ELECTROMAGNETIC TENSOR FIELD, NIJENHUIS TENSOR (III)

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In this paper we have determined the conditions for satisfying the identities in Nijenhuis tensor, with respect to the electromagnetic tensor field in a four dimensional space-time.

1. Introduction. The electromagnetic field f is said to be of the first class if $Kk \neq 0$, and the relation

$$ {}^{(4)}X + 2K\, {}^{(2)}X + k\, X = 0 \tag{1.0} $$

is the characteristic equation for the first class.

There are 64 independent N's whose values contain the Lie brackets, K, k and the terms of the type ${}^{(r)}X\,({}^{(s)}Y . K)$. We could only obtain the identities in the N's which contain either the Lie brackets or the terms of the type $({}^{(r)}Y . K)\,{}^{(s)}X$, which is obvious because we have only one relationship between K and k. This induces us to find another relationship between K and k such that we may obtain the required identities (i. e. identities free from Lie brackets and the terms of the type $(({}^{(r)}Y . K)\,{}^{(s)}X)$. To obtain such conditions we let the right hand side of the obtained identities be equal to zero so that we have a system of equations. Finally these equations yield some conditions.

In recent papers [1], [2], the following identities in N's for the first class have been obtained :

$$ \begin{aligned} & {}^{(2)}N({}^{(3)}X, {}^{(3)}Y) + {}^{(3)}N({}^{(2)}X, {}^{(3)}Y) + {}^{(3)}N({}^{(3)}X, {}^{(2)}Y) - k\{{}^{(1)}N(X, {}^{(3)}Y) \\ & + {}^{(1)}N({}^{(3)}X, Y) + N({}^{(3)}X, {}^{(1)}Y) + N({}^{(1)}X, {}^{(3)}Y)\} \\ & = -2K\,{}^{(3)}[{}^{(2)}X, {}^{(3)}Y] - 2Kk\,{}^{(2)}[{}^{(2)}X, Y] - 2K\,{}^{(3)}[{}^{(3)}X, {}^{(2)}Y] + 2K\,{}^{(2)}[{}^{(3)}X, {}^{(3)}Y] \\ & + k\,[{}^{(3)}X, {}^{(3)}Y] - 4K^{2}\,{}^{(2)}[{}^{(2)}X, {}^{(3)}Y] - 2Kk\,{}^{(2)}[X, {}^{(2)}Y] - k\,{}^{(3)}[X, {}^{(2)}Y] \\ & - k\,{}^{(3)}[{}^{(3)}X, Y] - k^{2}\,{}^{(2)}[X, Y] \end{aligned} \tag{1.1} $$

$$ \begin{aligned} & {}^{(1)}N({}^{(3)}X, {}^{(3)}Y) + {}^{(2)}N({}^{(3)}X, {}^{(2)}Y) + {}^{(2)}N^{(2)}(X, {}^{(3)}Y) + {}^{(3)}N({}^{(1)}X, {}^{(3)}Y) + {}^{(3)}N({}^{(3)}X, {}^{(1)}Y) \\ & - k\{N(X, {}^{(3)}Y) + N({}^{(3)}X, Y)\} + {}^{(3)}N({}^{(2)}X, {}^{(2)}Y) + 2K\{{}^{(1)}N({}^{(1)}X, {}^{(2)}Y) + {}^{(1)}N({}^{(3)}X\ {}^{(1)}Y)\} \\ & = -4K^{2}\,{}^{(1)}[{}^{(2)}X, {}^{(2)}Y] - 2K\{{}^{(2)}[{}^{(2)}X, {}^{(3)}Y] + {}^{(2)}[{}^{(3)}X, {}^{(2)}Y] + {}^{(3)}[{}^{(2)}X, {}^{(2)}Y] \\ & - {}^{(2)}[{}^{(3)}X, {}^{(2)}Y] - {}^{(2)}[{}^{(2)}X, {}^{(3)}Y]\} - 2Kk\{{}^{(1)}[{}^{(2)}X, Y] + {}^{(1)}[X, {}^{(2)}Y]\} \\ & - k\{{}^{(2)}[X, {}^{(3)}Y] + {}^{(2)}[{}^{(3)}X, Y] + k\,{}^{(1)}[X, Y] + {}^{(1)}[{}^{(2)}X, {}^{(2)}Y] - [{}^{(3)}X, {}^{(2)}Y] - [{}^{(2)}X, {}^{(3)}Y]\} \end{aligned} \tag{1.2} $$

$$ \begin{aligned} & N({}^{(3)}X, Y) + {}^{(3)}N(X, Y) + {}^{(2)}N({}^{(1)}X, Y) + {}^{(1)}N({}^{(2)}X, Y) + 2K\{N({}^{(1)}X, Y) + {}^{(1)}N(X, Y)\} \\ & = -2(Y, K)\,{}^{(3)}X + 2\,({}^{(1)}Y, K)\,{}^{(1)}X - (Y, k)\,{}^{(1)}X + ({}^{(1)}Y, k)\,X \end{aligned} \tag{1.3} $$

$$ \begin{aligned} & N(X, {}^{(3)}X) + 2k\{N(X, {}^{(1)}Y) + {}^{(1)}N(X, Y)\} + {}^{(3)}N(X, Y) + {}^{(3)}N(X, {}^{(1)}Y) + {}^{(1)}N(X, {}^{(2)}Y) \\ & = 2(X, K)\,{}^{(3)}Y + (X, k)\,{}^{(1)}Y - 2\,({}^{(1)}X, K)\,{}^{(2)}Y - ({}^{(1)}X, k)\,Y \end{aligned} \tag{1.4} $$

$$
(1.5)\quad \begin{aligned} &^{(1)}N(^{(2)}X, Y) + 2K\,^{(1)}N(^{(1)}X, Y) + {}^{(2)}N(^{(2)}X, Y) + {}^{(3)}N(^{(1)}X, Y) - k\,N(X, Y) \\ &= 2(^{(1)}Y, K)\,^{(3)}X + \{4K(Y, K) - (Y, k)\}\,^{(3)}X + (^{(1)}Y, k)\,^{(1)}X + 2k\,(Y, K)\,X \end{aligned}
$$

$$
(1.6)\quad \begin{aligned} &^{(2)}N(^{(3)}X, Y) + {}^{(3)}N(^{(2)}X, Y) - k\,\{N(^{(1)}X, Y) + {}^{(1)}N(X, Y)\} \\ &= \{(4K(Y, K) - (Y, k)\}\,^{(3)}X + \{^{(1)}Y, k - 4K(^{(1)}Y, K)\}\,^{(2)}X + 2k\,(Y, K)\,^{(2)}X - 2k\,(^{(1)}Y, K)X \end{aligned}
$$

$$
(1.7)\quad \begin{aligned} &^{(3)}N(^{(3)\prime}X, Y) + 2K\,^{(1)}N(^{(3)}X, Y) + 2K\,^{(3)}N(^{(1)}X, Y) + (4K^2 - k)\,^{(1)}N(^{(1)}X, Y) \\ &- k\,\{2KN(X, Y) + {}^{(2)}N(X, Y) + N(^{(2)}X, Y)\} \\ &= (^{(1)}Y, k)\,^{(2)}X + 2k\,(Y, K)\,^{(2)}X + 2\,\{K(^{(1)}Y, k) - k\,(^{(1)}Y, K)\}\,^{(1)}X, + k\,(X, k) \end{aligned}
$$

$$
(1.8)\quad \begin{aligned} &K\,\{^{(3)}N(^{(2)}X, Y) + {}^{(2)}N(^{(3)}X, Y)\} + Kk\,\{^{(1)}N(X, Y) + N(^{(1)}X, Y)\} \\ &+ k\,\{^{(2)}N(^{(1)}X, Y) + {}^{(3)}N(X, Y) + N(^{(2)}X, Y) + {}^{(1)}N(^{(2)}X, Y)\} \\ &= [4K^2\,(Y, K) - K(Y, K) - 2k(Y, K)]\,^{(3)}X + k\,[2K(Y, K) - (Y, k)]\,^{(1)}X \\ &+ [K(^{(1)}Y, k) - 4K^2\,(^{(1)}Y, K) + 2k(^{(1)}Y, K)]\,^{(2)}X + k[^{(1)}Y, k - 2K\,(^{(1)}Y, K)]X. \end{aligned}
$$

2. We see that the identities contain either the LIE brackets or the terms of the type $(^{(r)}X, k)\,^{(s)}Y$. Further we note that K and k both occur in the right hand side of the above identities. This is obvious, since we have only one relationship (1.0) between K and k via X for first class. Thus with only one relationship between K and k we can't get the identities in N's for first class which are free from LIE brackets as well as of the terms of the type $(^{(r)}X.\,k)\,^{(s)}Y$.

As in the case of second class, we have a relationship which only involves K, viz. $^{(3)}X + 2K^{(1)}\ X = 0$, it was possible to obtain identities in the N's which are free from LIE brackets as well as the terms of the type $(^{(r)}X\,.\,K)^{(s)}Y$.

Thus to obtain the relationship between K and k via X which will make the identities in N's free from LIE brackets as well as the terms of the type $(^{(r)}X\,.\,K)^{(s)}\ Y$, we put the right hand side of the obtained identities equal to zero and get a system of equations that will finally yield the required conditions.

Considering the equations (1.3) and (1.4), we at once see that the required condition is :

$$
(2.0)\qquad 2^{(r+2)}\,X\,(^{(s)}Y\,.\,K) + {}^{(r)}X\,(^{(s)}Y\,,k) = 0\,.
$$

Similarly for equations (1.1) and (1.2)

$$
(2.1)\quad \begin{aligned} &2K^{(r+2)}\,[^{(s)}X\,,\,^{(t)}Y] + k^{(r)}\,[^{(r)}X\,,\,^{(t)}Y] = 0, \\ &2K^{(r)}\,[^{(s+2)}X\,,\,^{(t)}Y] + k^{(r)}\,[^{(s)}X\,,\,^{(t)}Y] = 0, \\ &2K^{(r)}\,[^{(s)}X\,,\,^{(t+2)}Y] + k^{(r)}\,[^{(1)}X\,,\,^{(t)}Y] = 0, \end{aligned}
$$

these make the right hand side of (1.1), (1.2), (1.3) and (1.4) zero. Putting (2.0) and (2.1) together we can say that :

«If we replace - k by $2K$ and apply the f- operation twice either on LIE brackets or on X or on Y such that the total number of f- operations on these does not exceed three then the right hand side of (1.1) - (1.4) is equal to zero and we get the identities in N's which are free from LIE brackets as well as the terms of the type $(^{(r)}X.K)\,^{(s)}Y$ ».

Putting the right hand side of the equations (1.5), (1.6), (1.7) and (1.8) equal to zero, we get :

$$(2.2)\qquad 2\,({}^{(1)}Y.K)\,{}^{(3)}X + \{(4K\,(Y.K) - Y.k\}\,{}^{(2)}X + ({}^{(1)}Y.k){}^{(1)}X + 2k\,(Y.K)\,X = 0,$$

$$(2.3)\qquad \{4K(Y.K) - Y.k\}\,{}^{(3)}X + \{{}^{(1)}(Y.k) - 4K({}^{(1)}Y.K)\}\,{}^{(2)}X + 2k(Y.K){}^{(1)}X - 2k({}^{(1)}Y.K)X = 0,$$

$$(2.4)\qquad ({}^{(1)}Y.k)\,{}^{(3)}X + 2k\,(Y.K)\,{}^{(2)}X + 2\,\{K({}^{(1)}Y.k) - K({}^{(1)}Y,K)\}\,{}^{(1)}X + k(Y.k)X = 0,$$

$$(2.5)\qquad \{4K^2\,(Y.K) - K(Y.K) - 2k(Y.K)\}\,{}^{(3)}X + \{K({}^{(1)}Y.k) - 4K^2\,({}^{(1)}Y.K) + 2k\,({}^{(1)}Y.K)\}\,{}^{(2)}X$$
$$+\, k\,\{2K(Y.K) - Y.k\}\,{}^{(1)}X + k\,\{{}^{(1)}Y\,.\,k) - 2K\,({}^{(1)}Y\,.\,K)\}\,X = 0.$$

The condition that these four equations be consistent is given by the following equation :

$$(2.6)\qquad \begin{bmatrix} 2({}^{(1)}Y.K) & 4K(Y.K) - Y.k & {}^{(1)}Y.k & 2k(Y.K) \\ 4k(Y.K) - Y.k & {}^{(1)}Y.k - 4K({}^{(1)}Y.K) & 2k(Y.K) & -2k({}^{(1)}Y.K) \\ {}^{(1)}Y.k & 2Y(y.K) & \begin{matrix} 2K({}^{(1)}Y.k) \\ -2k({}^{(1)}Y.K) \end{matrix} & k(Y.k) \\ \begin{matrix} 4K^2(Y.K) \\ -K(Y.K) - 2k(Y.K) \end{matrix} & \begin{matrix} K({}^{(1)}Y.k) + 2k({}^{(1)}Y.K) \\ -4K^2\,({}^{(1)}Y.\,K) \end{matrix} & \begin{matrix} 2Kk(Y.K) \\ -k(Y.k) \end{matrix} & \begin{matrix} k({}^{(1)}Y.k) \\ -2Kk({}^{(1)}Y.K) \end{matrix} \end{bmatrix} = 0$$

3. Discussion. The equation (2.6) gives the required condition for the identities (1.5) - (1.8) to be free from the LIE brackets as well as the terms of the type $({}^{(r)}X.K){}^{(s)}Y$. We notice that there can be in all 24 such conditions which we shall not mention here for want of space. Also the equation (2.0) can take six different values.

REFERENCES

[1] MISHRA, R. S. : *Electromagnetic tensor field, Nijenhuis tensor* (under publication) (1969).

[2] MISHRA, R. S. and POKHARIYAL G.P. : *Electromagnetic tensor field, Nijenhuis tensor* (II), (1969).

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ÖZET

Bu araştırmada dört boyutlu uzay-zaman kontinuumundaki elektromanyetik tensör alanına göre NIJENHUIS tensörünün bâzı özdeşlikler gerçeklemesi için gerek şartlar belirtilmiştir.