CERTAIN THEOREMS ON THE GENERALIZED FOURIER TRANSFORM

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A particular kernel, first considered by BHATNAGAR [2] as a generalisation of the FOURIER transform discussed by WATSON [1] is examined and various of its properties as well as the differential equation it satisfies are given.

1. Introduction. The function $\tilde{\omega}_{\mu,\nu}(x)$ was defined by G. N. WATSON [2, (i)] in 1931 by the integral relation [1])

$$\tilde{\omega}_{\mu,\nu}(x) = x^{\frac{1}{2}} \int_0^\infty J_\mu(t)\, J_\nu\left(\frac{x}{t}\right) t^{-1}\, dt$$

$$= \frac{x^{\nu+\frac{1}{2}}\, 2^{-2\nu-1}\, \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu\right)}{\Gamma(\nu+1)\, \Gamma\left(1+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\; F_3\left(\nu+1,\; 1-\frac{\mu}{2}+\frac{\nu}{2}, 1+\frac{\mu}{2}+\frac{\nu}{2}\,;\, \frac{x^2}{16}\right)$$

+ another term with μ and ν interchanged ;

$$-R\left(\mu+\frac{3}{2}\right) < 0 < R\left(\nu+\frac{2}{3}\right).$$

He showed (without proof) that it is a symmetric FOURIER kernel.

Later K. P. BHATNAGAR [2, (i), (ii)] in 1953 and 1954 investigated in some detail the properties of this kernel and extended it to n parameters and defined

$$\tilde{\omega}_{\mu_1,\dots,\mu_n}(x) = x^{\frac{1}{2}} \int_0^\infty \dots \int_0^\infty J_{\mu_1}(t) \dots J_{\mu_{n-1}}(t_{n-1})$$

$$\times J_{\mu_n}\left(\frac{x}{t_1 \dots t_{n-1}}\right)(t_1 \dots t_{n-1})^{-1}\, dt_1 \dots dt_{n-1}$$

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1) The integral $\int_0^\infty J_\mu(t)\, J_\nu\left(\frac{x}{t}\right) t^p\, dt$ was originally evaluated by C. V. H. RAO. See Messenger of Maths., 47, (1918), 134-7. Also see Bessel Functions by WATSON, (1922) 437.

$$= \int_0^\infty \tilde{\omega}_{\mu_1, \ldots, \mu_{n-1}}\left(\frac{x}{t}\right) J_{\mu_n}(t)^{-\frac{1}{2}}\, dt,$$

where $R\left(\mu_K + \frac{1}{2}\right) \geq 0$, $(K = 1, 2, \ldots, n)$ and the μ's may be permuted among themselves. In 1961, CHARLES FOX [8] showed that the G-Function as defined by C. S. MEIJER [4] is a symmetric FOURIER kernel. For certain values of the parameters the kernel degenerates into the kernels of BHATNAGAR, but he did not investigate the properties of this kernel. In the present paper, the author has proved certain properties involving the generalized transforms. The following results can be proved easily :

(1) $\tilde{\omega}_\mu(x) = \sqrt{x}\, J_\mu(x)$, $\tilde{\omega}_{\mu,\mu+1}(x) = J_{2\mu+1}(2\sqrt{x})$, $R(\mu) > -1$.

(2) $\tilde{\omega}_{\mu_1, \ldots, \mu_n}(x) = 0(x^{\mu_r + \frac{1}{2}})$, $r = 1, 2, \ldots, n$ for small x,

$$= x^{\frac{1-n}{2n}} [\cos (2nx^{\frac{1}{n}} + \alpha)(A + 0(x^{-\frac{2}{n}})) + \sin (2nx^{\frac{1}{n}} + \alpha)\, 0(x^{\frac{1}{n}})],$$

(3) $$\tilde{\omega}_{\mu_1, \ldots, \mu_r}(x) = \int_0^\infty \tilde{\omega}_{\mu_1, \ldots, \mu_{n-1}}(xt)\, J_{\mu_n}(t^{-1})\, t^{-\frac{3}{2}}\, dt$$

$$= \int_0^\infty \tilde{\omega}_{\mu_1, \ldots, \mu_r}(xt)\, \tilde{\omega}_{\mu_{r+1}, \ldots, \mu_n}(t^{-1}) \frac{dt}{t}, \quad R(\mu_r) > -1,\ r = 1, 2, \ldots, n.$$

(4) $\tilde{\omega}_{v-1, v, \mu-1, \mu}(x) = x^{-\frac{1}{4}}\, \tilde{\omega}_{2v-1, 2\mu-1}(2^2 x^{\frac{1}{2}})$,

(5) $\tilde{\omega}_{v - \frac{\mu}{3}, v - \frac{2}{3}, v}(x) = x^{-\frac{1}{6}} J_{2v-2}(3x^{\frac{1}{3}})$,

(6) $\tilde{\omega}_{v - \frac{2(n-1)}{n}, \ldots, v - \frac{2}{v}, v}(x) = x^{\frac{1}{n} - \frac{1}{2}} J_{nv-n+1}(nx^{\frac{1}{n}})$,

(7) $\tilde{\omega}_{\mu_1, \mu_1 - 1, \ldots, \mu_n - \mu_{n-1}}(x) = 2^{\frac{n}{2} - 1} x^{-\frac{1}{4}}\, \tilde{\omega}_{2\mu_1 - 1, \ldots, 2\mu_{n-1}}(2^n x^{\frac{1}{2}})$,

(8) $2\mu \frac{d}{dz} [z^{v - \frac{1}{2}}\, \tilde{\omega}_{\mu,v}(z)] = z^{v - \frac{1}{2}} [\tilde{\omega}_{\mu-1, v-1}(z) + \tilde{\omega}_{\mu+1, v-1}(z)]$,

(9) $2\mu \frac{dz}{d} [z^{-v - \frac{1}{2}}\, \tilde{\omega}_{\mu,v}(2)] = -z^{-v - \frac{1}{2}} [\tilde{\omega}_{\mu+1, v+1}(z) + \tilde{\omega}_{\mu-1, v+1}(z)]$,

(10) $\sum_{n=0}^{\infty} \varepsilon_n \tilde{\omega}_{2n, \mu_1, \ldots, \mu_p}(x) = \frac{x^{\frac{1}{2}}}{\mu_1 \ldots \mu_p}$, $R(\mu_r) > 0$, $r = 1, 2, \ldots, p$.

$$(11)\quad \sum_{n=0}^{\infty} n^2\, \tilde{\omega}_{2n,\mu_1,\ldots,\mu_p}(x) = \frac{x^{\frac{5}{2}}}{8\mu_1(\mu_p^{\,2}-4)\ldots\mu_p(\mu_p^{\,2}-4)},\quad R(\mu_r) < 2,\ r = 1, 2, 3, \ldots, p.$$

$$(12)\quad \sum_{n=0}^{\infty} (2n+1)\, \tilde{\omega}_{2n+1,\mu_1,\ldots,\mu_p}(x) = \frac{x^{\frac{3}{2}}}{2(\mu_1^{\,2}-1)\ldots(\mu_p^{\,2}-1)},\quad R(\mu_r) > 1,\ r = 1, 2, \ldots, p.$$

$$(13)\quad \sum_{n=0}^{\infty} (2n)(2n+1)(2n+2)\tilde{\omega}_{2n+1\,\mu_1,\ldots,\mu_p}(x)$$

$$= \frac{x^{\frac{7}{2}}}{2(\mu_1^{\,2}-1)\ldots(\mu_p^{\,2}-1)(\mu_1^{\,2}-9)\ldots(\mu_p-9)},\quad R(\mu_r) < 3,\ r = 1, 2, \ldots, p.$$

$$(14)\quad \sum_{n=0}^{\infty} (2n+1)^3\, \tilde{\omega}_{2n+1,\mu_1,\ldots,\mu_p}(x)$$

$$= \frac{x^{\frac{3}{2}}}{2(\mu_1^2-1)\ldots(\mu_p^2-1)}\left[1 + \frac{x^2}{(\mu_1^2-9)\ldots(\mu_p^2-9)}\right],\quad R(\mu_r) > 3.$$

$$(15)\quad \sum_{n=0}^{\infty} \frac{(m+2n)\,\Gamma(m+n)}{n!}\, \tilde{\omega}_{m+2n,\mu_1,\ldots,\mu_p}(x)$$

$$= \frac{\Gamma\left(\frac{\mu_1-m}{2}\right)\ldots\Gamma\left(\frac{\mu_p-m}{2}\right)x^{m+\frac{1}{2}}}{2^{mp+p+m}\,\Gamma\left(\frac{\mu_1+m}{2}+1\right)\ldots\Gamma\left(\frac{\mu_p+m}{2}+1\right)},\quad R(\mu_r) > m,\ r = 1, 2, \ldots, p,\ m = 1, 2, 3, \ldots$$

The infinite series above are convergent as can be seen easily by taking the asymptotic values for large n.

The MELLIN transform of $\tilde{\omega}_{\mu_1,\ldots,\mu_n}(x)$ is

$$(16)\qquad 2^{n\left(s-\frac{1}{2}\right)}\ \frac{\Gamma\left(\frac{\mu_1}{2}+\frac{s}{2}+\frac{1}{4}\right)\ldots\Gamma\left(\frac{\mu_n}{2}+\frac{s}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu_1}{2}-\frac{s}{2}+\frac{3}{4}\right)\ldots\Gamma\left(\frac{\mu_n}{2}-\frac{s}{2}-\frac{3}{4}\right)}.$$

2. **Theorem 1.** *If* $1 \leq p \leq 2$, $j = 1, 2$; $f_j(t) \in L_p(0, \infty)$ *and*

$$(2.1) \qquad F_j(x) = \int_0^\infty \tilde{\omega}_{\mu_1,\ldots,\mu_n}(xy) f_j(y)\, dy$$

then

$$(2.2) \qquad \frac{1}{r}\int_0^\infty \left(\frac{x}{r}\right) f_2(sx)\, dx = \frac{1}{s}\int_0^\infty F_2\left(\frac{x}{s}\right) f_1(rx)\, dx$$

provided the integrals exist and $f_j(t)$ *is continuous.*

Proof. Here

$$r^{-1}\int_0^\infty F_1\left(\frac{x}{r}\right) f_2(sx)\, dx = \int_0^\infty f_2(rs\mu)\, F_1(\mu)\, d\mu$$

$$= \int_0^\infty f_2(rs\mu) \int_0^\infty \tilde{\omega}_{\mu_1,\ldots,\mu_n}(\mu y) f_1(y)\, dy\, d\mu$$

$$= \int_0^\infty f_1(y)\, dy \int_0^\infty \tilde{\omega}_{\mu_1,\ldots,\mu_n}(\mu y) f_2(rs\mu)\, d\mu$$

$$= (rs)^{-1}\int_0^\infty f_1(y)\, F_2\left(\frac{y}{rs}\right) dy$$

$$= \frac{1}{s}\int_0^\infty f_1(rx)\, F_2\left(\frac{x}{s}\right) dx$$

which proves the theorem.

Assuming that the integrals in (2.1) are absolutely convergent, the change in order of integration will be justified, if the integrals in (2.2) are absolutely convergent.

In particular if we take $r = s = 1$, then we arrive at MITRA's theorem [5] as

$$\int_0^\infty F_1(x) f_2(x)\, dx = \int_0^\infty f_1(x) F_2(x)\, dx$$

which may be called the *Parseval theorem* for the generalized transform $\tilde{\omega}_{\mu_1,\ldots,\mu_n}(x)$. Result (3) can be proved by an application of PARSEVAL's theorem.

3. **Theorem 2.** *Let*

(*i*) $f(x) \doteqdot \Phi(p)$

(*ii*) $t^m f(t)$ *be* $R_{\mu_1,\ldots,\mu_n}$;

then $x^{\lambda+m} f(x) \doteqdot \frac{p^{1-\lambda-m}}{2^{1\lambda-2m-\frac{n}{2}}} \int_0^\infty t^{-m}\Phi(t)$

$$\times G_{2n}^{n2}\left(\frac{4^{2-n}}{p^2t^2}\middle| \begin{matrix} 1-\frac{\lambda+m}{2}, & \frac{1}{2}-\frac{\lambda+m}{2}, \frac{2-m}{2}, \frac{3-m}{2} \\ \frac{\mu_1+3}{2},\ldots & \ldots \frac{\mu_n+3}{2}, \frac{3}{4}-\frac{\mu_1}{2},\ldots,\frac{3}{4}-\frac{\mu_n}{2} \end{matrix}\right) dt \tag{3.1}$$

provided $f(x)$, $x^m f(x)$ *and* $x^{\lambda+2m} f(x)$ *are continuous and obsolutely integrable in* $(0, \infty)$ *ond*

$$R(\mu_1,\ldots,\mu_n) \geqq -\frac{1}{2},\ R(\nu_1,\ldots,\nu_n) \geqq -\frac{1}{2},\ R(\lambda+2) > 0.$$

Proof. Let $p^m \tilde{\omega}_{\mu_1,\ldots,\mu_n}(p) \doteqdot F(x)$, then

$$x^\lambda F\left(\frac{1}{x}\right) \doteqdot p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{px})\, x^{\frac{\lambda-1}{2}+m}\, \tilde{\omega}_{\mu_1,\ldots,\mu_n}(x)\, dx$$

$$R\left(\mu_r+\lambda+m+\frac{3}{2}\right) > 0,\quad R\left(\nu_r+\lambda+m+\frac{3}{2}\right) > 0,\quad R(\lambda+2) > 0,$$

$$R(1-\lambda-2m) > 0.$$

or,

$$x^\lambda F\left(\frac{1}{x}\right) \doteqdot \frac{p^{1-m-\lambda}}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\Gamma(\lambda+m+s)}{\Gamma(-m-s+2)}\, 2^{n\left(\frac{1}{2}-s\right)} \frac{\Gamma\left(\frac{\mu_1-s}{2}+\frac{3}{4}\right)\cdots}{\Gamma\left(\frac{\mu_1+s}{2}+\frac{1}{4}\right)\cdots}$$

$$\times \frac{\cdots\Gamma\left(\frac{\mu_n-s}{2}+\frac{3}{4}\right)}{\cdots\Gamma\left(\frac{\mu_n+s}{2}+\frac{1}{4}\right)} p^{-s}\, de \tag{3.2}$$

or,

$$x^\lambda F\left(\frac{1}{x}\right) \doteqdot \frac{p^{1-m-\lambda}}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\lambda+2m-2+\frac{n}{2}}$$

$$\times \frac{\Gamma\left(\frac{\lambda+m+s}{2}\right)\Gamma\left(\frac{\lambda+m+s+1}{2}\right)\Gamma\left(\frac{\mu_1-s}{2}+\frac{3}{\mu}\right)\cdots\Gamma\left(\frac{\mu_n-s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{2-m-s}{2}\right)\Gamma\left(\frac{s-m-3}{2}\right)\Gamma\left(\frac{\mu_1+s}{2}+\frac{1}{4}\right)\quad\Gamma\left(\frac{\mu_n-s}{2}+\frac{1}{4}\right)} 2^{s(2-n)} p^{-s}\, ds$$

or,

$$x^\lambda F\left(\frac{1}{x}\right) \doteqdot \frac{2^{\lambda+2m+\frac{n}{2}-1}}{p^{m+\lambda-1}}$$

$$\times G_{2n}^{n2}\left(\frac{2^{2(2-n)}}{p^2}\middle| \begin{matrix} 1-\frac{\lambda+m}{2}, \frac{1}{2}-\frac{\lambda+m}{2}, \frac{2-m}{2}, \frac{3-m}{2} \\ \frac{\mu_1}{2}+\frac{3}{4},\ldots,\frac{\mu_n}{2}+\frac{3}{4}, \quad, \frac{3}{4}-\frac{\mu_1}{2},\ldots,\frac{3}{4}-\frac{\mu_n}{2} \end{matrix}\right) \tag{3.3}$$

(3.4) Let (*i*) $(ap)^m \tilde{\omega}_{\mu_1, \ldots, \mu_n}(ap) \doteqdot F\left(\frac{x}{a}\right)$

(3.5) (*ii*) $\Phi(p) \doteqdot f(x)$.

Applying GOLDSTEIN's theorem to (3.4) and (3.5), we get

$$\int_0^\infty f(t)\,(at)^m\,\tilde{\omega}_{\mu_1,\ldots,\mu_n}(at)\,\frac{dt}{t} = \int_0^\infty \Phi(t)\,F\left(\frac{t}{a}\right)\frac{dt}{t}. \tag{3.6}$$

On writing x for a and multiplying both sides by x^λ, it follows

$$x^\lambda \int_0^\infty f(t)\,(xt)^m\,\tilde{\omega}_{\mu_1,\ldots,\mu_n}(xt)\,\frac{dt}{t} = \int_0^\infty \Phi(t)\,F\left(\frac{t}{x}\right)\left(\frac{x}{t}\right)^\lambda t^{\lambda-1}\,dt\,.$$

Interpreting with the help of (3.3), we get

$$x^{\lambda+m}\int_0^\infty f(t)\,t^m\,\tilde{\omega}_{\mu_1,\ldots,\mu_n}(xt) \doteqdot \frac{2^{\lambda+2m+\frac{n}{2}-1}}{p^{m+\lambda-1}}\int_0^\infty t^{-m}\,\Phi(t)$$

$$\times\, G^{n\,2}_{n\,2}\left(\frac{4^{2-n}}{p^2 t^2}\middle|\begin{matrix} 1-\frac{\lambda+m}{2},\ \frac{1}{2}-\frac{\lambda+m}{2},\ \frac{2-m}{2},\ \frac{3-m}{2} \\ \frac{\mu_1+3}{2},\ldots,\frac{\mu_n+3}{2},\ \frac{3}{4}-\frac{\mu_1}{2},\ldots,\frac{3}{4}-\frac{\mu_n}{2} \end{matrix}\right) \tag{3.7}$$

By using the property that $t^m f(t)$ is $R_{\mu_1,\ldots,\mu_n}$, we get the desired theorem.

Corollary. In particular if we take $n = 2$, $m = \mu - \lambda - \frac{1}{2}$, we arrive at the following result :

$$x^{2\mu-\lambda} f(x) \doteqdot \frac{2^{2\mu-\lambda+1}\,\Gamma\left(\mu+\frac{3}{2}\right)\,\Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\lambda+3}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+2\right)\,\Gamma\left(\frac{\nu-\mu-1}{2}\right)}\,p^2\int_0^\infty t^{\lambda+1}\,\Phi(t)$$

$$\times\, {}_3F_2\left[\begin{matrix}\mu+\frac{3}{2},\ \frac{\mu+\nu+3}{2},\ \frac{\nu-\mu+3}{2}\,; \\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2\,;\end{matrix}\ p^2 t^2\right] dt \tag{3.8}$$

where $t^{\mu-\lambda-\frac{1}{2}} f(t)$ is $R_{\mu,\nu}$ and $\Phi(p)$ is the operational image of $f(x)$.

Examples. Suppose $t^{\frac{\mu-\nu+1}{2}} J_{\frac{\mu+\nu}{2}}(t)$ is $R_{\mu,\nu}$; then from (3.8), we get

$$\int_0^\infty {}_2F_1\left(\begin{matrix}\frac{\lambda}{2}+1,\ \frac{\lambda+3}{2}\,; \\ \frac{\mu+\nu+2}{2}\,;\end{matrix}\ -\frac{1}{t^2}\right){}_3F_2\left(\begin{matrix}\mu+\frac{3}{2},\ \frac{\mu-\nu+3}{2}, \\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2\,;\end{matrix}\ t^2p^2\right)dt$$

$$= \frac{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{\nu-\mu-1}{2}\right)\Gamma(2\mu+2)}{2^{2\mu-\lambda+1}\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)\Gamma(\lambda+2)p^{2\mu+3}}\ {}_2F_1\left[\begin{matrix}\mu+1,\ \mu+\frac{3}{2}\ ; \\ \frac{\mu+\nu+2}{2}\ ;\end{matrix}\quad -\frac{1}{p^2}\right],$$

(3.9) $\qquad R(\lambda) > -3,\ R(\mu) > -1,\ R(\mu \pm \nu)\ \nu > 2,$

(2) Suppose $t^{n+\mu+\frac{1}{2}} K_n(t)$ is $R_{\mu,\nu}$ where $\nu = \mu + 2n$; then from (3.8), we get

$$\int_0^\infty \frac{t^{\lambda-1}}{(t^2-1)^{\frac{\lambda+n+2}{2}}} Q^n_{n+\lambda+1}\left(\frac{t}{\sqrt{t^2-1}}\right) {}_3F_2\left(\begin{matrix}\mu+\frac{3}{2},\ \mu+n+\frac{3}{2},\ n+\frac{3}{2}\ ; \\ \frac{\lambda+3}{2}\ ,\ \frac{\lambda}{2}+2\ ;\end{matrix}\quad p^2 t^2\right) dt$$

$$= \frac{\Gamma(2\mu+2)\,\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{2n-1}{2}\right)\sin[(2\mu+n+1)\mu]\sin[(2n+\mu+1)\mu]}{2^{2\mu-\lambda+1}\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\mu+n+\frac{3}{2}\right)\Gamma(\lambda+2)\sin[(2\mu+2n+1)\mu\sin[n+\lambda+1)\mu]}$$

(3.10) $$\times \frac{1}{p^2(p^2-1)^{\mu+1+\frac{n}{2}}} Q^n_{2\mu+n+1}\left(\frac{p}{\sqrt{p^2-1}}\right);$$

$$R(\mu+n) > -\frac{3}{2},\ R(\lambda+2) > 0,\ R(\mu+2n) > -\frac{3}{2},\ 0 > R(n) > -\frac{3}{4}\cdot$$

4. Differential equation satisfied by $\tilde{\omega}_{\mu_1,\ldots,\mu_n}(x)$

From result (16), we have

$$\tilde{\omega}_{\mu_1,\ldots,\mu_n}(x) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{2^{n\left(s-\frac{1}{2}\right)}\Gamma\left(\frac{\mu_1}{2}+\frac{1}{4}+\frac{s}{2}\right)\cdots\Gamma\left(\frac{\mu_n}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu_1}{2}-\frac{3}{4}+\frac{s}{2}\right)\cdots\Gamma\left(\frac{\mu_n}{2}+\frac{3}{4}-\frac{s}{2}\right)x^s}\, ds$$

(4.1) $$-R(\mu_1,\ldots,\mu_n)-\frac{1}{2} < C < \frac{1}{2}-\frac{1}{n}$$

Now let $y = x^{-\frac{1}{2}}\tilde{\omega}_{\mu_1,\ldots,\mu_n}(x)$ and $x\frac{d}{dx} = D$.

Therefore $$(\mu_1{}^2 - D^2)\, y = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} 2^{n\left(s-\frac{1}{2}\right)+2}\, x^{-s-\frac{1}{2}}$$

$$\times \frac{\Gamma\left(\frac{\mu_1}{2}+\frac{5}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\mu_2}{2}+\frac{1}{4}+\frac{s}{2}\right)\ldots\Gamma\left(\frac{\mu_n}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu_1}{2}-\frac{1}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\mu_1}{2}+\frac{3}{4}-\frac{s}{2}\right)\ldots\Gamma\left(\frac{\mu_n}{2}+\frac{3}{4}-\frac{s}{2}\right)}\, ds$$

Repeating the same process, where differentiation under the sign of integration is justifiable, we obtain

$$(\mu_1^2 - D^2)(\mu_2^2 - D^2)\dots(\mu_n^2 - D^2)\,y$$

$$= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{2^{n\left(s-\frac{1}{2}\right)+2n}\,\Gamma\left(\frac{\mu_1}{2}+\frac{5}{4}+\frac{s}{2}\right)\dots\Gamma\left(\frac{\mu_n}{2}+\frac{5}{4}+\frac{s}{2}\right)}{x^{s+\frac{1}{2}}\,\Gamma\left(\frac{\mu_1}{2}-\frac{1}{4}-\frac{s}{2}\right)\dots\Gamma\left(\frac{\mu_n}{2}-\frac{1}{4}-\frac{s}{2}\right)}\,ds$$

$$= x^2\,\frac{1}{2\pi i}\int_{C+2-i\infty}^{C+2+i\infty} \frac{2^{n\left(s-\frac{1}{2}\right)}\,\Gamma\left(\left(\frac{\mu_1}{2}\right)+\frac{1}{4}+\frac{s}{2}\right)\cdots\Gamma\left(\frac{\mu_n}{2}+\frac{1}{4}+\frac{s}{2}\right)}{x^{\frac{1}{2}+s}\,\Gamma\left(\frac{\mu_1}{2}+\frac{3}{\mu}-\frac{s}{2}\right)\cdots\Gamma\left(\frac{\mu_n}{2}+\frac{3}{4}-\frac{s}{2}\right)}\,ds,$$

(4.2) $$s = \sigma + it, - R(\mu_1, \dots, \mu_n) - \frac{1}{2} < c < \frac{1}{2} - \frac{1}{n}$$

There is no pole of the integrand between the lines $\sigma = c$ and $\mu = c + 2$. Hence shifting the line of integration from $\sigma = c + 2$ to $\mu = c$ from (4.1) and (4.2), we obtain

(4.3) $$(\mu_1^2 - D^2)(\mu_2^2 - D^2)\cdots(\mu_n^2 - D^2)\,y = x^2 y$$

which is the differential equation satisfied by

$$x^{+1/2}\,\tilde{\omega}_{\mu_1, \dots, \mu_n}(x).$$

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ÖZET

Bu travayda WATSON'un [1] incelemiş olduğu FOURIER dönüştürülmüşünün BHATNAGAR [2] tarafından verilen bir genelleştirilmişi incelenmekte ve bu çekirdeğin birçok özelikleri ile sağladığı diferansiyel denklem elde edilmektedir.

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