INFINITE SERIES OF H-FUNCTIONS [1)]

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The sum of an infinite series of H-functions is obtained through the process of expressing these functions as MELLIN-BARNES integrals and interchanging the order of integration and summation.

1. In this paper, we have summed a number of infinite series of H-functions, expressing the H-functions as MELLIN-BARNES type integrals and then interchanging the order of integration and summation.

2. The MELLIN-BARNES type integral [1] which we have employed is

$$H_{p,q}^{m,n}\left(x\left|\begin{matrix}\{(a_p, e_p)\}\\ \{(b_p, c_q)\}\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-c_j s)\prod_{j=1}^{n}\Gamma(a_j+e_j s)}{\prod_{j=m+1}^{q}\Gamma(b_j+c_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)}\,x^s\,ds, \tag{2,1}$$

where an empty product is interpreted as 1; p, q, m and n are integers satisfying $1\leq m\leq q$, $0\leq n\leq p$; $e_j\,(j=1,\dots,p)$, $c_j\,(j=1,\dots,q)$ are positive numbers and $a_j\,(j=1,\dots,p)$, $b_j\,(j=1,\dots,q)$ are complex numbers such that no pole of $\Gamma(b_h-c_h s)$ $(h=1,\dots,m)$ coincides with any pole of $\Gamma(a_i+e_i s)$ $(i=1,\dots,n)$;

$$e_i\,(b_h+\nu)\neq(a_i-\lambda)\,c_h \quad (\nu,\lambda=0,1,\dots\,;\,h=1,\dots,m\,;\,i=1,\dots,n) \tag{2.2}$$

and $\{(f_r, \gamma_r)\}$ stands for set of the parameters $(f_1, \gamma_1),\dots,(f_r, \gamma_r)$. Further, the contour L runs from $\sigma-i\infty$ to $\sigma+i\infty$ so that the points

$$S=\frac{b_h+\nu}{c_h}\quad(h=1,\dots,m\,;\,\nu=0,1,\dots), \tag{2,3}$$

which are poles of $\Gamma(b_h-c_h s)$ $(h=1,\dots,m)$, lie on the right and the points

$$S=\frac{a_i-\lambda}{e_i}\,(i=1,\dots,m\,;\,\lambda=0,1,\dots), \tag{2,4}$$

which are poles of $\Gamma(a_i+e_i s)$ $(i=1,\dots,n)$, lie to the left of L. We have used the following known formulae which are due to WHIPPLE [2]:

1) I wish to express my thanks to DR. S. MASOOD for his guidance in the preparation of this paper.

$$F\left[\begin{matrix}\alpha,\ \frac{\alpha}{2}+1,\ \beta,\ \gamma \\ \frac{\alpha}{2},\ \alpha-\beta+1,\ \alpha-\gamma+1\end{matrix};-1\right]=\frac{\Gamma(\alpha-\beta+1)\,\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha-\beta-\gamma+1)} \tag{2.5}$$

where $$Re\,(\alpha-2\beta-2\gamma)>-2$$

and

$$F\left[\begin{matrix}a,\ b \\ c\end{matrix};1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,. \tag{2.6}$$

3. The first summation is

$$\sum_{\gamma=0}^{\infty}\frac{(a_p-a)_\gamma}{\gamma!\,\Gamma(2a_p+e_p+\gamma)}\,H^{q,p}_{2p,2q}\left(x\left|\begin{matrix}\{(a_{p-1},e_{p-1})\},(a_p+\gamma,e_p);\{(a_{p-1}+e_{p-1},e_{p-1})\},\ (a+e_p,e_p) \\ \{(b_q,c_q)\};\{(b_q-c_q,c_q)\}\end{matrix}\right.\right)$$

$$=\frac{1}{\Gamma(a+a_p+e_p)}\,H^{q,p}_{2p,2q}\left(x\left|\begin{matrix}\{(a_p,e_p)\};\{(a_p+e_p,e_p)\} \\ \{(b_q,c_q)\};\{(b_q-c_q,c_q)\}\end{matrix}\right.\right) \tag{3.1}$$

where $$Re\,(a)<0,\ |\arg x|<\frac{1}{2}\left(\sum_{1}^{p}e_j-\sum_{p+1}^{2p}e_{p+j}+\sum_{1}^{q}c_j-\sum_{q+1}^{2q}c_{q+j}\right)\pi$$

and $$\sum_{1}^{p}e_j-\sum_{p+1}^{2p}e_{p+j}+\sum_{1}^{q}c_j-\sum_{q+1}^{2q}c_{q+j}>0.$$

Substituting on the left from (2.1), we get

$$\sum_{\gamma=0}^{\infty}\frac{(a_p-a)_\gamma}{\gamma!\,\Gamma(2a_p+e_p+\gamma)}\,\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{q}\Gamma(b_j-c_js)\prod_{j=1}^{p-1}\Gamma(a_j+e_js)\,\Gamma(a_p+\gamma+e_ps)\,x^s}{\prod_{j=1}^{q}\Gamma(b_j-c_j+c_js)\prod_{j=1}^{p-1}\Gamma(a_j+e_j-e_js)\,\Gamma(a+e_p-e_ps)}\,ds.$$

Changing the order of integration and summation, the series becomes

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{q}\Gamma(b_j-c_js)\prod_{j=1}^{p}\Gamma(a_j+e_js)\ F\left(\begin{matrix}a_p-a,\ a_p+e_ps \\ 2a_p+e_p\end{matrix};1\right)ds}{\prod_{j=1}^{q}\Gamma(b_j-c_j+c_js)\prod_{j=1}^{p-1}\Gamma(a_j+e_j-e_js)\,\Gamma(a+e_p-e_ps)\,\Gamma(2a_p+e_p)}\,.$$

On applying (2.6) and (2.1), the result is obtained. Now we use WHIPLLE's theorem to prove the following summation :

$$\sum_{\gamma=0}^{\infty} \frac{\left(\frac{1}{2} b_q - \frac{1}{2} c_q + 2\gamma\right)\left(\frac{1}{2} b_q - \frac{1}{2} c_q + \gamma\right)\left(1 - \frac{3}{2} b_q + \frac{1}{2} c_q\right)_\gamma (-1)^\gamma}{\gamma\,!\,\Gamma(2b_q - c_q + \gamma)}$$

(3.2)
$$H_{2p,2q}^{q,p}\left(x \left| \begin{matrix} \{(a_p, e_p)\}\,;\{a_p + e_p, e_p)\} \\ \{(b_{q-1}, c_{q-1})\}, (b_q + \gamma, c_q)\,; \\ \{(b_{q-1} - c_{q-1}, c_{q-1})\}, \left(1 - \frac{1}{2} b_q + \frac{1}{2} c_q + \gamma, c_q\right. \end{matrix} \right. \right) =$$

$$= H_{2p,2q}^{q,p}\left(x \left| \begin{matrix} \{(a_p, e_p)\}\,;\{(a_p + e_p, e_p)\} \\ \{(b_q, c_q)\}\,;\{b_q - c_q, c_q)\} \end{matrix} \right. \right),$$

where $\quad Re\left(\frac{3}{2} b_q - c_q\right) < 2.$

Proceeding as before and noting that $(\lambda + 2\gamma) = \dfrac{\lambda\left(\frac{1}{2}\lambda + 1\right)_\gamma}{\left(\frac{1}{2}\lambda\right)_\gamma}$ and using (2.5) and (2.1), we arrive at the result.

Similarly, using $F(1 - a_p + s : : h) = (1 - h)^{a_p - 1 - s}$, we have the summation

$$\sum_{\gamma=0}^{\infty} \frac{(-h)^\gamma}{\gamma\,!} H_{p,q}^{m,n}\left(x \left| \begin{matrix} \{a_{p-1}, e_{p-1})\}, (a_p - \gamma, 1) \\ \{(b_q, c_q\} \end{matrix} \right. \right)$$

(3.3)

$$= (1 - h)^{a_p - 1}\, H_{p,q}^{m,n}\left(\frac{x}{1-h} \left| \begin{matrix} \{(a_{p-1}, e_{p-1})\}, (a_p, 1) \\ \{(b_q, c_q)\} \end{matrix} \right. \right),$$

where $|h| < 1, |\arg x| < \frac{1}{2}\left(\sum_{1}^{n} e_j - \sum_{n+1}^{p} e_j + \sum_{1}^{m} c_j - \sum_{m+1}^{q} c_j\right)\pi.$

REFERENCES

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ÖZET

H-fonksiyonları cinsinden ifade edilmiş bir sonsuz serinin toplamı, H-fonksiyonlarını Mellin-Barnes tipi integraller olarak ifade etmek ve toplam ve integrasyon sıralarını değiştirmek suretiyle hesaplanmaktadır.