ON EXCEPTIONAL VALUES OF ENTIRE AND MEROMORPHIC FUNCTIONS[1]

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In connection with the NEVANLINNA theory for entire and meromorphic functions, some special values are defined by limiting operations and called *exceptional values*. Some properties of these values are obtained.

Let $F(z)$ be a meromorphic function and $T(r,F)$ be it's NEVANLINNA characteristic function. Let $N(r,a) = N(r,F-a)$, $N(r,F) = N(r,\infty)$ have their usual meaning as in the NEVANLINNA theory.

Define

$$\delta(a) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,F)},$$

$$\Delta(a) = 1 - \liminf_{r\to\infty} \frac{N(r,a)}{T(r,F)}.$$

If

$\delta(a) > 0$ then a is an exceptional value "N" (or *e. v. N*);

$\Delta(a) > 0$ then a is an exceptional value "V" (or *e. v. V*).

We define the proximate order $\varrho(r)$ and lower proximate order $\lambda(r)$ with respect to $T(r,F)$ by the following conditions:

(*i*) $\varrho(r)$ and $\lambda(r)$ are differentiable for $r \geq r_0$ except at isolated points at which $\varrho'(r-0)$, $\varrho'(r+0)$ and $\lambda'(r-0)$, $\lambda'(r+0)$ exist.

(*ii*) $\varrho(r) \to \varrho$, $\lambda(r) \to \lambda$ as $r \to \infty$.

(*iii*) $r\varrho'(r)\log r \to 0$ and $r\lambda'(r)\log r \to 0$ as $r \to \infty$.

(*iv*) $r^{\lambda(r)} T \leq (r) \leq r^{\varrho(r)}$ for $r \geq r_0$.

$T(r) = r^{\varrho(r)}$ for a sequence of values of $r \to \infty$.

$T(r) = r^{\lambda(r)}$ for a sequence of values of $r \to \infty$.

For the existence of such proximate orders see [1].

Now, we define

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a as an exceptional value "A" (or $e.v.A.$) if $\lim\limits_{r\to\infty} \inf \dfrac{n(r,a)}{r^{\lambda(r)}} = 0.$

and

a as an exceptional value "D" (or $e.v.D.$) if $\lim\limits_{r\to\infty} \inf \dfrac{n(r,a)}{r^{\varrho(r)}} = 0.$

We derive results concerning the $e.v.A$, $e.v.D$ and the other exceptional values.

Theorem 1

Let $F(z)$ be a meromorphic function of finite order ϱ.

(i) If a, b are $e.v.E.$, then they are $e.v.A.$, also.

(ii) If a, b are $e.v.N.$ with total defects, then they are $e.v.A.$ also.

(iii) $\lambda_1(a) < \lambda \Rightarrow a$ is $e.v.A.$

(iv) $\Delta(a) = 1 \Rightarrow a$ is $e.v.D.$

where

a is an axceptional value "E" (or $e.v.E.$) if $\lim\limits_{r\to\infty} \inf \dfrac{T(r,F)}{n(r,a)\cdot\Phi(r)} > 0$, where $\displaystyle\int_A^\infty \frac{dx}{x\cdot\Phi(x)} < \infty$,

and

$$\lim_{r\to\infty} \inf \frac{\log^+ + n(r,a)}{\log r} = \lambda_1(a).$$

Theorem 2

Let $F(z)$ be a meromorphic function of finite order ϱ and $\lambda(r)$ be lower proximate order with respect to $T(r,F)$. If $F(z)$ has two finite values as $e.v.S.$ then,

(i) $T(r,F') \sim 2T(r,F)$.

(ii) $\delta(F',\infty) = \Delta(F,\infty) = 0$ and $\delta(F',0) = \Delta(F',0) = 1$.

(iii) $F'(z)$ has "0" as $e.v.L.$ and no other $e.v.L.$

where,

a is an exceptional value "S" (or $e.v.S.$) if $\lim\limits_{r\to\infty} \dfrac{n(r,a)}{r^{\lambda(r)}} = 0$

and

a is an exceptional value "L" (or $e.v.L.$) if $\lim\limits_{r\to\infty} \dfrac{n(r,a)}{r^{\varrho(r)}} = 0.$

Theorem 3

Let $F(z)$ be a meromorphic function of order ϱ. If

(i) $T(r,F)/r^{\varrho}.\,L(r) \to 1$, as $r \to \infty$.

(ii) $n_L(r,x)/r^{\varrho}.\,L(r) \to 0$, as $r \to \infty$.

Then

$$n(r,x) / r^{\varrho}. L(r) \to \varrho, \text{ for all } x \neq x_1, x_2$$

where

$$n(r,x) = n(r,x_1) + n(r,x_2)$$

and $L(r)$ is continuous for all large r such that $L(cr) \sim L(r)$ for every fixed positive c.

Theorem 4

If

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(r)}{r^{\varrho(r)}} = \begin{matrix}K\\ J\end{matrix}, \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(r)}{r^{\lambda(r)}} = \begin{matrix}H\\ G\end{matrix},$$

then

$$G\left(\frac{1}{\lambda} - \frac{1}{\varrho}\frac{K}{G}\right) \leq \lim_{r\to\infty} \sup \frac{N(r)}{\log M(r)} - \lim_{r\to\infty} \inf \frac{N(r)}{\log M(r)} \leq H\left(\frac{1}{\lambda} - \frac{1}{\varrho}\frac{J}{H}\right).$$

Proof of Theorem 1

(i) Since a is $e.v.E.$

$$\lim_{r\to\infty} \inf \frac{T(r)}{n(r,a).\Phi(r).} > 0.$$

So

$$T(r) > A.n(r,a).\Phi(r).$$

Hence, $r^{\lambda(r)} > A.n(r,a).\Phi(r)$ for a sequence of values of $r \to \infty$, where $\lambda(r)$ is lower proximate order with respect to $T(r,F)$.

So

$$\frac{n(r,a)}{r^{\lambda(r)}} < \frac{A}{\Phi(r)}.$$

Thus

$$\lim_{r\to\infty} \inf \frac{n(r,a)}{r^{\lambda(r)}} = 0.$$

Hence, a is $e.v.A.$

(ii)

$$\delta(a) = 1 - \lim_{r\to\infty} \sup \frac{N(r,a)}{T(r,F)} = 1$$

so

$$\lim_{r\to\infty} \frac{N(r,a)}{T(r,F)} = 0.$$

Hence

$$N(r,a) < \varepsilon\, T(r,F) \quad \text{for} \quad r \geq r_0$$

but

$$T(r,F) = r^{\lambda(r)}$$

for a sequence of values of $r \to \infty$ and

$$N(2r,a) > n(r,a).\log 2.$$

So

$$n(r,a). \log 2 < N(2r,a)$$

$$< \varepsilon \cdot T(2r,F)$$

$$n(r,a). \log 2 < \varepsilon (2r)^{\lambda(2r)}$$

for a sequence of values of $r \to \infty$ and

$$n(r,a). \log 2 < \varepsilon\, r^{\lambda(r)} \cdot 2^{\lambda}.$$

hence

$$\lim_{r\to\infty} \text{iuf} \frac{n(r,a)}{r^{\lambda(r)}} = 0 .$$

So a is *e.v.A.*

(iii) $n(r,a) < r^{\lambda_1(a)+\Sigma}$ for a sequence of values of $r \to \infty$. Then

$$\frac{n(r,a)}{\gamma^{\lambda(r)}} < \frac{r^{\lambda_1(a)+\Sigma}}{\gamma^{\lambda(r)}} = \frac{1}{r^{\lambda(r)-\lambda_1(a)-\Sigma}}$$

so

$$\lim \ \inf \frac{n(r,a)}{\gamma^{\lambda(r)}} = 0$$

Hence a is *e.v.A.*

(iv)

$$A(a) = 1$$

so

$$1 - \lim_{r\to\infty} \inf \frac{N(r,a)}{T(r,F)} = 1 .$$

Then

$$\lim_{r\to\infty} \inf \frac{N(r,a)}{T(r,F)} = 0$$

i.e.,

$$N(r,a) < \varepsilon\, T(r,F)$$

for a sequence of values of $r \to \infty$. But

$$T(r,F) \leq r^{\rho(r)} \text{ for } r \geq r_0$$

so

$$\lim_{r\to\infty} \inf \frac{N(r,a)}{r^{\rho(r)}} = 0 \quad \Rightarrow \quad \lim_{r\to\infty} \inf \frac{n(r,a)}{r^{\rho(r)}} = 0.$$

Therefore a is *e.v.D.*

Proof of Theorem 2

(i) Let a and b be *e.v.S.* Then

$$\frac{n(r,a)}{r^{\lambda(r)}} \longrightarrow 0 \Rightarrow \frac{N(r,a)}{r^{\lambda(r)}} \longrightarrow 0$$

and

$$\frac{n(r,b)}{r^{\lambda(r)}} \longrightarrow 0 \Rightarrow \frac{N(r,b)}{r^{\lambda(r)}} \longrightarrow 0.$$

But

$$\lim_{r\to\infty} \sup \frac{N(r,a)}{T(r)} \leq \lim_{r\to\infty} \sup \frac{N(r,a)}{r^{\lambda(r)}} \lim_{r\to\infty} \sup \frac{r^{\lambda(r)}}{T(r)} = 0$$

because

$$\lim_{r\to\infty} \sup \frac{r^{\lambda(r)}}{T(r)} < \infty.$$

So

$$\lim_{r\to\infty} \sup \frac{N(r,a)}{T(r,F)} = 0$$

i.e

$$\delta(a) = 1.$$

Similarly

$$\delta(b) = 1.$$

But from the second fundamental theorem we get

$$T(r) < \bar{N}(r,a) + \bar{N}(r,b) + \bar{N}(r,\infty) + 0(\log r).$$

So

$$1 \leq \lim_{r\to\infty} \sup \frac{\bar{N}(r,a)}{T(r)} + \lim_{r\to\infty} \sup \frac{\bar{N}(r,b)}{T(r)} + \lim_{r\to\infty} \inf \frac{\bar{N}(r,\infty)}{T(r)}$$

and

$$\lim_{r\to\infty} \inf \frac{\bar{N}(r,\infty)}{T(r,F)} \geq 1.$$

But

$$\lim \sup \frac{\bar{N}(r,\infty)}{T(r,F)} \leq 1,$$

so

$$\bar{N}(r,\infty) \sim T(r,F), \text{ as } r \to \infty.$$

Now

$$\frac{T(r,F')}{T(r,F)} > \frac{N(r,F')}{T(r,F)} = \frac{N(r,F) + \bar{N}(r,F)}{T(r,F)},$$

so

$$\lim_{r\to\infty} \inf \frac{T(r,F')}{T(r,F)} \geq 2 \cdot$$

But

$$\lim_{r\to\infty} \sup \frac{T(r,F')}{T(r,F)} \leq 2 - \mu(\infty) - \delta(\infty)$$

hence

$$\mu(\infty) = \delta(\infty) = 0$$

and

$$\lim_{r\to\infty} \frac{T(r,F')}{T(r,F)} = 2 \cdot$$

(ii) It is known [2] that

$$\Delta(F',0). \lim_{r\to\infty} \inf \frac{T(r,F')}{T(r,F)} \geq \sum_{i=1}^{\infty} \delta(a_i) \cdot$$

so

$$\Delta(F',0) \geq 1.$$

Again

$$\Delta(F',0)\{2 - \delta(\infty) - \mu(\infty)\} \geq \{1 + (F',0) - \delta(F,0)\} \sum_{i=1}^{\infty} \delta(a_i),$$

so

$$2 \geq \{2 - \delta(F',0)\} . 2,$$

so

$$1 \geq 2 - \delta(F',0).$$

Hence

$$\delta(F',0) = 1.$$

Hence from [3] " 0 " is *e. .v.L.* of $F'(z)$.

Again

$$\frac{N(r,F')}{T(r,F')} \sim \frac{N(r,F')}{2T(r,F)} = \frac{N(r,F) + \bar{N}(r,F)}{2T(r,F)}$$

but

$$\bar{N}(r,F) \sim T(r,F), \text{ as } r \to \infty$$

so

$$\lim_{r\to\infty} \inf \frac{N(r,F')}{T(r,F')} \geq 1.$$

But

$$\operatorname{Lim}_{r\to\infty} \sup \frac{N(r,F')}{T(r,F')} \leq 1$$

so

$$\lim_{r\to\infty} \frac{N(r,F')}{T(r,F')} = 1.$$

Hence,

$$\delta(F', \infty) = 0 = \Delta(F', \infty)$$

so " ∞ " is not an *e.v.L.* of $F'(z)$. We shall show that $x\,(x \neq 0)$ is not *e.v.L.*

It is known [4] that

$$pq < \frac{N(r,F)}{T(r,F)} + q \sum_{i=1}^{p} \frac{N(r,a_i)}{T(r,F)} + \sum_{i=1}^{q} \frac{N(r,1/F'-b_i)}{T(r,F)} -$$

$$\left\{ (q-1)\frac{N(r,1/F')}{T(r,F)} + \frac{N(r,1/F'')}{T(r,F)} \right\} + 0(1).$$

Setting $p = 2$, $q = 1$,

$$\frac{2}{1} < \frac{\bar{N}(r,F)}{T(r,F)} + \frac{N(r,a)}{T(r,F)} + \frac{N(r,b)}{T(r,F)} + \frac{N(r,1/F'-x)}{T(r,F')} \cdot \frac{T(r,F')}{T(r,F)} + 0(1)$$

so

$$2 \leq \liminf_{r\to\infty} \frac{N(r,1/F'-x)}{T(r,F)} \cdot \limsup_{r\to\infty} \frac{T(r,F')}{T(r,F)} + \limsup_{r\to\infty} \frac{\bar{N}(r,F)}{T(r,F)}$$

$$+ \limsup_{r\to\infty} \frac{N(r,a)}{T(r,F)} + \limsup_{r\to\infty} \frac{N(r,b)}{T(r,F)}$$

so

$$2 \leq \{1 - \Delta(F', x)\}\, 2 + 1 + 0 + 0$$

so

$$\Delta(F', x) \leq 1/2.$$

But if x were an *e.v.L* for $F'(z)$, $\Delta(F',x) = 1$. Hence $x\,(\neq 0)$ can not be *e.v.L.* This shows that $F'(z)$ can have only "0" as *e.v.L.* and has no other *e.v.L.*

Proof of Theorem 3

$$\frac{n_1(r,x)}{r^{\varrho} L(r)} \longrightarrow 0 \Rightarrow \frac{N_1(r,x)}{r^{\varrho} L(r)} \longrightarrow 0 \text{ , as } r \to \infty.$$

Now,

$$\frac{T(r,F)}{r^{\varrho} L(r)} < \frac{N(r,x_1) + N(r,x_2) + N(r,x_3) + 0(\log r)}{r^{\varrho} L(r)}$$

so

$$\frac{T(r,F)}{r^{\varrho} L(r)} < \frac{N(r,x_3)}{r^{\varrho} L(r)} \cdot$$

Hence,

$$T(r,F) \sim N(r,x_3)$$

as $r \to \infty$ so,

$$\frac{N(r,x_3)}{r^{\varrho} L(x)} \longrightarrow 1 \Rightarrow \frac{n(r,x_3)}{r^{\varrho} L(r)} \longrightarrow \varrho \text{ , as } r \to \infty .$$

We omit the proof of Theorem 4.

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ÖZET

Tam ve meromorf fonksiyonların NEVANLINNA teorisi ile ilgili olarak, limit işlemi sâyesinde bâzı özel değerler tanımlanmaktadır. Tekil değer olarak adlandırılan bu değerler ile ilgili bâzı sonuçlar elde edilmiştir.