldots,(b_q,\beta_q),(1,1) \end{matrix} \right. \right] \operatorname{Cos} r\theta$$

$$= \sqrt{\pi}\; H^{m,\,n}_{p,\,q}\left[\frac{Z}{\sin^2\frac{\theta}{2}} \left| \begin{matrix} (a_1,\alpha_1),\ldots,(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots,(b_q,\beta_q) \end{matrix} \right. \right],$$

where $0 < \theta \leq \pi, \sum_1^p \alpha_j - \sum_1^q \beta_j \leq 0, \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$

and $|\arg Z| < \frac{1}{2}\lambda\pi$.

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The H-function introduced by Fox [1, 408] will be represented and defined in the following manner :

$$(1.3)\qquad H^{m,n}_{p,q}\left[Z\left|\begin{matrix}(a_1\,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}z^s ds,$$

where z is not equal to zero and an empty product is interpreted as having the value unity ; m, n, p and q are integers satisfying $1\leq m\leq q$, $0\leq n\leq p$; $\alpha_j(j=1,\ldots,p)$, $\beta_j(j=1,\ldots q)$ are positive numbers and $a_j\,(j=1,\ldots,p)$, $b_j\,(j=1,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_j-\beta_j s)$, $j=1,\ldots,m$ coincides with any pole of $\Gamma(1-a_j+\alpha_j s)$, $j=1,\ldots,n$. The poles of the integrand must be simple and those of $\Gamma(b_j-\beta_j s)$, $j=1,\ldots,m$ lie on one side of the contour L and those of $\Gamma(1-a_j+\alpha_j s)$, $j=1,\ldots,n$ must lie on the other side.

To prove (1.1) and (1.2) whose conditions of validity are given in section 2, we require the following FOURIER series established by MACROBERT [2, 79] and [3, 143].

$$(1.4)\qquad \frac{\sqrt{\pi}\,\Gamma(2-s)}{2\,\Gamma\left(\frac{3}{2}-s\right)}(\text{Sin}\,\theta)^{1-2s}=\sum_{r=0}^{\infty}\frac{(s\,;r)}{(2-s\,;r)}\sin(2r+1)\theta,$$

where $0\leq\theta\leq\pi$, $Re\,s\leq\frac{1}{2}$. Here $(s\,;0)=1$, $(s\,;r)=s(s+1)\ldots(s+r-1)$, $r=1,2,3,\ldots$

$$(1.5)\qquad \frac{\sqrt{\pi}\,\Gamma(1-s)}{\Gamma\left(\frac{1}{2}-s\right)}\left(\sin\frac{\theta}{2}\right)^{-2s}=1+2\sum_{r=0}^{\infty}\frac{(s\,;r)}{(1-s\,;r)}\cos r\theta,$$

where $0<\theta\leq\pi$,

Using (1.3) , the expression on the left side of (1.1) can be written as

$$\sum_{r=0}^{\infty}\frac{1}{2\pi i}\int_L\frac{\Gamma\left(\frac{3}{2}-s\right)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)\,\Gamma(r+s)}{\Gamma(s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\,\Gamma(2+r-s)}Z^s ds.\ \sin(2r+1)\theta.$$

Here the contour L runs from $\sigma-i\infty$ to $\sigma+i\infty$. The conditions

$$0<\sigma<\frac{3}{2}$$

$$Re\,b_j>\sigma\,\beta_j\qquad,\ j=1,\ldots,m$$

$$Re\,a_j<\sigma\,\alpha_j+1\quad,\ j=1,\ldots,n$$

ensure that all the poles of $\Gamma\left(\frac{3}{2}-s\right)$ and $\Gamma(b_j-\beta_j s)$, $j=1,\ldots,m$ lie to the right of L and those of $\Gamma(r+s)$ and $\Gamma(1-a_j+\alpha_j s)$, $j=1,\ldots,n$ lie to the left of L, as required for the definition of the H-function on the left side of (1.1). The integral converges if $\sum_1^p\alpha_j-\sum_1^q\beta_j\leq 0$,

$$\sum_{1}^{n}\alpha_j-\sum_{n+1}^{p}\alpha_j+\sum_{1}^{m}\beta_j-\sum_{m+1}^{q}\beta_j\equiv\lambda>0 \text{ and } |\arg Z|<\frac{1}{2}\lambda\pi .$$

On changing the order of integration and summation which is easily seen to be justified, the above expression becomes

$$\frac{1}{2\pi i}\int_L \frac{\Gamma\left(\frac{3}{2}-s\right)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\Gamma(2-s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\left[\sum_{r=0}^{\infty}\frac{(s\,;r)}{(2-s\,;r)}\sin(2r+1)\theta\right]Z^s ds$$

and on using the relation (1.4), it takes the form

$$\frac{\sqrt{\pi}}{2}\sin\theta\,.\,\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j)s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\left(\frac{Z}{\sin^2\theta}\right)^s ds$$

which is just the expression on the right side of (1.1). (1.1) is the FOURIER sine series for the *H*-functions.

The FOURIER cosine series (1.2) is proved in an analogous manner by using (1.3) and (1.5). The conditions of validity of (1.2) are

$$0<\sigma<\frac{1}{2}$$

$$Re\, b_j>\sigma\,\beta_j \quad , \; j=1,\ldots,m$$

$$Re\, a_j<\sigma\,\alpha_j+1\,, \quad j=1,\ldots,n\,.$$

2. As the *H*-function is a very general function, we get, on specializing the parameters, many cases, some of which are known and others are believed to be new. Some interesting cases are given below :

(i) Using the relation [4, 199]

$$H^{m,n}_{p,q}\left[Z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\equiv Z^{-\mu}\,H^{m,n}_{p,q}\left[Z\left|\begin{matrix}(a_1+\mu\alpha_1,\alpha_1),\ldots,(a_p+\mu\alpha_p,\alpha_p)\\(b_1+\mu\beta_1,\beta_1),\ldots,(b_q+\mu\beta_q,\beta_q)\end{matrix}\right.\right]$$

in (1.1) and (1.2), we get

$$\sum_{r=0}^{\infty}H^{m+1,n+1}_{p+2,q+2}\left[Z\left|\begin{matrix}(1-r+\mu,1),(a_1+\mu\alpha_1,\alpha_1),\ldots,(a_p+\mu\alpha_p,\alpha_p),(2+r+\mu,1)\\\left(\frac{3}{2}+\mu,1\right),(b_1+\mu\beta_1,\beta_1),\ldots,(b_q+\mu\beta_q,\beta_q),1+\mu,1)\end{matrix}\right.\right]\sin(2r+1)\theta$$

(2.1)

$$=\frac{\sqrt{\pi}}{2}(\sin\theta)^{1+2\mu}\,H^{m,n}_{p,q}\left[\frac{Z}{\sin^2\theta}\left|\begin{matrix}(a_1+\mu\alpha_1,\alpha_1),\ldots,(a_p+\mu\alpha_p,\alpha_p)\\(b_1+\mu\beta_1,\beta_1),\ldots,(b_q+\mu\beta_q,\beta_q)\end{matrix}\right.\right]$$

and

$$H^{m+1,\,n}_{p+1,\,q+1}\left[Z\left|\begin{matrix}(a_1+\mu\alpha_1,\alpha_1),\ldots,(a_p+\mu\alpha_p,\alpha_p),(1+\mu,1)\\(\tfrac{1}{2}+\mu,1),(b_1+\mu\beta_1,\beta_1),\ldots,(b_q+\mu\beta_q,\beta_q)\end{matrix}\right.\right]+$$

(2.2)
$$+\,2\sum_{r=0}^{\infty}H^{m+1,\,n+1}_{p+2,\,q+2}\left[Z\left|\begin{matrix}(1-r+\mu,1),(a_1+\mu\alpha_1,\alpha_1),\ldots,(a_p+\mu\alpha_p,\alpha_p),(1+r+\mu,1)\\(\tfrac{1}{2}+\mu,1),(b_1+\mu\beta_1,\beta_1),\ldots,(b_q+\mu\beta_q,\beta_q),(1+\mu,1)\end{matrix}\right.\right]\cos r\theta$$

$$=\sqrt{\pi}\left(\sin\frac{\theta}{2}\right)^{2\mu}H^{m,\,n}_{p,\,q}\left[\frac{Z}{\sin^2\frac{\theta}{2}}\left|\begin{matrix}(a_1+\mu\alpha_1,\alpha_1),\ldots,(a_p+\mu\alpha_p,\alpha_p)\\(b_1+\mu\beta_1,\beta_1),\ldots,(b_q+\mu\beta_q,\beta_q)\end{matrix}\right.\right].$$

(ii) Using another relation [4, 199]

$$H^{m,\,n}_{p,\,q}\left[Z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\equiv H^{n,\,m}_{q,\,p}\left[\frac{1}{Z}\left|\begin{matrix}(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p)\end{matrix}\right.\right]$$

in (1.i) and (1.2), we get

(2.3)
$$\sum_{r=0}^{\infty}H^{n+1,\,m+1}_{q+2,\,p+2}\left[\frac{1}{Z}\left|\begin{matrix}(-\tfrac{1}{2},1),(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q),(0,1)\\(r,1),(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p),(-1-r,1)\end{matrix}\right.\right]\sin(2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sin\theta\,.\,H^{n,\,m}_{q,\,p}\left[\frac{\sin^2\theta}{Z}\left|\begin{matrix}(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p)\end{matrix}\right.\right]$$

and

$$H^{n,\,m+1}_{q+1,\,p+1}\left[\frac{1}{Z}\left|\begin{matrix}(\tfrac{1}{2},1),(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p),(0,1)\end{matrix}\right.\right]+$$

(2.4)
$$+\,2\sum_{r=0}^{\infty}H^{n+1,\,m+1}_{q+2,\,p+2}\left[\frac{1}{Z}\left|\begin{matrix}(\tfrac{1}{2},1),(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q),(0,1)\\(r,1),(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p),(-r,1)\end{matrix}\right.\right]\cos\gamma\theta$$

$$=\sqrt{\pi}\,H^{n,\,m}_{q,\,p}\left[\frac{\sin^2\frac{\theta}{2}}{Z}\left|\begin{matrix}(-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p)\end{matrix}\right.\right].$$

(iii) Again, using the relation [4, 199]

$$H^{m,n}_{p,\,q}\left[Z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\equiv c\,H^{m,\,n}_{p,\,q}\left[Z^c\left|\begin{matrix}(a_1,c\,\alpha_1),\ldots,(a_p,c\,\alpha_p)\\(b_1,c\,\beta_1),\ldots,(b_q,c\,\beta_q)\end{matrix}\right.\right],\ c>0$$

in (1 . 1) and (1.2) , we get

(2.5)
$$\sum_{r=0}^{\infty}H^{m+1,\,n+1}_{p+2,\,q+2}\left[Z^c\left|\begin{matrix}(1-r,c)\,(a_1,c\,\alpha_1),\ldots,(a_p,c\,\alpha_p),(2+r,c)\\(\tfrac{3}{2},c),(b_1,c\,\beta),\ldots,(b_q,c\,\beta_q),(1,c)\end{matrix}\right.\right]\sin(2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sin\theta\,.\,H^{m,\,n}_{p,\,q}\left[\frac{Z^c}{(\sin\theta)^{2c}}\left|\begin{matrix}(a_1,c\,\alpha_1),\ldots(a_p,c\,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,c\,\beta_q)\end{matrix}\right.\right]$$

and

$$H^{m+1,\,n}_{p+1,\,q+1}\left[Z^c\left|\begin{matrix}(a_1, c\,\alpha_1),\ldots,(a_p, c\,\alpha_p), (1, c)\\ \left(\frac{1}{2}, c\right), (b_1, c\,\beta_1),\ldots,(b_q, c\,\beta_q)\end{matrix}\right.\right] +$$

(2.6)
$$+\,2\sum_{r=0}^{\infty} H^{m+1,\,n+1}_{p+2,\,q+2}\left[Z^c\left|\begin{matrix}(1-r, c), (a_1, c\,\alpha_1),\ldots,(a_p, c\,\alpha_p), (1+r, c)\\ \left(\frac{1}{2}, c\right), (b_1, c\,\beta_1),\ldots,(b_q, c\,\beta_q), (1, c)\end{matrix}\right.\right]$$

$$= \sqrt{\pi}\, H^{m,\,n}_{p,\,q}\left[\frac{Z^c}{\left(\sin\frac{\theta}{2}\right)^{2c}}\left|\begin{matrix}(a_1, c\,\alpha_1),\ldots,(a_p, c\,\alpha_p)\\ (b_1, c\,\beta_1),\ldots,(b_q, c\,\beta_q)\end{matrix}\right.\right].$$

If we make use of the relation [1,199).

$$H^{m,\,n}_{p,\,q}\left[Z\left|\begin{matrix}(a_1, 1),\ldots,(a_p, 1)\\ (b_1, 1),\ldots,(b_q, 1)\end{matrix}\right.\right] \equiv G^{m,\,n}_{p,\,q}\left[Z\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right],$$

the formulae (1.1) and (1.2) reduce to the FOURIER series for MEI-JER'S G-functions obtained by KESARWANI [5, 149].

Further, on using the relation (4, 199)

$$H^{p,\,1}_{q+1,\,p}\left[Z\left|\begin{matrix}(1, 1), (b_1, 1),\ldots,(b_q, 1)\\ (a_1, 1),\ldots,(a_p, 1)\end{matrix}\right.\right] \equiv E(a_1,\ldots,a_p\,;\, b_1,\ldots,b_q\,;\, z),$$

where $E(.)$ denotes MACROBERT'S E-function [6, 203], the formulae (1.1) and (1.2) reduce to the FOURIER series for E-functions obtained by MACROBERT [2, 79, equs. (1) and (2)] .

3. From (1.1) and (1.2) , we easily deduce the integrals

$$\int_0^{\pi} \sin(2r+1)\,\theta\,\sin\theta\,.\, H^{m,\,n}_{p,\,q}\left[\frac{Z}{\sin^2\theta}\left|\begin{matrix}(a_1, \alpha_1),\ldots,(a_p, \alpha_p)\\ (b_1, \beta_1),\ldots,(b_q, \beta_q)\end{matrix}\right.\right] d\theta$$

(3.1)
$$= \sqrt{\pi}\, H^{m+1,\,n+1}_{p+2,\,q+2}\left[Z\left|\begin{matrix}(1-r, 1), (a_1, \alpha_1),\ldots,(a_p, \alpha_p), (2+r, 1)\\ \left(\frac{3}{2}, 1\right), (b_1, \beta_1),\ldots,(b_q, \beta_q), (1, 1)\end{matrix}\right.\right]$$

and

$$\int_0^{\pi} \cos r\,\theta\,.\, H^{m,\,n}_{p,\,q}\left[\frac{Z}{\sin^2\frac{\theta}{2}}\left|\begin{matrix}(a_1, \alpha_1),\ldots,(a_p, \alpha_p)\\ (b_1, \beta_1),\ldots,(b_q, \beta_q)\end{matrix}\right.\right] d\theta$$

(3.2)
$$= \sqrt{\pi}\, H^{m+1,\,n+1}_{p+2,\,r+2}\left[Z\left|\begin{matrix}(1-r, 1), (a_1, \alpha_1),\ldots,(a_p, \alpha_p), (1+r, 1)\\ \left(\frac{1}{2}, 1\right), (b_1, \beta_1),\ldots,(b_q, \beta_q), (1, 1)\end{matrix}\right.\right],$$

where $\sum_1^p \alpha_j - \sum_1^q \beta_j \leq 0, \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,$

$|\arg z| < \frac{1}{2}\lambda\pi$ and $r = 0, 1, 2, \ldots$.

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ÖZET

Bu araştırmanın gayesi FOX tarafından [1] tanımlanan *H*-fonksiyonları için iki FOURIER serisi açılımını bulmaktır. Elde edilen ifadeler (1.1) ve (1.2) de gösterilmiştir.