ON THE MEAN VALUES OF ENTIRE FUNCTIONS OF SLOW GROWTH REPRESENTED BY DIRICHLET SERIES

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Particular types of means are defined for an entire function expressed by means of a DIRICHLET series. Results concerning limits of these means in connection with the logarithmic and lower-logarithmic orders of these functions are obtained.

1. Introduction. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, where $\{a_n\}$ is a sequence of complex numbers, $s = \sigma + it$, $\lambda_1 \geq 0$, $\lambda_{n+1} < \lambda_n$ and $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$ represents an entire function. Set $M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|$ and $\mu(\sigma) = \max_{n\geq 1} \{|a_n| e^{\sigma\lambda_n}\}$ denote the maximum term in the expansion of $f(s)$. We shall always take $D = 0$. It is well known [1], that for functions of finite RITT-order ϱ,

$$\log M(\sigma) \sim \log \mu(\sigma). \tag{1.1}$$

The means of $f(s)$ are defined by

$$\nu_k(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^k \, dt, \; 0 < k < -\infty, \tag{1.2}$$

$$m_{\delta,k}(\sigma) = \frac{1}{e^{\delta\sigma}} \int_0^{\sigma} V_k(x)\, e^{\delta x}\, dx \tag{1.3}$$

$$= \lim_{T\to\infty} \frac{1}{2T e^{\delta\sigma}} \int_0^{\sigma} \int_{-T}^{T} |f(x+it)|^k e^{\delta x}\, dx\; dt,$$

$$(0 < \delta,\; k < \infty).$$

Similarly we define mean values of $f^{(m)}(s)$, $(m \geq 1)$, the m-th derivative of $f(s)$.

Now let the RITT-order of $f(s)$ be zero. We define the logarithmic order and lower-logarithmic order [4], by

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$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log \sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m(\sigma)}{\log \sigma} = \frac{\bar{\varrho}}{\bar{\lambda}}. \tag{1.4}$$

Correspondingly, we define the mean values of $f(s)$ of zero order by (1.2) and

$$\nu_{\delta,k}(\sigma) = \frac{1}{\sigma^{\delta}} \int_0^{\sigma} \nu_k(x)\, x^{\delta-1}\, dx\,,\ 1 < \delta < \infty$$

$$= \lim_{T\to\infty} \frac{1}{2T\sigma^{\delta}} \int_0^{\sigma}\int_{-T}^{T} |f(x+it)|^k x^{\delta-1}\, dx\, dt. \tag{1.5}$$

Similarly for $f^{(m)}(s)$, $(m \geq 1)$ we define

$$\nu_k(\sigma, f^{(m)}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f^{(m)}(\sigma+it)|^k\, dt,\ 0 < k < \infty, \tag{1.6}$$

$$V_{\delta,k}(\sigma, f^{(m)}) = \frac{1}{\sigma^{\delta}} \int_0^{\sigma} \nu_k(x, f^{(m)})\, x^{\delta-1}\, dx,\ 1 < k < \infty \tag{1.7}$$

$$= \lim_{T\to\infty} \frac{1}{2T\sigma^{\delta}} \int_0^{\sigma}\int_{-T}^{T} |f^{(m)}(x+it)|^k x^{\delta-1} dx\, dt.$$

In this paper we study some properties of $\nu_k(\sigma)$ and $\nu_{\delta,k}(\sigma)$ etc. for $k = 2$. We shall denote $\nu_k(\sigma)$ by $\nu(\sigma)$ and $\nu_{\delta,k}(\sigma)$ by $\nu_\delta(\sigma)$ for $k = 2$. The functions $\nu(\sigma, f^{(m)})$ and $\nu_\delta(\sigma, f^{(m)})$ carry the same meaning for $f^{(m)}(s)$.

2. Lemmas.

Lemma 1. *$\sigma^{\delta}\nu(\sigma)$ is a convex function of $\nu_\delta(\sigma)\sigma^{\delta}$.*

Proof. We have:

$$\frac{d(\sigma^{\delta}\nu(\sigma))}{d(\sigma^{\delta}\nu_\delta(\sigma))} = \frac{\frac{d}{d\sigma}(\sigma^{\delta}\nu(\sigma))}{\frac{d}{d\sigma}(\sigma^{\delta}\nu_\delta(\sigma))}$$

$$= \frac{\delta\,\sigma^{\delta-1}\nu(\sigma) + \sigma_\delta \frac{d\nu(\sigma)}{d\sigma}}{\sigma^{\delta-1}\nu(\sigma)}$$

$$= \delta + \sigma\,\frac{d\log\nu(\sigma)}{d\sigma}.$$

Since $\log \nu(\sigma)$ is a convex function of σ, [3], it follows that $\frac{d \log \nu(\sigma)}{d\sigma}$ and therefore $\sigma \frac{d \log \nu(\sigma)}{d\sigma}$ is positive increasing. Hence $\sigma^{\delta} \nu(\sigma)$ is a convex function of $\sigma^{\delta} \nu_{\delta}(\sigma)$.

Lemma 2. *$\nu_{\delta}(\sigma)$ increases with $\log \sigma$ and $\log \nu_{\delta}(\sigma)$ is a convex function of $\log \sigma$, for $\sigma > \sigma_0$.*

Proof. We have, by the definition of $\nu_{\delta}(\sigma)$,

$$\frac{d \log \nu_{\delta}(\sigma)}{d \log \sigma} = \sigma \frac{d}{d\sigma} \left[\log \left\{ \int_0^{\sigma} \nu(x) \,.\, x^{\delta-1} dx \right\} - \delta \log \sigma \right]$$

$$= \frac{\nu(\sigma)}{\nu_{\delta}(\sigma)} - \delta \,.$$

Since $\sigma^{\delta} \nu(\sigma)$ is a convex function of $\sigma^{\delta} \nu_{\delta}(\sigma)$, the right hand side above is a positive, indefinitely increasing function of σ for $\sigma > \sigma_0$ (say.) Hence

$$\frac{d^2 \log \nu_{\delta}(\sigma)}{d (\log \sigma)^2} = \sigma \frac{d}{d\sigma} \left(\frac{\nu(\sigma)}{\nu_{\delta}(\sigma)} \right)$$

$$> 0 \text{ for } \sigma > \sigma_0 \,.$$

Hence $\log \nu_{\delta}(\sigma)$ is a convex increasing function of $\log \sigma$ and lemma 2 follows.

Lemma 3. $\lim\limits_{\sigma \to \infty} \dfrac{\log \nu_{\delta}(\sigma)}{\sigma} = \infty.$

Proof. Since $\nu(\sigma)$ is always a positive and increasing function of σ, we have

$$\nu_{\delta}(\sigma) = \frac{1}{\sigma^{\delta}} \int_0^{\sigma} \nu(x)\, x^{\delta-1} dx$$

$$\geq \frac{1}{\sigma^{\delta}} \int_{\sigma/2}^{\sigma} \nu(x)\, x^{\delta-1} dx$$

$$\geq \frac{\nu(\sigma/2)}{\delta} \left[1 - \left(\frac{1}{2}\right)^{\delta} \right].$$

Since $\lim\limits_{\sigma \to \infty} \dfrac{\log \nu(\sigma)}{\sigma} = \infty$, we have $\lim\limits_{\sigma \to \infty} \dfrac{\log \nu_{\delta}(\sigma)}{\sigma} = \infty$ and lemma 3 follows.

Next we prove:

Theorem 1. *For $\nu(\sigma)$ defined as in* **1.**

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log \nu(\sigma)}{\log \sigma} = \begin{matrix} \varrho \\ \lambda \end{matrix} . \tag{2.1}$$

Proof. We have

$$\nu(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt$$

$$\leq \{M(\sigma)\}^2$$

or

$$\log \nu(\sigma) \leq 2 \log M(\sigma)$$

or

$$\frac{\log\log \nu(\sigma)}{\log \sigma} \leq 0(1) + \frac{\log\log M(\sigma)}{\log \sigma}.$$

Therefore

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \nu(\sigma)}{\log \sigma} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log \sigma} = \frac{\bar{\varrho}}{\bar{\lambda}}. \tag{2.2}$$

Again, proceeding as in [2], it can be seen very easily that

$$\nu(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} \tag{2.3}$$

for all $\sigma < \infty$. Since the series on the right hand side is of positive terms,

$$\nu(\sigma) \geq |a_n|^2 e^{2\sigma\lambda_n}, \; n \geq 1, \; \sigma < \infty.$$

Hence, for all $\sigma < \infty$

$$\nu(\sigma) \geq \{\mu(\sigma)\}^2$$

or

$$\frac{\log\log \nu(\sigma)}{\log \sigma} \geq 0(1) + \frac{\log\log \mu(\sigma)}{\log \sigma}$$

or

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \nu(\sigma)}{\log \sigma} \geq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\log \sigma} = \frac{\bar{\varrho}}{\bar{\lambda}}. \tag{2.4}$$

Combining (2.2) and (2.4) we get (2.1).

Theorem 2. *If $f(s)$ is of logarithmic order $\bar{\varrho}$ and lower logarithmic order $\bar{\lambda}$ $(1 \leq \bar{\lambda}, \bar{\varrho} \leq \infty)$ then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \nu(\sigma)}{\log \sigma} = \frac{\bar{\varrho}}{\bar{\lambda}}. \tag{2.5}$$

Proof. Since $\nu(\sigma)$ is an increasing function of σ,

$$\frac{\nu(\sigma)}{\delta}\left\{1-\left(\frac{1}{2}\right)^{\delta}\right\} \leq \frac{1}{(2\sigma)^{\delta}} \int_{\sigma}^{2\sigma} \nu(x)\, x^{\delta-1}\, dx$$

$$\leq \frac{1}{(2\sigma)^{\delta}} \int_{\sigma}^{2\sigma} \nu(x)\, x^{\delta-1}\, dx = \nu_{\delta}2(\sigma)$$

Hence $\log \nu(\sigma) + 0(1) \leq \log \nu_{\delta}(2\sigma)$

or

$$\frac{\log\log \nu(\sigma)}{\log \sigma} + 0(1) \leq \frac{\log\log \nu_{\delta}(2\sigma)}{\log (2\sigma)}\, \frac{\log 2\sigma}{\log \sigma}$$

or

(2.6) $$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log \nu(\sigma)}{\log \sigma} \leq \lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log \nu_{\delta}(\sigma)}{\log \sigma}$$

Conversely we have

$$\nu_{\delta}(\sigma) = 0(1) + \frac{1}{\sigma^{\delta}} \int_{\sigma_1}^{\sigma} \nu(x)\,.\, x^{\delta-1}\, dx\,,\ \sigma > \sigma_1$$

$$\leq 0(1) + \frac{1}{\sigma^{\delta}} \int_{\sigma_1}^{2\sigma} \nu(x)\, x^{\delta-1}\, dx$$

$$\leq 0(1) + \frac{\nu(2\sigma)}{\delta}\,[(2)^{\delta} - 0(1)].$$

Proceeding as above we get

(2.7) $$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log \nu_{\delta}(\sigma)}{\log \sigma} \leq \lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log \nu(\sigma)}{\log \sigma}\,.$$

Combining (2.1), (2.6) and (2.7) we get (2.5).

Theorem 3. *If $f(s)$ is an entire function of logarithmic order $\bar{\varrho}$ and lower logarithmic order $\bar{\lambda}\,(1 \leq \bar{\lambda},\ \bar{\varrho} \leq \infty)$ then*

(2.8) $$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log \{\nu(\sigma)\,/\,\nu_{\delta}(\sigma)\}}{\log \sigma} = {\bar{\varrho} \atop \bar{\lambda}}\,.$$

Proof. Let

$$\lim_{\sigma\to\infty} \sup \frac{\log \{\nu(\sigma)\,/\,\nu_{\delta}(\sigma)\}}{\log \sigma} = L.$$

Let $1 \leq L < \infty$. Then, given $\varepsilon > 0$

(2.9) $$\frac{\nu(\sigma)}{\nu_0(\sigma)} < \sigma^{(L+\varepsilon)} \text{ for all } \sigma > \sigma_0 \,.\,^{1})$$

Now

$$\log\{t^\delta \nu_\delta(t)\} = \log\left\{\int_0^t \nu(x)\, x^{\delta-1}\, dx\right\}.$$

Differentiating on both sides

$$\frac{\frac{d}{dt}\{t^\delta \nu_\delta(t)\}}{t^\delta \nu_\delta(t)} = \frac{\nu(t)\, t^{\delta-1}}{\int_0^t \nu(x)\, x^{\delta-1}\, dx} = \frac{\nu(t)\, t^{\delta-1}}{\nu_\delta(t)\, t^\delta}.$$

Integrating from σ_0 to σ on both sides

(2.10) $$\log\{\sigma^\delta \nu_\delta(\sigma)\} = \log\{\sigma_0{}^\delta \nu_\delta(\sigma)\} + \int_{\sigma_0}^{\sigma} \frac{\nu(x)}{\nu_\delta(x)} \frac{dx}{x}.$$

Hence using (2.9)

$$\log\{\sigma^\delta \nu_\delta(\sigma)\} < 0(1) + \frac{\sigma^{L+\varepsilon}}{L+\varepsilon}$$

or, by lemma 3,

$$\log \nu_\delta(\sigma)\,\{1 + 0(1)\} < 0(1) + \frac{\sigma^{L+\varepsilon}}{L+\varepsilon}\,, \quad \sigma > \sigma_0$$

or

$$\frac{\log\log \nu_\delta(\sigma)}{\log \sigma} \leq (L+\varepsilon)\ \{1 + 0(1)\},\ \sigma > \sigma_0$$

or

(2.11) $$\limsup_{\sigma\to\infty} \frac{\log\log \nu_\delta(\sigma)}{\log \sigma} = \bar{\varrho} \leq L$$

which obviously holds if $L = \infty$.

Again we have by (2.10)

$$\log\{(2\sigma)^\delta \nu^\delta(2\sigma)\} = \log\{(\sigma_0{}^\delta \nu_\delta(\sigma_0)\} + \int_{\sigma_0}^{2\sigma} \frac{\nu(x)}{\nu_\delta(x)} \frac{dx}{x}$$

$$> 0(1) + \int_{\sigma}^{2\sigma} \frac{\nu(x)}{\nu_\delta(x)} \frac{dx}{x}.$$

[1]) σ_0 need not be the same at each occurence.

Since $\frac{\nu(x)}{\nu_\delta(x)}$ is an increasing function of x

$$\log \{ (2\sigma)^\delta \nu_\delta(2\sigma) \} > 0(1) + \frac{\nu(\sigma)}{\nu_\delta(\sigma)} \log 2$$

$$> 0(1) + \sigma^{(L-\varepsilon)} \log 2$$

for a sequence of values of σ tending to infinity, or

(2.12) $$\limsup_{\sigma\to\infty} \frac{\log\log \nu_\delta(\sigma)}{\log \sigma} \geqq L \cdot$$

If $L = \infty$, then by taking an arbitrary large number in place of $(L - \varepsilon)$ we get

$$\limsup_{\sigma\to\infty} \frac{\log\log \nu_\delta(\sigma)}{\log \sigma} = \infty \cdot$$

Hence combining (2.11) and (2.12) we get

$$\limsup_{\sigma\to\infty} \frac{\log\log \nu_\delta(\sigma)}{\log \sigma} = \bar{\varrho} = L$$

or

$$\limsup_{\sigma\to\infty} \frac{\log \{\nu(\sigma) / \nu_\delta(\sigma)\}}{\log \sigma} = \bar{\varrho} \cdot$$

Similarly we can show that

$$\liminf_{\sigma\to\infty} \frac{\log \{\nu(\sigma) / \nu(\sigma)\}}{\log \sigma} = \bar{\lambda}$$

and thus the proof of Theorem 3 is completed.

Corrolary. If $f(s)$ is of finite logarithmic order ϱ ,

(2.13) $$\log \nu(\sigma) \sim \log \nu_\delta(\sigma) \sim 2 \log M(\sigma)$$

we have from (2.8)

$$(\bar{\lambda} - \varepsilon) \log \sigma < \log \nu(\sigma) - \log \nu_\delta(\sigma) < (\bar{\varrho} + \varepsilon) \log \sigma \text{ , } \sigma > \sigma_0$$

or

$$\frac{(\bar{\lambda} - \varepsilon) \log \sigma}{\log \nu_\delta(\sigma)} + 1 < \frac{\log \nu(\sigma)}{\log \nu_\delta(\sigma)} < \frac{(\bar{\varrho} + \varepsilon) \log \sigma}{\log \nu_\delta(\sigma)} + 1 \cdot$$

Taking limits as $\sigma \to \infty$ we get the result in view of Lemma 3 and Theorem 1.

3. In this section we shall investigate the relationship between the means of $f(s)$ and its derivatives $f^{(m)}(s)$. We prove :

Theorem 4. *If $\nu_\delta(\sigma)$ and $\nu_\delta(\sigma, f^{(1)})$ denote the mean values of $f(s)$ and its first derivative then we have for $\sigma > \sigma_0$,*

(3.1) $$\nu_\delta(\sigma, f^{(1)}) \geq \nu_\delta(\sigma) \left\{ \frac{\log \nu_\delta(\sigma)}{2\sigma \log \sigma} \right\}^2 .$$

Proof. We have

$$\nu_\delta(\sigma, f^{(1)}) = \lim_{T\to\infty} \frac{1}{2T\sigma^\delta} \int_0^\sigma \int_{-T}^{T} |f^{(1)}(x+it)|^2 x^{\delta-1} \, dx \, dt , \; 1 < \delta < \infty$$

$$= \lim_{T\to\infty} \frac{1}{2T\sigma^\delta} \int_\sigma^0 \int_{-T}^{T} \left| \lim_{\varepsilon\to 0} \frac{f(x+it) - f(\overline{x-\varepsilon x}+it)}{\varepsilon x} \right|^2 x^{\delta-1} \, dx \, . \, dt$$

$$\nu_\delta(\sigma, f^{(1)}) = \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{1}{2T\varepsilon^2\sigma^\delta} \int_\sigma^\sigma \int_{-T}^{T} \left\{ \left| \frac{f(x+it) - f(\overline{x-\varepsilon x}+it)}{x} \right|^2 x^{\delta-1} \right\} dx \, dt$$

$$\geq \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{1}{2T\varepsilon^2\sigma^{\delta+2}} \int_0^\sigma \int_{-T}^{T} \left[\left\{ \Big| |f(x+it)| - |f(\overline{x-\varepsilon x}+it)| \Big| \right\}^2 x^{\delta-1} \right] dx \, dt.$$

Now by MINKOWSKI's inequality

$$\int_{-T}^{T} \left\{ \Big| f(x+it) \Big| - \Big| f(\overline{x-\varepsilon x}+it) \Big| \right\}^2 dt$$

$$\geq \left\{ \left(\int_{-T}^{T} |f(x+it)|^2 \, dt \right)^{\frac{1}{2}} - \left(\int_{-T}^{T} |f(\overline{x-\varepsilon x}+it)|^2 \, dt \right)^{\frac{1}{2}} \right\}^2 ,$$

hence

$$\nu_\delta(\sigma, f^{(1)}) \geq \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{1}{2T\varepsilon^2\sigma^{\delta+2}} \left[\int_0^\sigma \left\{ \left(\int_{-T}^{T} |f(x+it)|^2 \, dt \right)^{\frac{1}{2}} \right. \right.$$

$$\left. \left. - \left(\int_{-T}^{T} |f(\overline{x-\varepsilon x}+it)|^2 \, dt \right)^{\frac{1}{2}} \right\}^2 x^{\delta-1} \, dx \right].$$

Again, by MINKOWSKI's inequality

$$\nu_\delta(\sigma, f^{(1)}) \geq \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{1}{2T\varepsilon^2\sigma^{\delta+2}} \left[\left\{ \int_0^\sigma \int_{-T}^{T} |f(x+it)|^2 x^{\delta-1}\, dx\, dt \right\}^{\frac{1}{2}} \right.$$

$$\left. - \left\{ \int_0^\sigma \int_{-T}^{T} |f(\overline{x-\varepsilon x}+it)|^2 x^{\delta-1}\, dx\, dt \right\}^{\frac{1}{2}} \right]^2$$

$$= \lim_{\varepsilon\to 0} \left[\frac{\{\nu_\delta(\sigma)\}^{\frac{1}{2}} - \{\nu_\delta(\sigma-\varepsilon\sigma)\}^{\frac{1}{2}}}{\varepsilon\,\sigma} \right]^2 .$$

Let us set $g(\sigma) = \dfrac{\log \nu_\delta(\sigma)}{\log \sigma}$. Then, since by lemma 2 $\log \nu_\delta(\sigma)$ is a convex increasing function of $\log \sigma$ for $\sigma > \sigma_0$, it follows that $g(\sigma)$ is a positive increasing function of σ for $\sigma > \sigma_0$. Therefore

$$\nu_\delta(\sigma, f^{(1)}) \geq \lim_{\varepsilon\to 0} \left[\frac{\sigma^{g(\sigma)/2} - (\sigma-\varepsilon\sigma)^{g(\sigma-\varepsilon\sigma)2}}{\varepsilon\,\sigma} \right]^2$$

$$\geq \lim_{\varepsilon\to 0} \left[\frac{\sigma^{g(\sigma)/2}\,(1-(1-\varepsilon)^{g(\sigma)/2})}{\varepsilon\,\sigma} \right]^2$$

$$\geq \left[\sigma^{g(\sigma)/2} \frac{g(\sigma)}{2\sigma} \right]^2$$

$$= \nu_\delta(\sigma) \left\{ \frac{\log \nu_\delta(\sigma)}{2\sigma \log \sigma} \right\}^2 .$$

Hence Theorem 4 follows.

We give some applications of Theorem 4.

(*i*) If $1 < \bar{\lambda} < \varrho < \infty$ then

(3.2) $$\nu_\delta(\sigma) < \nu_\delta(\sigma, f^{(1)}) < \nu_\delta(\sigma, f^{(2)}) < \ldots$$

for sufficiently large values of σ.

From (3.1) we have

$$\log \{\nu_\delta(\sigma, f^{(1)}) / \nu_\delta(\sigma)\}^{\frac{1}{2}} \geq \log\log \nu_\delta(\sigma) - \log(2\sigma) - \log\log\sigma .$$

Hence

$$\lim_{\sigma\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{\log \{\nu_\delta, f^{(1)}) / \nu_\delta(\sigma)\}^{\frac{1}{2}}}{\log \sigma} \geq \frac{\varrho - 1}{\bar{\lambda} - 1} .$$

Therefore for any $\varepsilon < 0$ and all large σ,

$$\nu_\delta(\sigma, f^{(1)}) > \nu_\delta(\sigma),\ \sigma^{(2\lambda-1-\varepsilon)}$$

Since $\bar{\lambda} > 1$ and ε can be taken arbitrarily small we get

$$r_\delta(\sigma , f^{(1)}) > \nu_\delta(\sigma).$$

Writing this inequality for $f^{(1)}(s)$, $f^{(2)}(s)$,... and combining we get (3.2).

(*ii*) For $< \bar{\lambda} \leq \bar{\varrho} < \infty$,

$$\text{(3.3)} \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \{\nu_\delta(\sigma , f^{(m)}) / \nu_\delta(\sigma)\}^{\frac{1}{2m}}}{\log \sigma} \geq \frac{\bar{\varrho} - 1}{\bar{\lambda} - 1} .$$

Writing (3.1) for the function $f^{(n-1)}(s)$ we have

$$\frac{\nu_\delta(\sigma , f^{(m)})}{\nu_\delta(\sigma)} \geq \left\{ \frac{\log \nu_\delta(\sigma)}{2\sigma \log \sigma} \right\}^{2m}$$

Writing this for $m = 1, 2, 3, \ldots\ldots m$ and multiplying the resulting inequalities, we get

$$\frac{\nu_\delta(\sigma, f^{(m)})}{\nu_\delta(\sigma)} \geq \left\{ \frac{\log \nu_\delta(\sigma)}{2\sigma \log \sigma} \right\}^{2m}$$

since $\nu_\delta(\sigma) < \nu_\delta(\sigma , f^{(1)}) < \nu_\delta(\sigma , f^{(2)}) < \ldots$, for $\bar{\lambda} > 1$.

Hence we have

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \{\nu_\delta(\sigma , f^{(m)}) / \nu_\delta(\sigma)\}^{\frac{1}{2m}}}{\log \sigma} \geq \frac{\bar{\varrho} - 1}{\bar{\lambda} - 1}$$

and (3.3) follows.

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ÖZET

Bir DIRICHLET serisi ile verilen bir tam fonksiyon için bâzı ortalamalar tanımlanmakta ve bu ortalamalara bağlı bâzı ifadelerin alt ve üst limitlerinin, fonksiyonun logaritmik ve alt-logaritmik derecesi denen bâzı büyüklüklerle ilgileri araştırılmaktadır.