SURFACES ON WHICH THE LAGUERRE LINES FORM AN HEXAGONAL THREE - WEB

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The aim of this paper is lo study those surfaces on which the LAGUERRE lines (G-lines) form an hexagonal three-web. The following results have been obtained :

In 1. a necessary and sufficient condition, for the G-liues of a pseudo-spherical surface to form an hexagonal three-web, is given. Furthermore, the three families of G-lines together with the two families of asymptotic lines form an hexagonal 5-web on such surfaces.

In 2. a criterion for the surfaces of constant mean curvature on which the G-lines form an hexagonal three-web is also given.

In 3. a theorem concerning the surfaces on which the two families of G-lines form an isothermally orthogonal net having a mean curvature W, of the form $W = W[U(u) + V(v)]$, where u and v are the isothermic parameters of these G-lines is proved. A characterization of DUPIN's Cyclides by means of G-lines is given in 4. and it is shown that the G-lines on DUPIN's Cyclides form an hexagonal three-web.

In 5. the developable surfaces on which the two families of G-lines together with a family of lines of curvature form an hexagonal three-weh are considered and it is shown that any cone satisfies this condition. More generally, two families of G-lines together with a family of lines of curvature will form an hexagonal three-web on the tangential developable of a space curve if and only if there exists a relation, between ρ and τ, of the form

$$\frac{\rho}{\tau} = (ps + q)^{5/3} \quad (p\,,\, q = \text{const.}),$$

ρ , τ and s being, respectively, the curvature, the torsion and the arc length of the space curve. Finally, in 6. a result concerning parallel surfaces is mentisoned

0. Laguerre Lines (G-Lines).

A G-line on a surface is defined as follows [1] : Let S be a real surface and let C be a line drawn on S. C is said to be a G-line if and only if at every poiut P on C, the normal plane of S containing the tangent line PT to C, cuts the surface S in a line superosculated by its circle of curvature at P.

From the above definition, one can see that a line C on S will be a G-line if and only if the relation

$$\mathfrak{L} = \dot{\varrho}_n - 2\,\varrho_g\,\tau_g = 0 \quad \left(\dot{\varrho}_n = \frac{d\,\varrho_n}{d\,s}\right), \tag{0.1}$$

holds all along C, [1, 140], where $\mathfrak{L}$ is LAGUERRE's direction function and ϱ_n, ϱ_g, τ_g and s are, respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc length of C.

Now, let a G-line on a surface S be given in function of any parameter t, by

$$u = u(t) \quad , \quad v = v(t)$$

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u and v being the parameters on S. Then the differential equation (0.1) can be expressed in function of the coefficients of the two fundamental forms of S and their partial derivatives, namely

$$\begin{aligned}\{L_u - 2(L\Gamma^1_{11} + M\Gamma^2_{11})\}\, du^3 + 3\{L_v - 2(L\Gamma^1_{12} + M\Gamma^2_{12})\}\, du^2\, dv \\ + 3\{N_u - 2(M\Gamma^1_{12} + N\Gamma^2_{12})\}\, du\, dv^2 + \{N_v - 2(M\Gamma^1_{22} + N\Gamma^2_{22})\}\, dv^3 = 0,\end{aligned} \tag{0.2}$$

where L, M and N are the coefficients of the second fundamental form of S and Γ^k_{ij} $(i, j, k = 1,2)$ are its CHRISTOFFEL symbols [2, 107].

The differential equation (0.2) being of the first order and third degree, in general, three G-lines pass through each point of the surface. Therefore it will be convenient to study the three-web formed by its G-lines on a surface and see under what conditions this three web will be an hexagonal one. It is, however, difficult to find a general solution for this problem and we have had to limit our analysis and consider only special classes of surfaces.

1. The pseudo-spherical surfaces on which the G-lines form an hexagonal three-web.

The G-lines on a pseudo-spherical surface do not, in general, form an hexagonal three-web. The following theorem gives us a necessary and sufficient condition for this to occur.

Theorem 1.1. *The necessary and sufficient condition for the G-lines on a pseudo-spherical surface to form an hexgonal three-web is that $\omega(u,v)$ be a function of t satisfying the differential equation*

$$\frac{d^2\omega}{dt^2} + a\, e^t \sin\omega = 0\,, \quad (a = \text{const.})$$

where $\omega(u,v)$ is the angle between the asymptotic lines of the pseudo-spherical surface and $uv = \varepsilon\, e^t$, $(\varepsilon = \mp 1)$.

Proof. The differential equation of the G-lines of any surface is (0.2). Suppose that the asymptotic lines of the surface are taken as parametric lines. Then, if we make $L = 0$, $N = 0$ in (0.2) we obtain the differential equation

$$\Gamma^2_{11}\, du^3 + 3\,\Gamma^2_{12}\, du^2\, dv + 3\,\Gamma^1_{12}\, du\, dv^2 + \Gamma^1_{22}\, dv^3 = 0. \tag{1.1}$$

Now let the surface be pseudo-spherical : then its asymptotic lines form a TCHEBYCHEF net [3, 180], and therefore the coefficients, $E(u, v)$ and $G(u, v)$, of the first fundamental form of the pseudo-spherical surface, are of the form $E = E(u)$ and $G = G(v)$. Then, by a suitable transformation of the parameters [4, 150], the first fundamental form of the surface reduces to

$$ds^2 = du^2 + 2\cos\omega\, du\, dv + dv^2. \tag{1.2}$$

We then have

$$E = G = 1 \quad , \quad F = \cos\omega,$$

$$\Gamma^1_{12} = \Gamma^2_{12} = 0 \quad , \quad \Gamma^2_{11} = -\frac{1}{\sin\omega}\cdot\frac{\partial\omega}{\partial u}\,, \quad \Gamma^1_{22} = -\frac{1}{\sin\omega}\cdot\frac{\partial\omega}{\partial v}$$

and for the GAUSSIAN curvature K, of the pseudo-spherical surface, we find

$$K = -\frac{1}{\sin\omega}\cdot\frac{\partial^2\omega}{\partial u\,\partial v}\,. \tag{1.3}$$

Setting the above values for Γ^k_{ij} in (1.1) this equation reduces to

$$\frac{\partial\omega}{\partial u}\, du^3 + \frac{\partial\omega}{\partial v}\, dv^3 = 0\,. \tag{1.4}$$

Solving this differential equation in $\frac{dv}{du}$ and putting $p(u, v) = \frac{dv}{du} = -\sqrt[8]{\frac{\omega_u}{\omega_v}}$, we see that the three families of G-lines on a pseudo-spherical surface are given by

$$\begin{aligned} &dv - p(u, v)\, du = 0 \quad , \\ &dv - j\, p\, (u,v)\, du = 0 \quad , \\ &dv - j^2\, p(u, v)\, du = 0 \quad , \end{aligned} \tag{1.5}$$

where j is a complex cube root of unity.

We now introduce the three differential forms

$$\begin{aligned} \sigma_1 &= g_1\,(u, v)\,.\,(dv - p\, du) \quad , \\ \sigma_2 &= g_2\,(u, v)\,.\,(dv - j\,.\,p\, du) \quad , \\ \sigma_3 &= g_3\,(u, v)\,.\,(dv - j^2\,.\,p\, du) \end{aligned}$$

with $g_i\,(u, v) \neq 0$, $(i = 1, 2, 3)$, and determine $g_i\,(u, v)$ so as to satisfy the condition $\sum_{i=1}^{3} \sigma_i = 0$.

Then the normed differential forms satisfying the condition $\sum_{i=1}^{3} \sigma_i^* = 0$ are

$$\begin{aligned} \sigma_1^* &= -p\, du + dv \quad , \\ \sigma_2^* &= -j^2\, p\, du + j\, dv, \\ \sigma_3^* &= -j\, p\, du + j^2\, dv. \end{aligned} \tag{1.6}$$

The three families of G-lines which are given by (1.5), will form an hexagonal three-web, if and only if the dependency γ, of the three-web is an exact differential [5, 164].

For the element of area, Ω, of the three-web, we can take the exterior product $[\sigma_1^*, \sigma_2^*] = \Omega = j\,(j-1)\,p\,(u, v)\,[du\,, dv]$. The exterior differentials of the differential forms (1.6) differ from Ω by scalar functions, say $h_i\,(u,v)$ $(i = 1, 2, 3)$, and the h_i's satisfy the relation $\sum_{i=1}^{3} h_i = 0$. Denoting exterior differentiation by $\mathfrak{d}$, we obtain

$$\mathfrak{d}\,\sigma_1^* = \frac{p_v}{j\,(j-1)\,p}\,\Omega = h_1\,\Omega\,, \qquad \mathfrak{d}\,\sigma_2^* = \frac{j}{j-1}\,\frac{p_v}{p}\,\Omega = h_2\,\Omega$$

and hence

$$h_1 = \frac{1}{j\,(j-1)}\,\frac{p_v}{p}\,, \qquad h_2 = \frac{j}{j-1}\,\frac{p_v}{p}\,.$$

On the other hand, γ is defined as $\gamma = h_2\,\sigma_1^* - h_1\,\sigma_2^*$, [5, 163]. Inserting the above values for h_1, h_2, σ_1^* and σ_2^* in the expression of γ, we find that $\gamma = (\ln p)_v\, dv$. The condition that γ be an exact differential is

$$(\ln p)_{uv} = 0, \tag{1.7}$$

and therefore the function $p\,(u, v)$ is of the form $p\,(u, v) = Z\,(u)\,.\,S\,(v)$. Since, $p\,(u, v) = -\sqrt[8]{\frac{\omega_u}{\omega_v}}$, we see that the function $\omega\,(u, v)$ satisfies the partial differential equation

$$\frac{\partial \omega}{\partial u} - Z_1\,(u)\,.\,S_1\,(v)\,\frac{\partial \omega}{\partial v} = 0\,,\ [S_1\,(v) = S^8\,(v),\, Z_1\,(u) = -Z^8\,(u)]. \tag{1.8}$$

The general solution of the differential equation (1.8), is of the form

(1.9) $$\omega(u,v) = f\{\lambda(u) + \mu(v)\}$$

Therefore our problem reduces to the problem of determining the three arbitrary functions f, λ and μ, appearing in (1.9) so as to satisfy the differential equation

(1.10) $$K = -\frac{1}{\sin\omega}\cdot\frac{\partial^2\omega}{\partial u\partial v} = a\,,$$

which is found by putting $K = \text{const} = a$ in (1.3). If we put $t = \lambda(u) + \mu(v)$, the differential equation (1.10) takes the form

(1.10′) $$\lambda'(u)\,.\,\mu'(v) = -\frac{a\sin f(t)}{f''(t)}$$

where $f''(t) = \dfrac{d^2 f}{dt^2}$; $\lambda'(u) = \dfrac{d\lambda}{du}$, $\mu'(v) = \dfrac{d\mu}{dv}$ and $K = a < 0$.

Using the property of equation (1.10′) and making the necessary calculations we first find

(1.11) $$\lambda(u) = k\ln(c_1 u + c_2) \quad , \quad (c_1\,,\ c_2 = \text{const.})$$

and

(1.12) $$\mu(v) = k\ln(c_3 v + c_4)\ ,\ (c_3\,.\ c_4 = \text{const.}).$$

Therefore $t = \lambda(u) + \mu(v) = k\{\ln[(c_1 u + c_2)(c_3 v + c_4)]\}$. It is clear that we can make $c_2 = c_4 = 0$ by choosing a suitable origin on the pseudo-spherical surface. Accordingly $t = k\ln c_1 c_3 + k\ln uv$ and hence we get

(1.13) $$\omega(u,v) = \omega\{\ln uv\}.$$

Finally the differential equation (1.10) reduces to

(1.14) $$\frac{d^2\omega(t)}{dt^2} + a\,e^t\sin\omega(t) = 0\ ,\quad (t = \ln uv)$$

and thus the proof of the theorem is completed.

Now, let us determine the G-lines for these pseudo-spherical surfaces. Using the value of $\omega(u,v)$, which is given by (1.13), we find that $p(u,v) = -\sqrt[3]{\frac{v}{u}}$. Substituting this value for $p(u,v)$ in (1.5), we obtain the three differential equations

$$dv + \sqrt[3]{\frac{v}{u}}\,du = 0\quad,\quad dv + j\,.\sqrt[3]{\frac{v}{u}}\,du = 0\ ,\quad dv + j^2\,.\sqrt[3]{\frac{v}{u}}\,du = 0.$$

Integrating these differential equations, we find the three families of G-lines of the surfaces under consideration, namely

(1.15) $$\begin{aligned} u^{2/3} + v^{2/3} &= k_1 = \text{const.}\ , \\ u^{2/3} + j\,v^{2/3} &= k_2 = \text{const.}\ , \\ u^{2/3} + j^2\,v^{2/3} &= k_3 = \text{const.}\ , \end{aligned}$$

It can be easily seen that the three families of G-lines, given by (1.15), together with the two families of asymptotic lines for these pseudo-spherical surfaces form an hexagonl 5-web. Moreover, we see from (1.13) that the family $\omega(u,v) = \text{const.}$ together with the asymptotic lines form an hexagonal three-web on these surfaces.

2. Surfaces of constant mean curvature on which the G-lines form an hexagonal three-web.

On a surface of constant mean curvature the three families of G-lines cut each other at an angle of 120°, [1, 145], but they do not, in general, form an hexagonal three-web. We shall now prove the following theorem :

Theorem 2.1. *The G-lines of a surface of constant mean curvature will form an hexagonal three-web if and only if the surface is applicable to a surface of revolution.*

Proof. Let S be a surface of constant mean curvature and let S be referred to its minimal lines. Then, since $E = G = 0$ and $F \neq 0$, the Christoffel symbols Γ^k_{ij} $(i, j, k = 1,2)$ take the form

$$\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{22} = 0 \quad , \quad \Gamma^1_{11} = (\ln F)_u , \quad \Gamma^2_{22} = (\ln F)_v . \tag{2.8}$$

On the other hand, the Gaussian curvature K and the mean curvature W of S, are of the form

$$K = -\frac{LN}{F^2} + W^2 , \quad W = \frac{M}{F} = \text{const.} \tag{2.9}$$

Using the Mainardi-Codazzi relations, we find that

$$L_v = 0 \quad , \quad N_u = 0. \tag{2.10}$$

By virtue of (2.8), (2.10) and (0.2) the differential equation of the G-lines on S reduces to

$$L\left(\ln \frac{L}{F^2}\right)_u du^3 - N\left(\ln \frac{F^2}{N}\right)_v dv^3 = 0 . \tag{2.11}$$

If we assume the G-lines to form an hexagonal three-web on S, we find that

$$\left(\ln \sqrt[3]{\frac{L}{N} \cdot \frac{\left(\ln \frac{L}{F^2}\right)_u}{\left(\ln \frac{F^2}{N}\right)_v}}\right)_{uv} = 0, \tag{2.12}$$

where we put $\sqrt[3]{\frac{L}{N} \cdot \frac{\left(\ln \frac{L}{F^2}\right)_u}{\left(\ln \frac{F^2}{N}\right)_v}} = p\,(u,v)$ in (1.7).

With the aid of (2.10), it is easy to see that the condition (2.12) may be written in the form

$$\frac{\left(\ln \frac{LN}{-F^2}\right)_u}{\left(\ln \frac{LN}{-F^2}\right)_v} = \frac{b(v)}{a(u)}$$

and therefore we obtain

$$\frac{LN}{-F^2} = T\,[\alpha\,(u) + \beta\,(v)] = K - W^2 \quad , \quad (W = \text{coust.})$$

where $a(u)$, $b(v)$, $\alpha(u)$, $\beta(u)$, and $T[\alpha(u) + \beta(v)]$ are arbitrary functions.

Therefore the minimal lines $u = \text{const.}$, $v = \text{const.}$ and the family $K = \text{const.}$ form an hexagonal three-web on S. But this means that the surface S is applicable to a surface of revolution [12,34].

Conversely, we assume the surface S to be applicable to a surface of revolution. In this case the minimal lines $u =$ const., $v =$ const. together with the family $K =$ conts. form an hexagonal three-web on S. Consequently, K must be of the form $K = K[U(u) + V(v)]$, $U(u)$ and $V(v)$ being arbitrary functions [6,29].

Therefore since, by (2.9), $F^2 = -\dfrac{L(u) \cdot N(v)}{K[U(u) + V(v)] - W^2}$ we see that the hexagonality condition, (2.12), is satisfied,. This means that the G-lines of S form an hexagonal three web.

The G-lines of a minimal ruled surface (right helicoid) form an hexagonal three-web because every minimal ruled surface is applicable to a surface of revolution. The helicoid $x = u \cos v$, $y = u \sin v$, $z = cv = c \int \sqrt{\dfrac{u^2 + c^2}{u^2 - c^2}} \cdot \dfrac{du}{u}$, being a minimal surface, gives a second example for the above theorem ; since every helicoid is applicable to some surface of revolution.

3. Surfaces on which the two families of G-lines form an isothermally orthogonal net and have a mean curvature of the form $W = W[U(u + V(v)]$ u and v being the isothermic parameters of these G-lines.

If a surface is referred to its orthogonal parametric lines, the three directions of G-lines, through the same point on the surface, are given [7, 106] by

$$(3.1) \qquad (r_1 - 2\,g\,t)\cos^3\varphi + 3\,(r_2 - 2\,\bar{g}\,t)\cos^2\varphi \sin\varphi + 3\,(\bar{r}_1 + 2\,g\,t)\cos\varphi \sin^2\varphi + (\bar{r}_2 + 2\,\bar{g}\,t)\sin^3\varphi = 0,$$

where g and $\bar{g}$ are the geodesic curvatures of the parametric lines, t is the geodesic torsion of the parametric lines $v =$ const. and $r_1, r_2, \bar{r}_1, \bar{r}_2$ are the invariant derivatives of r and $\bar{r}$ respectively.

Since $tg\,\varphi = \sqrt{\dfrac{G}{E}} \cdot \dfrac{dv}{du}$ in an orthogonal coordinate system the equation (3.1) takes the form

$$(3.2) \qquad E^{3/2}(r_1 - 2\,g\,t)\,du^3 + 3\,E\sqrt{G}\,(r_2 - 2\,\bar{g}\,t)\,du^2\,dv + 3\,G\sqrt{E}\,(\bar{r}_1 + 2\,g\,t)\,du\,dv^2 + G^{3/2}(\bar{r}_2 + 2\,\bar{g}\,t)\,dv^3 = 0.$$

The necessary and sufficient conditions for the lines $v =$ const. $u =$const. and $u + v =$ const. to be the G-lines of the surface, are, according to (3.2)

$$(3.3) \qquad r_1 - 2\,gt = 0 \quad ,$$

$$(3.4) \qquad \bar{r}_2 + 2\,\bar{g}t = 0 \quad ,$$

and

$$(3.5) \qquad \sqrt{E}\,(r_2 - 2\,\bar{g}t) = \sqrt{G}\,(\bar{r}_1 + 2\,gt)$$

respectively. By using the conditions (3.3), (3.4) and (3.5) we find that

$$\sqrt{E}\,(r_2 + \bar{r}_2) = \sqrt{G}\,(r_1 + \bar{r}_1).$$

This equality may be written as

$$(3.6) \qquad \sqrt{E}\,W_2 = \sqrt{G}\,W_1 \quad , \quad \left(W_1 = \frac{W_u}{\sqrt{E}}, \quad W_2 = \frac{W_v}{\sqrt{G}} \right),$$

where W_1 and W_2 are the invariant derivatives of W in the direction of the parametric lines.

We now prove the following theorem :

Theorem 3.1. *If the two families of G-lines of a surface form an isothermally orthogonal net, then the three families of G-lines of such a surface will form an hexagonal three-web, if and only if the surface has a mean curvature W, of the form $W = W[U(u) + V(v)]$, where u and v are the isothermic parameters.*

Proof. Let the two families of G-lines which are assumed to form an isothermally orthogonal net, be taken as parametric lines on the surface. Then the conditions (3.3), (3.4) and

$$\left(\ln\frac{G}{E}\right)_{uv} = 0 \quad , \quad F = 0 \tag{3.7}$$

are satisfied.

Suppose that the G-lines form an hexagonal three-web on the surface in question. Then, the lines $v = \text{const.}$, $u = \text{const.}$ and $u + v = \text{const.}$ may be taken as the three families of G-lines and therefore (3.5) and (3.6) are, also, satisfied. With the aid of (3.6) and (3.7), we find that

$$\left(\ln\frac{G}{E}\right)_{uv} = \left(\ln\frac{W_v}{W_u}\right)_{uv} = 0,$$

and therefore we have

$$W = W[U(u) + V(v)]. \tag{3.8}$$

This means that the family of curves for which $W = W[U(u) + V(v)] = \text{const.}$ forms an hexagonal three-web with the two families of G-lines which were assumed to form an isothermally orthogonal net.

Conversely, we assume the mean curvature, W, to be of the form $W = W[U(u) + V(v)]$ and prove that the three families of G-lines form an hexagonal three-web. Let the two families of G-lines which are assumed to form an isothermally orthogonal net be taken as the lines $v = \text{const.}$ and $u = \text{const.}$ In this case the conditions (3.3) and (3.4) are satisfied. We then find, from (3.2), that the differential equation of the third family of the G-lines is

$$E\sqrt{G}\,(r_2 - 2\,\bar{g}t)\,du + G\sqrt{E}\,(\bar{r}_1 + 2\,gt)\,dv = 0\,.$$

Using the relations (3.3) and (3.4), this differential equation may be written in the form

$$E\sqrt{G}\;W_2\,du + G\sqrt{E}\;W_1\,dv = 0\,,$$

and then we find

$$E\,W_v\,du + G\,W_u\,dv = 0. \tag{3.9}$$

Since $W = W[U(u) + V(v)]$ and the parametric lines form an isothermally orthogonal net, the differential equation (3.9) takes the form $A'(u)\,du + B'(v)\,dv = 0$, and therefore we obtain

$$A(u) + B(v) = \text{Const.}$$

as the third family of G-lines. But the families $v = \text{const.}$, $u = \text{const.}$ and $A(u) + B(v) = \text{const.}$ form an hexagonal three-web. Hence the proof of the theorem is completed.

4. Dupin's Cyclides.

We first prove the following theorem concerning DUPIN's Cyclides.

Theorem 4.1. *The necessary and sufficient condition that the two families of G-lines of a surface coincide with the lines of curvature is that the surface be a Dupin's Cyclide.*

Proof. Suppose that the two families of G-hnes on a surface coincide with the lines of curvature. If we refer the surface to its lines of curvature, then at any point on the surface the directions of LAGUERRE are given, [9, 291]. by the equation

(4.10) $$r_1 \cos^3\varphi + 3 r_2 \cos^2\varphi \sin\varphi + 3\bar{r}_1 \cos\varphi \sin^2\varphi + \bar{r}_2 \sin^3\varphi = 0,$$

φ being the angle between the tangents of a G-line and $v =$ const.

Now, since the two families of G-lines coincide with the lines of curvature, the equation (4.10) will be satisfied for $\varphi = 0$ and $\varphi = \pi/2$. In this case we have

(4.11) $$r_1 = 0 \quad , \quad \bar{r}_2 = 0 \quad ,$$

at every point of the surface. But these conditions characterize DUPIN's Cyclides [10, 141].

Conversely, we assume that the surface in question is a DUPIN's Cyclide and refer it to its lines of curvature. Since $r_1 = 0$ and $\bar{r}_2 = 0$ for a DUPIN Cyclide, we find from (4.10) that the LAGUERRE directions, at any point on the surface, are given by the equation

$$(r_2 \cos\varphi + \bar{r}_1 \sin\varphi) \,.\, \sin\varphi \cos\varphi = 0$$

Therefore the two directions of LAGUERRE coincide with $\varphi = 0$ and $\varphi = \frac{\pi}{2}$; that is, with the lines of curvature. This proves the theorem.

Theorem 4.2. *The G-lines on Dupin's Cyclides form an hexagonal three-web.*

Proof. Let us refer a DUPIN Cyclide to its lines of curvature. By theorem (4.1) the two families of G-lines on a DUPIN Cyclide coincide with the lines of curvature. On the other hand, the lines of curvature on a DUPIN Cyclide form an isothermally orthogonal net. Therefore the two families of G-lines which are coincident with the lines of curvature, also form an isothermally orthogonal net. Furthermore, from (4.11) we see that the principal curvatures r, $\bar{r}$ are of the form $r = r(v)$, $\bar{r} = \bar{r}(u)$. Therefore the mean curvature, W, of a DUPIN Cyclide takes the form $2W = \bar{r}(u) + r(v)$. Consequently, DUPIN's Cyclides satisfy all the conditions of theorem (3.1) and therefore the G-lines on DUPIN's Cyclides form an hexagonal three-web.

5. Developable surfaces.

Let S be a developable surface and let S be referred to its lines of curvature. Then $F = 0$, $M = 0$ and by a suitably chosen transformation of the parameters u and v, the coefficients of the first and the second fundamental form of S may be put into the form [11, 232].

(5.1) $$G = 1 \,, \quad \sqrt{E} = e = v \,.\, f(u) + \lambda(u) \;, \; N = 0,$$

where $f(u)$ and $\lambda(u)$ are arbitrary functions of u alone.

Furthermore, $K = r\bar{r} = 0$ $(\bar{r} = 0)$ and $r = \frac{1}{e}$. If the ratio of the two functions $f(u)$ and $\lambda(u)$, appearing in (5.1), is constant, the developable surface is a cone ; if $f(u) = 0$, the developable surface is a cylinder [11, 232].

We now prove a theorem concerning cones.

Theorem 5.1. *The two families of G-lines together with a family of lines of curvature (different from the generators) on a cone form an hexagonal three-web.*

Proof. If a surface is referred to its lines of curvature, the differential equation of the G-lines takes the form

(5.2) $$E^{3/2} r_1 \, du^3 + 3 E \sqrt{G} r_2 \, du^2 \, dv + 3 \, G \sqrt{E} \, \bar{r}_1 \, du \, dv^2 + G^{3/2} \bar{r}_2 \, dv^3 = 0,$$

where r_1, r_2, $\bar{r}_1$ and $\bar{r}_2$ have their usual meaning. Since $\bar{r} = 0$ $(r \neq 0)$ for a developable surface, the differential equation (5.2) reduces to

(5.3) $$(\sqrt{E} \; r_1 \, \mathrm{du} + 3 \sqrt{G} \; r_2 \, dv) . \, du^2 = 0.$$

Therefore two families of G-lines coincide with a family of lines of curvature, that is, with the lines $u =$ const. The differential equation of the third family of the G-lines is obtained by putting $r_1 = \dfrac{r_u}{\sqrt{E}}$ and $r_2 = \dfrac{r_v}{\sqrt{G}}$ in the differential equuation (5.3). Namely

(5.4) $$r_u \, du + 3 \, r_v \, dv = 0.$$

Thus we find that the two families of G-lines and a family of lines of curvature (different from the generators) of a developable surface are

(5.5) $$\begin{aligned} du &= 0 \, , \\ r_u \, du + 3 \, r_v \, dv &= 0 \, , \\ dv &= 0 \, , \end{aligned}$$

respectively. These three families of curves will form an hexagonal three-web if and only if

(5.6) $$\left(\ln \frac{r_u}{3 r_v} \right)_{uv} = 0$$

Setting $r = \dfrac{1}{e} = \dfrac{1}{v \, . f(u) + \lambda(u)}$ in (5.6), we obtain

(5.7) $$\lambda(u) = a_1 f(u) + b_1 \; , \quad \lambda' . f' \neq 0 \;\; (a_1 \, , \; b_1 = \text{const.})$$

If $b_1 = 0$ we have $\lambda(u)/f(u) = a_1 =$const. and therefore the developable surface is a cone. When $b_1 \neq 0$, we obtain a class of developable surfaces which includes the cones as a particular case. If $f = 0$, the developable surface is a cylinder and we see from (5.5) that the three families of G-lines coincide.

Theorem 5.2. *The two families of G-lines together with a family of lines of curvature (different from the generators) on the tangential developable of a space curve will form an hexagonal three-web if and only if there exists a relation, between ϱ and τ, of the form*

$$\frac{\varrho}{\tau} = (ps + q)^{5/3} \, , \quad (p, q = \text{const}),$$

where ϱ, τ and s are, respectively, the curvature, the torsion and the arc length of the space curve.

Proof. The vector equation of the tangential developable of a space curve, $\vec{x}(s)$, is

(5,8) $$\vec{y}(u, s) = \vec{x}(s) + u \, \vec{a}_1(s) \, , \quad \left(\vec{a}_1 = \frac{d \vec{x}}{ds} \right)$$

The coefficients of the first and second fundamental forms are

(5.9) $$E = F = 1\,, \quad G = 1 + \varrho^2 u^2\,, \quad L = M = 0\,, \quad N = \varrho\,\tau\,u.$$

The lines of curvature on the surface (5.8) arc

(5.10) $$s = \text{const}, \qquad u + s = \text{const}.$$

Substituting the values (5.9) in the differential equation (0.2), we obtain the differential equation of the G-lines for the tangential developable (5.9), namely

$$\left\{-\frac{1}{3u}\,du + \left[\left(\ln\frac{\tau}{\varrho}\right)_s - \frac{2}{u}\right] ds\right\}.\,ds^2 = 0.$$

Therefore two families of the G-lines on a tangential developable coincide with the generators (s=const.). The two families of G-lines and a family of lines of curvature on a tangential developable are, respectively,

(5.11) $$\begin{aligned} ds &= 0, \qquad (G \text{ line} = \text{line of curvature}) \\ du &+ [6 - 3\,u\,f(s)]\,ds = 0, \\ du &+ ds = 0, \end{aligned}$$

where $f(s) = \left(\ln\frac{\tau}{\varrho}\right)_s$. If we put $\left(\frac{\varrho}{\tau}\right)^3 = \varphi'$, $\left(f = -\frac{1}{3}\,\frac{\varphi''}{\varphi'}\right)$, in $(5.11)_2$ we find that $u = \frac{c_3}{\varphi'} - 6\,\frac{\varphi}{\varphi'}$ (c_3 = const.) Since, by $(5.11)_1$ and $(5.11)_3$ $s = \text{const.} = c_1$ and $u + c_1 = \text{const.} = c_2$ we obtain from $(5.11)_2$ that

(5.12) $$c_3 = (c_2 - c_1)\,.\,\varphi'(c_1) + 6\,\varphi(c_1).$$

The three-web, given by the web function (5.12), will be an hexagonal one if and only if

$$\frac{\partial^2}{\partial c_1\,\partial c_2}\left(\ln\frac{\partial c_3/\partial c_1}{\partial c_3/\partial c_2}\right) = \frac{\partial^2}{\partial c_1\,\partial c_2}\left(\ln\frac{(6\varphi - c_1\,\varphi')' + c_2\,\varphi''}{\varphi'}\right) = 0.$$

This condition may be written as $(6\,\varphi - c_1\,\varphi')' = a\,\varphi''$ (a = const). Integrating this differential equation we have

$$\varphi = b + k\,(a + s)^6 \quad (b, k = \text{const.})$$

or we find

$$\left(\frac{\varrho}{\tau}\right)^3 = \varphi' = 6\,k\,(a + s)^5 = (ps + q)^5\,,\ (p, q = \text{const}).$$

Therefore we obtain the required condition

$$\frac{\varrho}{\tau} = (ps + q)^{5/3},$$

which proves the theorem.

We remark that when $p = 0$, we obtain the tangential developable of a general helix.

6. Paralel surfaces.

Let S be a surface and let S^* be a surface parallel to S. DARBOUX has proved that: *The G-lines of S correspond to those of S^*.*

According to the above theorem, if the G-lines of a surface S form an hexagonal three-web, the G-lines on all surfaces which are parallel to S will also form an hexagonal three-web.

Therefore all surfaces which are parallel to a surface on which the G-lines form an hexagonal three-web, may be added to the class of surfaces on which the G-lines form an hexagonal three-web.

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ÖZET

Bu makalede LAGUERRE çizgilerinin (G-çizgileri) bir üclü altıgen doku teşkil ettiği yüzeyler araştırılmış ve aşağıdaki neticeler elde edilmiştir :

1. de basit ve negatif Gauss eğriliğini haiz yüzeylerin G-çizgilerinin bir üclü altıgen doku teşkil etmeleri için gerek ve yeter bir şart verilmiştir. Fazla olarak, böyle yüzeyler üzerindeki üç G-çizgisi ailesi ile iki asimptotik çizgi ailesi bir 5-li altıgen doku teşkil ederler.

2. de, üzerinde G-çizgilerinin üçlü altıgen bir doku teşkil ettikleri sâbit ortalama eğriliğini haiz yüzeyler icin bir kriter verilmiştir.

3. de, üzerinde iki G-çizgisi ailesinin dik izometrik bir sistem teşkil ettiği ve W ortalama eğriliği $W = W[U(u) + V(v)]$ şeklinde olan yüzeylere ait bir teorem ispatlanmıştır. Burada u ve v izometrik bir sistem teşkil eden iki G-çizgisi ailesinin parametreleridir.

4. de DUPIN Sikhdlerinin G-çizgileri yardımıyle bir temsili verilmiş ve DUPIN Siklidleri üzerindeki G-çizgilerinin bir üçlü altıgen doku teşkil ettikleri gösterilmiştir.

5. de, üzerinde iki G-çizgisi ailesi ile bir eğrilik çizgisi ailesinin üçlü altıgen bir doku teşkil ettikleri açılabilir yüzeyler gözönüne alınmış ve konilerin bu şartı sağladıkları gösterilmiştir. Daha genel olarak, bir uzay eğrisinin teğetleri ile teşkil edilen açılabilir yüzeyin iki G-cizgisi ailesi ile bir eğrilik çizgisi ailesinin bir üçlü altığen doku teşkil etmeleri için gerek ve yeter şartın

$$\frac{pQ}{\tau} = (ps + q)^{5/3} \qquad (p,q = \text{sâbit})$$

den ibaret olduğu ispat edilmiştir. Burada ρ , τ ve s, sırasiyle, uzay eğrisinin eğriliği, burulması ve yay uzunluğudur.