ON THE NORMAL ELLIPTIC RULED QUINTIC SURFACE IN FOUR DIMENSIONAL SPACE

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For the normal elliptic ruled quintic surface R^5 in four dimensions, we construct first a coordinate system such that the surface is invariant under a group of 20 permutations of the coordinates. In terms of this we obtain the GRASSMANN coordinates of the general plane cutting R^5 in a cubic curve, and of the general generator, as elliptic functions of order five, and two parametrisations of the surface itself in elliptic functions of two variables. This leads to a set of cubic equations for the surface, and the quintic equation of the primal generated by the planes of the cubic curves on it, depending homogeneously on the two paramers $\lambda = \wp'(\sigma)$, $\mu = \wp'(2\sigma)$, where σ is a primitive fifth part of a period of the elliptic functions; and also to explicit equations for the CREMONA transformation by cubics through R^5, and its inverse by quadrics through a normal elliptic quintic curve.

1. Some preliminary properties of ${}^1R_2{}^5$.

It is familiar [3] that the general normal elliptic ruled quintic surface ${}^1R_2{}^5$ in four dimensions (which we denote for brevity by R^5) has on it an elliptic ∞^1 family of elliptic plane cubics $\{C^3\}$, each of which is the residual section of R^5 by the prime joining any pair of an involution among the generators; the planes of the curves $\{C^3\}$ generate a quintic primal $R_3{}^5$, and are a system dual to that of the generators of R^5, each consisting of all lines (planes) that meet five general planes (lines) of the other. It is also known [2] that if 2Ω is the lattice of periods of the appropriate elliptic functions, we can assign to each generator a parameter w (mod 2Ω), and to each C^3 (and its plane) a parameter u (mod 2Ω), in such a way that writing $\equiv$ for congruence (mod 2Ω)

(i) Five generators $l(w_i)$ $(i = 0, 1, 2, 3, 4)$ belong to a linear complex not containing all the generators, i. e. meet a plane which is not that of one of the cubics $\{C^3\}$, if and only if

$$\sum_{i=0}^{4} w_i \equiv 0.$$

(ii) The cubic $C(u)$ and two generators $l(w_1)$, $l(w_2)$ are a prime section of R^5 if and only if $w_1 + w_2 + u \equiv 0$.

(iii) The unique intersection of the cubics $C(u_1)$, $C(u_2)$ is on the generator $l(w)$, where $w \equiv u_1 + u_2$.

(iv) Parametrising each curve $C(u)$ by assigning to each point of it the parameter w of the generator $l(w)$ through that point, the points w_1, w_2, w_3 of the curve $C(u)$ are collinear if and only if $w_1 + w_2 + w_3 \equiv u$.

As every point of R^5 lies on two of the curves $\{C^3\}$, and every two of these curves meet in one point, we can assign to each point, of the surface the unordered pair (u, u') of parameters of the two curves $C(u)$, $C(u')$ through it, so that the equation, in terms of this parametrisation, of the generator $l(w)$ is $u + u' \equiv w$. There is also on R^5 what is called the focal curve, the envelope of $\{C^3\}$, with equation $u = u'$, of order ten, quadrisecant to the generators, and touching each curve $C(u)$ in the point (u, u). We now prove :

Theorem 1. *The plane π joining the intersections by pairs of three cubics $C(u_i)$ $(i = 1,2,3)$ contains the generator $l(w)$, where $w + u_1 + u_2 + u_3 = 0$; and conversely, every plane through a generator meets the surface residually in three points, the intersections by pairs of three curves of $\{C^3\}$.*

Proof. Let $w_i \equiv u_j + u_k$, where, l, j, k is any permutation of 1, 2, 3 ; the plane π contains the points w_j, w_k of the curve $C(u_i)$, and hence also the third point w collinear with these, where by (iv) $w_j + w_k + w = u_i$, i.e. $w + u_i + u_j + u_k \equiv 0$. Thus π contains at least three points of the generator $l(w)$, its intersections with the three curves. Conversely, the primes through a generator trace residually on R^5 a net of elliptic quartics, algebraically equivalent to $\{C^3\} + \{l\}$, and hence of grade 3, since $\{C^3\}$, $\{l\}$ form a base for algebraic equivalence of curves on R^5, with intersection matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$; i.e. any plane π through $l(w)$ meets R^5 residually in three points ; if one of these is (u_1, u_2), since π contains the points w, $w_3 = u_1 + u_2$ of $C(u_1)$, it contains also by (iv) the point $w_2 \equiv u_1 - w - w_3 \equiv u_1 + u_3$, where $u_1 + u_2 + u_3 + w \equiv 0$; i.e. π contains the point (u_1, u_3), and similarly it contains the point (u_1, u_3). This completes the proof.

We now recall the familiar figure of fifteen planes and ten nodes of the SEGRE cubic primal [1] ; the planes π_{ij} (where ij is any of the fifteen unordered pairs of six symbols 0, 1, 2, 3,4,5) meet by sixes in ten points, associated with the ten bisections (ijk, lmn) of the six symbols (i. e. partitions of them into an unordered pair of unordered triads), π_{ij} containing each of the four points (ijk, lmn) for which i, j are in the same triad ; the five planes $\pi_{ij}, \pi_{ik}, \pi_{il}, \pi_{im}, \pi_{in}$ are associated, which means that every line meeting four of them meets all five, and they are the common members of six linearly independent linear complexes of planes ; the lines meeting these five planes are an ∞^2 system $\{i\}$, and include a pencil in each of the other ten planes, that in π_{jk} having its vertex at (ijk, lmn). Each of the six systems $\{i\}$ generates the same cubic primal ; and the lines common to the system $\{i\}$ and any linear complex not containing all of them generate an R^5, five of whose cubic curves C^3 are in the five associated planes $\pi_{ij}, \dots, \pi_{in}$, and which has one generator in each of the other ten planes. Any set of five (distinct) associated planes determines the whole figure uniquely and any two such figures are projectively equivalent; the ten points are the intersections of the five planes by pairs, and the other ten planes join sets of four of the ten points. We now prove :

Theorem 2. *On R^5 five cubics $C(u_i)$ $(i = 0,\dots,4)$ lie in five associated planes if and only if $\sum_{i=0}^{4} u_i = 0$; and in this case the other ten planes of the* SEGRE *configuration are those joining the intersections by pairs of sets of three of the five cubics.*

Proof. Let $\sum_{=0}^{4} u_i \equiv 0$. Denote by π_{i5} the plane of the cubic $C(u_i)$, by P_{ij} the point (u_i, u_j) which is the intersection π_{i5}, π_{j5}, and by π_{ij} the plane $P_{lm}\, P_{mk}\, P_{kl}$; by Theorem 1, π_{ij} contains the generator $l(w_{ij})$, where $w_{ij} \equiv -(u_k + u_l + u_m) = u_i + u_j$; and this generator also contains P_{ij}. Associating P_{ij} with the bisection $(ij5, klm)$, we see that all the point - plane incidences of the SEGRE configuration are verified, and the direct theorem is proved. The converse is trivial, since the fifth associated plane of four given planes is unique.

We note also that R^5 has a curve C^3 in each of the planes $\pi_{05}, \ldots, \pi_{45}$, and a generator $l(w_i \cdot)$ in each of the other ten planes ; and that all its generators meet $\pi_{05}, \ldots, \pi_{45}$, and are thus in the system $\{5\}$ associated with the SEGRE figure. The generators are thus the intersection of five linearly independent linear complexes, four of which intersect in the line system $\{5\}$, and they are those lines of $\{5\}$ that meet the plane of any of the curves $\{C^3\}$ other than $C(u_0), \ldots, C(u_4)$.

2. A self - dual configuration associated with R^5.

Now let σ be a primitive fifth of a period in the period lattice 2Ω, i. e. let 5σ (but not σ) be an element of 2Ω ; then $0, \sigma, 2\sigma, 3\sigma, 4\sigma$ form a subgroup of order 5 in the additive group of residue classes (mod 2Ω). There are six such subgroups, the 24 mutually incongruent primitive fifths of periods being the four non zero elements in each of these. In Theorem 2, let $u_i = i\sigma$ $(i = 0, \ldots, 4)$, satisfying $\sum_{i=0}^{4} u_i \equiv 0$; then treating i as a residue class (mod 5),

$$u_{i-1} + u_{i+1} \equiv u_{i-2} + u_{i+2} \equiv 2u_i ,$$

i.e.

$$w_{41} \equiv w_{23} \equiv 2u_0 , \; w_{02} \equiv w_{34} \equiv 2u_1 , \; w_{13} \equiv w_{40} \equiv 2u_2 ,$$
$$w_{24} \equiv w_{01} \equiv 2u_3 , \; w_{30} \equiv w_{12} \equiv 2u_4 ,$$

and the ten lines in which R^5 meets the ten planes π_{ij} $(i, j = 0, \ldots, 4)$ coincide by pairs in

$$(1) \qquad \begin{cases} l_{05} = l(2u_0) = P_{41} P_{23} = \pi_{41} \cdot \pi_{23} \\ l_{15} = l(2u_1) = P_{02} P_{34} = \pi_{02} \cdot \pi_{34} \\ l_{25} = l(2u_2) = P_{13} P_{40} = \pi_{13} \cdot \pi_{40} \\ l_{35} = l(2u_3) = P_{24} P_{01} = \pi_{24} \cdot \pi_{01} \\ l_{45} = l(2u_4) = P_{30} P_{12} = \pi_{30} \cdot \pi_{12}. \end{cases}$$

Theorem 3. *The five lines* (1) *are five associated lines ; the ten further lines which with these make up the fifteen line figure, consisting of six associated sets of five, dual to the* SEGRE *figure of fifteen planes considered above, are the lines* $l_{jk} = P_{ij} P_{ik}$, *where* $j + k \equiv 2i$ *(mod 5) ; and the fifteen points in which these lines are concurrent by threes* [1] *are the ten points* $P_{i\cdot}$, *and five points* P_{ii}, $(i = 0, \ldots, 4)$, *where* $P_{ii} = (u_i, u_i)$, *the point of contact of* $C(u_i)$ *with the focal curve* $u \equiv u'$ *envelope of the family* $\{C^3\}$ *on* R^5.

Proof. Two of the planes π_{ij} $(i, j = 0, \ldots, 5)$ meet in a line if and only if they have no common suffix ; three planes π_{ij}, π_{kl}, π_{mn} lie in a prime, and meet by pairs in three lines and all three in a point, if and only if (ij, kl, mn) is a syntheme, i. e. a partition of all six symbols into an unordered triad of unordered pairs. The point P_{00} is the intersection of π_{05}, π_{41}, π_{23}, since it is on the curve $C(u_0)$ in π_{05}, and on the generator $l(2u_0) = \pi_{41} \cdot \pi_{23}$. Moreover on $C(u_0)$, P_{00} is collinear with $P_{01} P_{04}$, and also with $P_{02} P_{03}$, by (iv). Similarly, for $i = 1, 2, 3, 4$, $P_{ii} = \pi_{i5} \cdot l_{i5}$, and in π_{i5} it is collinear with each of the pairs $P_{ij} P_{ik}$ for which $j + k \equiv 2i$ (mod 5), since this congruence is unaltered by the cyclic permutation of 0, 1, 2, 3, 4. Thus denoting $P_{ij} P_{ik}$ by l_{jk}, for all i, j, k satisfying $j + k \equiv 2i$ (mod 5), we see that the following 15 triads of lines are concurrent in the points named :

$l_{05}\, l_{14}\, l_{23} : P_{00}$	$l_{05}\, l_{12}\, l_{34} : P_{41}$	$l_{05}\, l_{13}\, l_{24} : P_{23}$
$l_{45}\, l_{20}\, l_{34} : P_{11}$	$l_{15}\, l_{23}\, l_{40} : P_{02}$	$l_{45}\, l_{24}\, l_{30} : P_{34}$
$l_{25}\, l_{31}\, l_{40} : P_{22}$	$l_{25}\, l_{34}\, l_{01} : P_{13}$	$l_{25}\, l_{30}\, l_{41} : P_{40}$
$l_{35}\, l_{42}\, l_{01} : P_{33}$	$l_{35}\, l_{40}\, l_{12} : P_{24}$	$l_{05}\, l_{41}\, l_{02} : P_{01}$
$l_{45}\, l_{03}\, l_{12} : P_{44}$	$l_{45}\, l_{01}\, l_{03} : P_{30}$	$l_{45}\, l_{02}\, l_{13} : P_{12}$

which is the complete system of concurrences for the fifteen line figure. This proves the theorem.

We note that the nomenclature is not symmetrical with respect to all permutations of the symbols 0,1,2,3,4,6. We have however the following correspondences between the ten bisections (*ijk*, *lmn*) and ten of the fifteen synthemes, the bisection representing a point in the fifteen plane figure, and the syntheme representing the same point in the fifteen line figure :

(2)

(05,12,34) : (145,023)	(05,13,24) : (235,401)
(15,23,40) : (205,134)	(15,24,30) : (345,012)
(25,34,01) : (315,240)	(25,30,41) : (405,123)
(35,40,12) : (425,301)	(35,41,02) : (015,234)
(45,01,23) : (035,412)	(45,02,13) : (125,340).

The remaining five synthemes

(3) (05, 41, 23), (15, 02, 34), (25, 13, 40), (35, 24, 01), (45, 30, 12)

corresponding to the points P_{ii} ($i = 0, 1, 2, 3, 4$) form what we may call a block of synthemes, i. e. a set of five synthemes no two of which have a common pair, so that between them they contain all the fifteen pairs. There are six such blocks ; every syntheme belongs to just two blocks, and every two blocks have just one syntheme in common ; the group S_6 of all permutations on 0, 1, 2, 3, 4, 5 also permutes the six blocks in every possible way ; and the subgroup of S_6 that stabilises a given block, say (3), is isomorphic with the symmetric group S_5, being the image under an outer automorphism of S_6 of the subgroup S_5 that stabilises one of 0, 1, 2, 3, 4, 5 ; we may denote this subgroup of S_6, which stabilises the block (3) of synthemes, by S_5' . The whole configuration constructed in Theorem 3 is invariant under S_5', which permutes the five synthemes of the stabilised block (3) in all possible ways, and also stabilises the correspondence (2) between the remaining ten synthemes and the ten bisections.

The configuration is also self dual ; for not only by (1) is each line l_{i5} the intersection of two of the fifteen planes,

$$l_{05} = \pi_{41} . \pi_{23} \,,\, l_{15} = \pi_{02} . \pi_{34} \,,\, l_{25} = \pi_{13} . \pi_{40} \,,\, l_{35} = \pi_{24} . \pi_{01} \,,\, l_{45} = \pi_{30} . \pi_{12} \,,$$

but also, as was seen in the proof of Theorem 3, each plane π_{i5} is the join of two intersecting lines :

$$\pi_{05} = l_{41}\, l_{03} \,,\, \pi_{15} = l_{02}\, l_{04} \,,\, \pi_{25} = l_{43}\, l_{40} \,,\, \pi_{35} = l_{21}\, l_{01} \,,\, \pi_{45} = l_{00}\, l_{42}.$$

The fifteen primes of the fifteen plane figure (each containing three of the fifteen planes) are the ten primes Σ_{ij}, dual to the points P_{ij}, and each containing six of the fifteen lines), and five further primes Σ_{ii}, each containing the triad of lines corresponding to one of the synthemes (3).

We shall see later that the R_*^5, related as described to the configuration, with a given lattice of periods 2Ω, and the multiples of a given fifth of a period corresponding in the specified way to the five lines and five planes, is unique. Anticipating this result, we see that if we set up duality in space, in which l_{i5} corresponds to π_{j5}, where $i \equiv 2j$ (mod 5), $l\,(w_i)$ corresponds to the plane of $C\,(w_i)$ ($l = 0, 1, 2, 3, 4$), and hence $l\,(w)$ to the plane of $C\,(w)$ for all w. Thus as P_{ii} is the limiting intersection of π_{j5}, with an ultimately coincident plane, in the elliptic family of planes containing the curves $\{C^3\}$, Σ_{ii} is the limiting join of l_{i5} to an ultimately coincident generator, i. e. Σ_{ii} is the prime containing the pencil of tangent planes to R^5 at points of l_{i5}.

We have kept in the suffix 5 to emphasise the symmetry of the configuration under S'_5; but as the line system (5) is singled out as containing the generators of R^5, we shall from now on write π_i, l_i for π_{i5}, l_{i5} respectively.

3. Introduction of coordinates

The simplest representation of the fifteen plane figure in terms of coordinates is by the use of six linear forms in the homogeneous coordinates, whose sum is identically zero, and any five which can be taken to be the coordinates themselves; if these are $x_0, x_1, x_2, x_3, x_4, x_5$, the equations of the fifteen planes are

$$x_i + x_j = x_k + x_l = x_m + x_n = 0,$$

where (ij, kl, mn) runs over all the fifteen synthemes on 0, 1, 2, 3, 4, 5; and the fifteen primes each containing three of the planes are $x_i + x_j = 0$, where (ij) runs over the fifteen pairs. The equation of the SEGRE cubic primal containing the fifteen planes is $\sum_{i=0}^{5} x_i^3 = 0$, or indeed the vanishing of any symmetrical cubic form in $(x_0, \ldots, x_5)$ since in virtue of the relation $\sum_{i=0} x_i = 0$, these all reduce to constant mltiples of any one of them. The common point of the three planes in $x_0 + x_1 = 0$ is $(1, -1, 0, 0, 0, 0)$; i. e. the fifteen common points of these triads of planes are the intersections of all but two of the primes $x_i = 0$ $(1 = 0, \ldots, 5)$. The ten points of concurrence by sixes of the fifteen planes (nodes of the SEGRE cubic primal) are $(1, 1, 1, -1, -1, -1)$ in all ten bisections, the six planes through any one of these points being given by the six synthemes in which each pair has one symbol in common with each triad in the bisection. The six sets of five associated planes correspond in this notation to the six blocks of synthemes on the coordinates, and the coordinates accordingly to the six blocks of synthemes in the notation of the last section. Thus as the subgroup of S_6 under which the configuration described is invariant, was in the previous notation that which stabilises a particular block of synthemes, it now stabilises one of the six linear forms, say x_5, which we accordingly suppress, and take $(x_0, \ldots, x_4)$ as homogeneous coordinates. Taking the five planes of the curves $C(u_i)$ $(i=0, 1, 2, 3, 4)$ to be those represented by the synthemes in the block (3), we have for the equations of these planes, and the coordinates of their points of intersection by pairs:

$$(4)\begin{cases} \pi_0 : x_4 + x_1 = x_2 + x_3 = 0 & P_{23} : (1, -1, 1, 1, -1) & P_{31} : (1, 1, -1, -1, 1) \\ \pi_1 : x_0 + x_2 = x_3 + x_4 = 0 & P_{34} : (-1, 1, -1, 1, 1) & P_{42} : (1, 1, 1, -1, -1) \\ \pi_2 : x_1 + x_3 = x_4 + x_0 = 0 & P_{40} : (1, -1, 1, -1, 1) & P_{13} : (-1, 1, 1, 1, -1) \\ \pi_3 : x_2 + x_4 = x_0 + x_1 = 0 & P_{01} : (1, 1, -1, 1, -1,) & P_{41} : (-1, -1, 1, 1, 1) \\ \pi_4 : x_3 + x_0 = x_1 + x_2 = 0 & P_{12} : (-1, 1, 1, -1, 1) & P_{30} : (1, -1, -1, 1, 1) \end{cases}$$

The remaining ten planes are identifiable from the points in each as

$$(4')\quad\begin{cases}\pi_{23}: x_1+x_2=x_3+x_4=0 & \pi_{41}: x_1+x_3=x_2+x_4=0\\ \pi_{34}: x_2+x_3=x_4+x_0=0 & \pi_{02}: x_2+x_4=x_3+x_0=0\\ \pi_{40}: x_3+x_4=x_0+x_1=0 & \pi_{13}: x_3+x_0=x_4+x_1=0\\ \pi_{01}: x_4+x_0=x_1+x_2=0 & \pi_{24}: x_4+x_1=x_0+x_2=0\\ \pi_{12}: x_0+x_1=x_2+x_3=0 & \pi_{30}: x_0+x_2=x_1+x_3=0\end{cases}$$

and the remaining points $P_{00},\dots,P_{44}$ as the vertices of the simplex of reference. The fifteen lines are

$$(5)\quad\begin{cases}l_0: x_1=-x_2=-x_3=x_4 & l_{03}: x_1=x_2=-x_3=-x_4 & l_{41}: x_1=-x_2=x_3=-x_4\\ l_1: x_2=-x_3=-x_4=x_0 & l_{04}: x_2=x_3=-x_4=-x_0 & l_{02}: x_2=-x_3=x_4=-x_0\\ l_2: x_3=-x_4=-x_0=x_1 & l_{10}: x_3=x_4=-x_0=-x_1 & l_{13}: x_3=-x_4=x_0=-x_1\\ l_3: x_4=-x_0=-x_1=x_2 & l_{01}: x_4=x_0=-x_1=-x_2 & l_{24}: x_4=-x_0=x_1=-x_2\\ l_4: x_0=-x_1=-x_2=x_3 & l_{12}: x_0=x_1=-x_2=-x_3 & l_{00}: x_0=-x_1=x_2=-x_3\end{cases}$$

The ten primes, each containing six of these lines, are $\Sigma_{23}: x_4=x_1=0$, $\Sigma_{41}: x_2=x_3=0$, and those obtained from these by the cyclic permutation of 0, 1, 2, 3, 4 ; and the five primes Σ_{ii} are given by the vanishing of the sum of all the coordinates except x_i ($i=0,\dots,4$).

The whole figure of fifteen points, fifteen lines, fifteen planes, and fifteen primes, is clearly invariant under all permutations of 0, 1, 2, 3, 4, i.e. in the notation before we suppressed x_5, under the subgroup S_5 that stabilises the symbol 5 in S_6 ; this is what we expect, as in the notation of the last section the figure was invariant under the subgroup S'_5 of S_6 that stabilises a particular block of synthemes ; and as the pairs in each notation correspond to the synthemes in the other, the individual symbols in each correspond to the six blocks of synthemes in the other.

In its relation to R^5 however, the figure is only invariant under the subgroup $S_5\cap S'_5$ of S that stabilises both the particular symbol 5, and the block (3) of synthemes, which we may denote by 5′. In both notations in fact both the symbol 5 and the block 5′ of synthemes are singled out, one by the symmetry of the 15 point, 15 line, 15 plane, and 15 prime figure, the other as denoting the particular sets of five associated lines and five associated planes in the figure that are generators of R^5, and planes of cubic curves on R^5. But $S_5\cap S'_5$ stabilises also a particular one-one correspondence between the individual symbols and the blocks of synthemes ; each symbol $i=0, 1, 2, 3, 4$ determines uniquely the pair ($i5$), the syntheme in the block 5′ containing this pair, and the other block containing this syntheme, which we denote by i' ; conversely, each block i' determines uniquely the syntheme common to this block and 5′ and the symbol i that is paired wih 5 in this syntheme. Thus $S_5\cap S'_5$ permutes the symbols 0, 1, 2, 3, 4 and the blocks 0′, 1′, 2′, 3′, 4′ in the same way, and from now on the two notations are equivalent.

$S_5\cap S'_5$ is of order 20, and contains the cyclic group C_5 generated by the cyclic permutation (01234) of the five symbols ; there are 36 such subgroups C_5 in S_6, one for each pair i, j'. The other elements of $S_5\cap S'_5$ are

(1243)	(2304)	(3410)	(4021)	(0132)
(1342)	(2403)	(3014)	(4120)	(0231)
(23) (41)	(34) (02)	(40) (13)	(01) (24)	(12) (30)

where ($ijkl$) denotes the cyclic permutation of i, j, k, l, and (ij) (kl) the simultaneous interchange of the two pairs. The three rows of the table are the cosets of C_5 in $S_5\cap S'_5$, and are also conjugacy classes in $S_5\cap S'_5$. We note that regarding the five symbols as residue clases (mod

5), these permutations are the linear transformations of i into $ai + b$, where $a \neq 0$, b are also residue classes (mod 5) ; the subgroup C_5 consists of the translations, of i into $i + b$, and the three cosets correspond to $a = 2, 3, 4$ respectively. The subgroup C_4 generated by (1243), the first column of the table, is induced on the notation by replacing σ by $2\sigma, 3\sigma, 4\sigma$ respectively, which doubles, trebles, or quadruples each symbol (mod 5).

The elements of the subgroup D_5, union of C_5 with the last coset, applied to the coordinates, represent projective transformations of R^5 into itself, replacing u, w by $\pm u + i\sigma$, $\pm w + 2i\sigma$ respectively, for $i = 0, 1, 2, 3, 4$.

4. Parametrisation of the planes $\pi(u)$

In terms of this coordinate system, we shall now parametrise the surface R^5. The first step is to obtain the Grassmann coordinates of the plane $\pi(u)$ containing the curve $C(u)$, as elliptic functions of u. As well as the familiar WEIERSTRASS function $\wp u$, we shall make use of the quasi-elliptic function ζu, in the modified form $\zeta_0 u = \zeta u - \eta u/\omega$, where $2\omega = 5\sigma$, and 2η is the period constant of ζu associated with the period 2ω of $\wp\, u$, i. e. $\zeta(u + 2\omega) = \zeta u + 2\eta$, so that $\zeta_0\, u$ is simply periodic, satisfying $\zeta_0(u + 5\sigma) = \zeta_0 u$ identically in u. (Owing to unavailability of type, we use the German $\mathfrak{p}$ in place of the more usual symbol for the WEIERSTRASS Function). We shall define also $\wp_i u = \wp(u - i\sigma)$, $\zeta_i u = \zeta_0(u - i\sigma)$, for $i = 0, 1, 2, 3, 4$ (writing $\wp_0 u$ for $\wp u$ when the symmetry of the formulae demands it.) We define also the constants

$$\alpha = \wp(\sigma),\ \beta = \wp(2\sigma),\ \lambda = \wp'(\sigma),\ \mu = \wp'(2\sigma),\ \theta = \zeta_0(\sigma),\ \varphi = \zeta_0(2\sigma).$$

On account of the addition theorems there are a number of relations between these. In the first place, from the addition theorem

$$\wp(u + v) + \wp u + \wp v = \frac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2$$

for $\wp u$, on putting $(u, v) = (\sigma, 2\sigma)$ and $(\sigma, 3\sigma)$ we obtain

$$\alpha + 2\beta = \frac{1}{4}\left(\frac{\lambda - \mu}{\alpha - \beta}\right)^2, \quad 2\alpha + \beta = \frac{1}{4}\left(\frac{\lambda + \mu}{\alpha - \beta}\right)^2,$$

and adding and subtracting these,

$$\lambda^2 + \mu^2 = 6(\alpha + \beta)(\alpha - \beta)^2, \quad \lambda\mu = (\alpha - \beta)^3. \tag{6}$$

Next, from the addition formula for ζu, which is satisfied also by $\zeta_0 u$, as the linear terms in the latter trivially cancel, namely

$$\zeta_0(u + v) - \zeta_0 u - \zeta_0 v = \frac{1}{2} \cdot \frac{\wp' u - \wp' v}{\wp u - \wp v},$$

again putting $(u, v) = (\sigma, 2\sigma)$ and $(\sigma, 3\sigma)$ we obtain

$$\theta + 2\varphi = \frac{1}{2}\,\frac{\lambda - \mu}{\beta - \alpha}, \qquad 2\theta - \varphi = \frac{1}{2}\,\frac{\lambda + \mu}{\beta - \alpha}$$

whence

$$3\theta + \varphi = \frac{\lambda}{\beta - \alpha}, \quad \theta - 3\varphi = \frac{\mu}{\beta - \alpha}, \quad 10\,\theta = \frac{3\lambda + \mu}{\beta - \alpha}, \quad 10\,\varphi = \frac{\lambda - 3\mu}{\beta - \alpha}. \tag{7}$$

We have also a relation which will be useful in the sequel

$$5(\lambda\varphi + \mu\theta) = \frac{1}{2}\,\frac{\lambda^2 + \mu^2}{\beta - \alpha} = 3(\beta^2 - \alpha^2). \tag{8}$$

We note that the substitution of 2σ for σ interchanges (α, β) and permutes $(\lambda, \mu, -\lambda, -\mu)$ and $(\theta, \varphi, -\theta, -\varphi)$ cyclically, and that the relations (6), (7), (8) are invariant under this substitution.

Now the intersection of the generator l_0 with the curve $C(u)$, i.e. with the plane $\pi(u)$, has coordinates $(f(u), 1, -1, -1, 1)$, where $f(u)$ is an even elliptic function of order 2, since each point of the line corresponds to two values $\pm u \pmod{2\Omega}$; $f(u)$ is infinite at $P_{00}(u = 0)$, and has the values 1 at $P_{41}(u = \pm\sigma)$ and -1 at $P_{23}(u = \pm 2\sigma)$. This means that

$$f(u) = \frac{2\wp u - (\alpha + \beta)}{\alpha - \beta}.$$

The corresponding functions on l_1, l_2, l_3, l_4 are similarly infinite at $u = \sigma, 2\sigma, 3\sigma, 4\sigma$ respectively ; thus the coordinates of the intersections of $\pi(u)$ with the five lines l_0, l_1, l_2, l_3, l_4 are the rows of the matrix

$$(9)\qquad \begin{bmatrix} 2\wp_0 u - (\alpha+\beta) & \alpha-\beta & \beta-\alpha & \beta-\alpha & \alpha-\beta \\ \alpha-\beta & 2\wp_1 u - (\alpha+\beta) & \alpha-\beta & \beta-\alpha & \beta-\alpha \\ \beta-\alpha & \alpha-\beta & 2\wp_2 u - (\alpha+\beta) & \alpha-\beta & \beta-\alpha \\ \beta-\alpha & \beta-\alpha & \alpha-\beta & 2\wp_3 u - (\alpha+\beta) & \alpha-\beta \\ \alpha-\beta & \beta-\alpha & \beta-\alpha & \alpha-\beta & 2\wp_4 u - (\alpha+\beta) \end{bmatrix}.$$

These five points are of course coplanar. To verify this analytically by showing that all the quartic minors in the matrix vanish identically in u would probably be excessively laborious, but is not necessary.

We are now in a position to prove

Theorem 4. *The* GRASSMANN *coordinates p_{ij} of the plane $\pi(u)$ containing the cubic curve $C(u)$ on R^5 are proportional to*

$$(10)\qquad \begin{cases} p_{23}(u) = -p_{32}(u) = \lambda(\wp_0 u - \beta)/(\wp_0 u - \alpha) = (\alpha-\beta)(\zeta_1 u - \zeta_4 u - \theta - \varphi) \\ p_{34}(u) = -p_{43}(u) = \lambda(\wp_1 u - \beta)/(\wp_1 u - \alpha) = (\alpha-\beta)(\zeta_2 u - \zeta_0 u - \theta - \varphi) \\ p_{40}(u) = -p_{04}(u) = \lambda(\wp_2 u - \beta)/(\wp_2 u - \alpha) = (\alpha-\beta)(\zeta_3 u - \zeta_1 u - \theta - \varphi) \\ p_{01}(u) = -p_{10}(u) = \lambda(\wp_3 u - \beta)/(\wp_3 u - \alpha) = (\alpha-\beta)(\zeta_4 u - \zeta_2 u - \theta - \varphi) \\ p_{12}(u) = -p_{21}(u) = \lambda(\wp_4 u - \beta)/(\wp_4 u - \alpha) = (\alpha-\beta)(\zeta_0 u - \zeta_3 u - \theta - \varphi) \\ p_{41}(u) = -p_{14}(u) = \mu(\wp_0 u - \alpha)/(\wp_0 u - \beta) = (\alpha-\beta)(\zeta_3 u - \zeta_2 u - \theta + \varphi) \\ p_{02}(u) = -p_{20}(u) = \mu(\wp_0 u - \alpha)/(\wp_1 u - \beta) = (\alpha-\beta)(\zeta_4 u - \zeta_3 u - \theta + \varphi) \\ p_{13}(u) = -p_{31}(u) = \mu(\wp_2 u - \alpha)/(\wp_2 u - \beta) = (\alpha-\beta)(\zeta_0 u - \zeta_4 u - \theta + \varphi) \\ p_{24}(u) = -p_{42}(u) = \mu(\wp_3 u - \alpha)/(\wp_3 u - \beta) = (\alpha-\beta)(\zeta_1 u - \zeta_0 u - \theta + \varphi) \\ p_{30}(u) = -p_{03}(u) = \mu(\wp_4 u - \alpha)/(\wp_4 u - \beta) = (\alpha-\beta)(\zeta_2 u - \zeta_1 u - \theta + \varphi) \end{cases}$$

Proof. The coordinates p_{ij} are proportional to the cubic minors formed from any three rows of the matrix (9), since for general u no three of the five points are collinear ; or equally, of course, as the matrix is symmetrical, from any three of the five columns. The ten-by-ten matrix of cubic minors is symmetrical and of rank 1, so that the ten diagonal elements are proportional to the squares of the ten elements in any one row or column. This means that the coordinates p_{ij} that we are seeking are proportional to the square roots of the ten diagonal minors of (9). In particular, the minor

$$D_{23}(u) = \begin{vmatrix} 2\wp_0 u - (\alpha+\beta) & \alpha-\beta & \alpha-\beta \\ \alpha-\beta & 2\wp_1 u - (\alpha+\beta) & \beta-\alpha \\ \alpha-\beta & \beta-\alpha & 2\wp_4 u - (\alpha+\beta) \end{vmatrix}$$

is clearly an elliptic function of u, with at most double poles at the points $u = 0, \pm\sigma$; it has no pole at the origin however, since the expansion of the diagonal elements at the origin gives

$$\begin{vmatrix} 2u^{-2} - (\alpha + \beta) + \ldots & \alpha - \beta & \alpha - \beta \\ \alpha - \beta & \alpha - \beta - 2\lambda u + \ldots & \beta - \alpha \\ \alpha - \beta & \beta - \alpha & \alpha - \beta + 2\lambda u + \ldots \end{vmatrix}$$

$$= -8\lambda^2 + O(u^2).$$

The coefficient of $\mathfrak{p}_1 u$ is not zero at $u = \sigma$, so that there is in fact a pole of order 2 there, and similarly at $u = -\sigma$. At $u = 2\sigma$ the corresponding expansion is

$$\begin{vmatrix} \beta - \alpha + 2\mu(u - 2\sigma) + \ldots & \alpha - \beta & \alpha - \beta \\ \alpha - \beta & \alpha - \beta - 2\lambda(u - 2\sigma) + \ldots & \beta - \alpha \\ \alpha - \beta & \beta - \alpha & \beta - \alpha - 2\mu(u - 2\sigma) + \ldots \end{vmatrix}$$

$$= O(u - 2\sigma)^2,$$

so that $D_{23}(u)$ has a double zero at $u = 2\sigma$, and similarly at $u = -2\sigma$. These double poles and zeros, and the value $-8\lambda^2$ at the origin, show that

$$D_{23}(u) = -8\lambda^2 \left(\frac{\mathfrak{p}u - \beta}{\mathfrak{p}u - \alpha} \right)^2 .$$

Omitting the factor -8, which will clearly be present in the same way in all the cubic minors to be considered, this gives the square of the first expression for $p_{23}(u)$ in (10). The second expression follows from the fact that $p_{23}(u)$, being an elliptic function of order 2 with poles at $u = \pm\sigma$, must be of the form $A(\zeta_0(u - \sigma) - \zeta_0(u + \sigma) + C)$, for some constants A, C; since it vanishes at $u = 2\sigma$, $C = -(\theta + \varphi)$; and since its value at the origin is λ,

$$A(3\theta + \varphi) = -\lambda, \quad A = \alpha - \beta, \text{ by } (7)$$

The expressions for $p_{41}(u)$ are obtained from these by the substitution of 2σ for σ, which as we have seen simultaneously substitutes $\beta, \alpha, \varphi, -\theta, \mu, -\lambda$ for $\alpha, \beta, \theta, \varphi, \lambda, \mu$ respectively, and permutes the suffixes 1, 2, 4, 3, and the corresponding rows and columns of the matrix (9), cyclically. Finally, the remaining coordinates are found from these two by the substitution of $u - \sigma$, $u - 2\sigma$, $u - 3\sigma$, $u - 4\sigma$ in turn for u, corresponding to the cyclic permutation of the suffixes 0, 1, 2, 3, 4, and of the corresponding rows and columns of the matrix (9). This completes the proof of Theorem 4.

The second forms of the coordinates $p_{ij}(u)$ in (10) are most convenient for differentiation, since for any constants, a, b the functions $\zeta_0(u - a) - \zeta_0(u - b)$ and $\zeta(u - a) - \zeta(u - b)$ differ only by a constant, and $\zeta' u = -\mathfrak{p} u$. We thus have immediately

$$\left.\begin{array}{ll} p_{23}'(u) = (\alpha - \beta)(\mathfrak{p}_4 u - \mathfrak{p}_1 u) & p_{41}'(u) = (\alpha - \beta)(\mathfrak{p}_2 u - \mathfrak{p}_3 u) \\ p_{34}'(u) = (\alpha - \beta)(\mathfrak{p}_0 u - \mathfrak{p}_2 u) & p_{02}'(u) = (\alpha - \beta)(\mathfrak{p}_3 u - \mathfrak{p}_4 u) \\ p_{40}'(u) = (\alpha - \beta)(\mathfrak{p}_1 u - \mathfrak{p}_3 u) & p_{13}'(u) = (\alpha - \beta)(\mathfrak{p}_4 u - \mathfrak{p}_0 u) \\ p_{01}'(u) = (\alpha - \beta)(\mathfrak{p}_2 u - \mathfrak{p}_4 u) & p_{24}'(u) = (\alpha - \beta)(\mathfrak{p}_0 u - \mathfrak{p}_1 u) \\ p_{12}'(u) = (\alpha - \beta)(\mathfrak{p}_3 u - \mathfrak{p}_0 u) & p_{30}'(u) = (\alpha - \beta)(\mathfrak{p}_1 u - \mathfrak{p}_2 u) \end{array}\right\} \quad (11)$$

Though it is not strictly necessary, it is instructive to verify the GRASSMANN equations, which are the necessary and (provided the whole matrix is not zero) sufficient conditions for a skew symmetric matrix p_{ij} $(i,j = 0,\ldots,4)$ to be the GRASSMANN coordinates of a plane (or a line) in four dimensions. These are the vanishing of the five Pfaffian forms $P_0, \ldots, P_4$, where $P_0 = p_{23}p_{14} + p_{31}p_{24} + p_{12}p_{34}$, and $P_1, \ldots, P_4$ are obtained from this by the cyclic permutation of 0, 1, 2, 3, 4. We shall write these

$$P_i = \sum\nolimits_i p_{jk}\, p_{lm}, \quad (i = 0,\ldots,4), \quad (12)$$

where $\sum_i$ denotes for $i = 0,\ldots,4$ the summation over the appropriate three permutations of the four suffixes other than i. It is obviously sufficient to verify one of these relations as an identity in u, since whatever function of u P_0 may be, P_1, P_2, P_3, P_4 are the same function of $u-\sigma$, $u-2\sigma$, $u-3\sigma$, $u-4\sigma$ respectively.

Now from the first form of (10), $p_{23}(u)\,p_{14}(u) = -\lambda\mu$. $p_{31}(u)$ has simple poles at $u = 0, -\sigma$, and simple zeros at $u = \sigma, -2\sigma$; and $p_{24}(u)$ has simple poles at $u = 0$, σ, and simple zeros at $u = -\sigma$, 2σ. Thus the product $p_{31}(u)\,p_{24}(u)$ has a double pole at the origin and simple zeros at $u = \pm 2\sigma$, the other pole of each factor being cancelled by a zero of the other. Its expansion at the origin has the leading term $(\alpha-\beta)^2u^{-2}$, the residue of each factor there being $-(\alpha-\beta)$, by the second form of (10). Thus $p_{31}(u)\,p_{24}(u) = (\alpha-\beta)^2\,(\mathfrak{p}u-\beta)$; smilarly $p_{12}(u)\,p_{34}(u) = -(\alpha-\beta)^2\,(\mathfrak{p}u-\alpha)$; and consequently

$$\sum\nolimits_0 p_{jk}(u)\,p_{lm}(u) = -\lambda\mu + (\alpha-\beta)^2\,(\mathfrak{p}u-\beta) - (\alpha-\beta)^2\,(\mathfrak{p}u-\alpha)$$
$$= -\lambda\mu + (\alpha-\beta)^3 = 0$$

by (6). Thus, identically in u,

$$\sum\nolimits_i p_{jk}(u)\,p_{lm}(u) = 0 \qquad (i = 0,\ldots,4)\,; \tag{13}$$

and hence also, identically in u,

$$\sum\nolimits_i (p_{jk}(u)\,p_{lm}'(u) + p_{jk}'(u)\,p_{lm}(u)) = 0 \qquad (i = 0,\ldots,4) \tag{14}$$

$$\sum\nolimits_i (p_{jk}(u)\,p_{lm}''(u) + 2p_{jk}'(u)\,p_{lm}'(u) + p_{jk}''(u)\,p_{lm}(u)) = 0 \qquad (i = 0,\ldots,4). \tag{15}$$

It is perhaps also worth verifying that (10) gives the correct values for the coordinates of the planes $\pi_i = \pi(i\sigma)$ $(i = 0,\ldots,4)$, whose equations are known. At $u = 0$, $p_{12}(u), p_{13}(u), p_{24}(u), p_{34}(u)$ have simple poles with the residues $\alpha-\beta$, $\alpha-\beta$, $\beta-\alpha$, $\beta-\alpha$ respectively. Thus for the plane π_0,

$$p_{12} : p_{13} : p_{24} : p_{34} : \text{ any other } p_{ij} = 1 : 1 : -1 : -1 : 0,$$

which agrees with the values found from the quadratic minors of the coordinate matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

of the primes $x_1 + x_4 = 0$, $x_2 + x_3 = 0$ which intersect in π_0.

5. Parametrisation of the surface

The ordinary expression $x_i = \sum_i (p_{jk}\,q_{lm} + q_{jk}\,p_{lm})$ $(i = 0,\ldots,4)$ for the coordinates of the point of intersection of two planes with Grassmann coordinates p_{ij}, q_{ij}, where $\sum_i$ is the summation defined in (12), gives us at once for the point (u, u') on the surface R^5, which is the intersection of the planes $\pi(u)$, $\pi(u')$, the coordinates

$$x_i(u, u') = \sum\nolimits_i (p_{jk}(u)\,p_{lm}(u') + p_{jk}(u')\,p_{lm}(u)). \qquad (i = 0,\ldots,4) \tag{16}$$

For constant u', these are functions of u of order 5, with simple poles at the points $u = i\sigma$ $(i = 0,\ldots,4)$; all five of these functions of u however have double zeros at $u = u'$, by (13), (14); thus the quotient of any two of them, or if we prefer to keep the coordinates homogeneous symmetrical the quotient of each over the sum of all five, is a function of order 3, in accordance with the fact that $C(u')$, of which these functions of u furnish a parametrisation, is a cubic curve.

Now for any constant v, as u varies, the point $(u+v, u-v)$ describes a curve quadrisecant to the generators, and bisecant to the curves $\{C^5\}$, and hence of order 10. in fact, it meets the generator $l(w)$ in the points

$$\left(\frac{1}{2}w+v, \frac{1}{2}w-v\right) \quad \text{and} \quad \left(\frac{1}{2}w+v+\omega_i, \frac{1}{2}w-v+\omega_i\right) \qquad (i=1,\ 2,\ 3),$$

where ω_1, ω_2, ω_3 are the three primitive half periods, i. e. any two of $2\omega_1$, $2\omega_2$, $2\omega_3$ generate the period lattice 2Ω, and $\sum_{i=1}^{3}\omega_i = 0$; and it meets the curve $C(u)$ in the points $(u, u\pm 2v)$. Three of these curves we shall now parametrise, namely those for $v = 0, \pm\sigma$ and $\pm 2\sigma$, the first being the focal curve, envelope of the family C.

Theorem 5. *The coordinates of the point (u, u), the point of contact of $C(u)$ with the focal curve, are proportional to*

$$\left.\begin{aligned}
X_0(u) &= (\alpha-\beta)\,\mathfrak{p}_0u + (\lambda-\mu)\,(\zeta_1u-\zeta_4u) + (\lambda+\mu)\,(\zeta_3u-\zeta_2u) + (\alpha^2-\beta^2)\,/\,\mathrm{i}0\\
X_1(u) &= (\alpha-\beta)\,\mathfrak{p}_1u + (\lambda-\mu)\,(\zeta_2u-\zeta_0u) + (\lambda+\mu)\,(\zeta_4u-\zeta_3u) + (\alpha^2-\beta^2)\,/\,\mathrm{i}0\\
X_2(u) &= (\alpha-\beta)\,\mathfrak{p}_2u + (\lambda-\mu)\,(\zeta_3u-\zeta_1u) + (\lambda+\mu)\,(\zeta_0u-\zeta_4u) + (\alpha_2-\beta^2)\,/\,\mathrm{i}0\\
X_3(u) &= (\alpha-\beta)\,\mathfrak{p}_3u + (\lambda-\mu)\,(\zeta_4u-\zeta_2u) + (\lambda+\mu)\,(\zeta_1u-\zeta_0u) + (\alpha^2-\beta^2)\,/\,\mathrm{i}0\\
X_4(u) &= (\alpha-\beta)\,\mathfrak{p}_4u + (\lambda-\mu)\,(\zeta_0u-\zeta_3u) + (\lambda+\mu)\,(\zeta_2u-\zeta_1u) + (\alpha^2-\beta^2)\,/\,\mathrm{i}0
\end{aligned}\right\} \qquad (17)$$

Proof. We cannot of course simply substitute $u' = u$ in (16), since this makes all the coordinates vanish, by (13). But putting $u' = u + v$, and expanding as power series in v, we have by (13), (14), and (15)

$$\begin{aligned}
x_i(u, u+v) &= \textstyle\sum_i (p_{jk}(u)\,p_{lm}{}''(u) + p_{jk}{}''(u)\,p_{lm}(u))\,v^2 + O(v^3)\\
&= -2\textstyle\sum_i p_{jk}{}'(u)\,p_{lm}{}'(u)\,.\,v^2 + O(v^3) \qquad (i=0,\ldots,4),
\end{aligned}$$

so that in the limit as v tends to zero, the coordinates of (u, u) are proportional to $\sum_i p_{jk}{}'(u)\,p_{lm}{}'(u)$ $(i = 0,\ldots,4)$. Taking the derivatives in the form (11), omitting the factor $(\alpha-\beta)$ common to all of them, and removing a further factor -2 which will appear in the course of simplification, we define

$$\begin{aligned}
-2X_0(u) &= (\alpha-\beta)^{-2}\,\left(p_{23}{}'(u)\,p_{14}{}'(u) + p_{31}{}'(u)\,p_{24}{}'(u) + p_{12}{}'(u)\,p_{34}{}'(u)\right)\\
&= (\mathfrak{p}_4u-\mathfrak{p}_1u)(\mathfrak{p}_3u-\mathfrak{p}_2u) + (\mathfrak{p}_0u-\mathfrak{p}_4u)(\mathfrak{p}_0u-\mathfrak{p}_1u) + (\mathfrak{p}_3u-\mathfrak{p}_0u)(\mathfrak{p}_0u-\mathfrak{p}_2u)\\
&= \mathfrak{p}_4u\,\mathfrak{p}_1u + \mathfrak{p}_0u\,\mathfrak{p}_2u - \mathfrak{p}_1u\,\mathfrak{p}_3u - \mathfrak{p}_2u\,\mathfrak{p}_4u + \mathfrak{p}_3u\,\mathfrak{p}_0u\\
&\quad - \mathfrak{p}_2u\,\mathfrak{p}_3u + \mathfrak{p}_3u\,\mathfrak{p}_4u - \mathfrak{p}_4u\,\mathfrak{p}_0u - \mathfrak{p}_0u\,\mathfrak{p}_1u + \mathfrak{p}_1\,\mathfrak{p}_2u
\end{aligned} \qquad (18)$$

and $-2X_1(u),\ldots,-2X_4(u)$ consist of the same ten terms, with the obvious cyclic changes of sign.

Now $\mathfrak{p}_iu\,\mathfrak{p}_ju$ has double poles at $u = i\sigma, j\sigma$, and is thus a linear combination of $\mathfrak{p}_iu$, $\mathfrak{p}_ju$ ζ_iu, ζ_ju, and a constant term. In fact, as $\mathfrak{p}_4u$ has the expansion $\mathfrak{p}_4u = \beta + \mu(u-\sigma) + 0(u-\sigma)^2$ at $u = \sigma$, $\mathfrak{p}_1u\,\mathfrak{p}_4u$ has the expansion $\mathfrak{p}_1u\,\mathfrak{p}_4u = \beta(u-\sigma)^{-2} + \mu(u-\sigma)^{-1} + 0(1)$ there, and similarly $\mathfrak{p}_1u\,\mathfrak{p}_4u = \beta(u+\sigma)^{-2} - \mu(u+\sigma)^{-1} + 0(1)$ at $u = -\sigma$; thus

$$\mathfrak{p}_1u\,\mathfrak{p}_4u = \beta(\mathfrak{p}_1u + \mathfrak{p}_4u) + \mu(\zeta_1u - \zeta_4u) + C,$$

where C is a constant, to be determined by comparing the values of both sides at the origin. which gives $C = \alpha^2 - 2\alpha\beta + 2\mu\vartheta$. $\mathfrak{p}_2u\,\mathfrak{p}_3u$ is found from this by the substitution of 2σ for σ, with the corresponding interchanges of the constants, and the other products from these two by the cyclic permutation of 0, 1, 2, 3, 4. We obtain

$$\left.\begin{aligned}
&\wp_1 u\,\wp_4 u = \beta(\wp_1 u+\wp_4 u)+\mu(\zeta_1 u-\zeta_4 u)+C, &&\wp_2 u\,\wp_3 u=\alpha(\wp_2 u+\wp_3 u)-\lambda(\zeta_2 u-\zeta_3 u)+D\\
&\wp_2 u\,\wp_0 u = \beta(\wp_2 u+\wp_0 u)+\mu(\zeta_2 u-\zeta_0 u)+C, &&\wp_3 u\,\wp_4 u=\alpha(\wp_3 u+\wp_4 u)-\lambda(\zeta_3 u-\zeta_4 u)+D\\
&\wp_3 u\,\wp_1 u = \beta(\wp_3 u+\wp_1 u)+\mu(\zeta_3 u-\zeta_1 u)+C, &&\wp_4 u\,\wp_0 u=\alpha(\wp_4 u+\wp_0 u)-\lambda(\zeta_4 u-\zeta_0 u)+D\\
&\wp_4 u\,\wp_2 u = \beta(\wp_4 u+\wp_2 u)+\mu(\zeta_4 u-\zeta_2 u)+C, &&\wp_0 u\,\wp_1 u=\alpha(\wp_0 u+\wp_1 u)-\lambda(\zeta_0 u-\zeta_1 u)+D\\
&\wp_0 u\,\wp_3 u = \beta(\wp_0 u+\wp_3 u)+\mu(\zeta_0 u-\zeta_3 u)+C, &&\wp_1 u\,\wp_2 u=\alpha(\wp_1 u-\wp_2 u)-\lambda(\zeta_1 u-\zeta_2 u)+D
\end{aligned}\right\}(19)$$

$$C=\alpha^2-2\alpha\beta+2\mu\theta,\qquad D=\beta^2-2\alpha\beta-2\lambda\varphi.$$

Substituting from (19) in (18) we have

$$-2X_0(u)=2(\beta-\alpha)\wp_0 u+2(\lambda-\mu)(\zeta_4 u-\zeta_1 u)+2(\lambda+\mu)(\zeta_2 u-\zeta_3 u)+C-D,$$

which as

$$C-D=\alpha^2-\beta^2+2(\lambda\varphi+\mu\theta)=(\beta^2-\alpha^2)/5$$

by (8), verifies the expression for $X_0(u)$ in (17). The others arc obtained from this by cyclic permutation of 0, 1, 2, 3, 4, and Theorem 5 is accordingly proved.

The general linear combination of $X_0(u),\dots,X_4(u)$ has double poles at $u=0,\sigma,2\sigma,3\sigma,4\sigma$, and is thus of order 10, which accords with the fact that the focal curve, of which these functions supply the parametrisation, is of order 10. Each of the individual functions $X_i(u)$ however is of order 6, having a double pole at $u=i\sigma$ only, and simple poles at the other four points ; thus for $u=0,\sigma,2\sigma,3\sigma,4\sigma$ respectively the five functions are proportional to

$$(1,0,0,0,0),\ (0,1,0,0,0),\ (0,0,1,0,),\ (0,0,0,1,0),\ (0,0,0,0,1)$$

confirming that these five points on the focal curve are the vertices P_{ii} ($i=0,\dots,4$) of the simplex of reference.

The tangents to the focal curve at the points P_{ii} ($i=0,\dots,4$) are easily found. For that at P_{00} for instance, $x_1,\dots,x_4$ are proportional to the values at $u=0$ of the derivatives of $X_1(u),\dots,X_4(u)$ each of which contains a term in $\wp_0 u$ and no other term which is infinite at $u=0$; thus for the tangent, $x_1,\dots,x_4$ are proportional to the coefficients of $\wp_0 u$ in the derivatives of the four functions, i. e. to those of $-\zeta_0 u$ in the functions themselves. The tangent to the focal curve at P_{00} is accordingly

$$x_1:x_2:x_3:x_4=(\lambda-\mu):-(\lambda+\mu):(\lambda+\mu):(\mu-\lambda). \tag{20}$$

This is seen to lie in the plane $\pi_0: x_1+x_4=x_2+x_3=0$, as we expect, since the focal curve, being the envelope of the family $\{C^3\}$, touches at P_{00} the curve $C(0)$, lying in the plane π_0.

We next prove

Theorem 6 . *The coordinates of the point* $(u+\sigma, u-\sigma)$ *are proportional to*

$$\left.\begin{aligned}
Y_0(u)&=(\alpha-\beta)(\wp_0 u+\wp_1 u-\wp_2 u-\wp_3 u+\wp_4 u)+2\lambda(\zeta_4 u-\zeta_2 u+\zeta_3 u-\zeta_4 u)-K\\
Y_1(u)&=(\alpha-\beta)(\wp_1 u+\wp_2 u-\wp_3 u-\wp_4 u+\wp_0 u)+2\lambda(\zeta_2 u-\zeta_3 u+\zeta_4 u-\zeta_0 u)-K\\
Y_2(u)&=(\alpha-\beta)(\wp_2 u+\wp_3 u-\wp_4 u-\wp_0 u+\wp_1 u)+2\lambda(\zeta_3 u-\zeta_4 u+\zeta_0 u-\zeta_1 u)-K\\
Y_3(u)&=(\alpha-\beta)(\wp_3 u+\wp_4 u-\wp_0 u-\wp_1 u+\wp_2 u)+2\lambda(\zeta_4 u-\zeta_0 u+\zeta_1 u-\zeta_2 u)-K\\
Y_4(u)&=(\alpha-\beta)(\wp_4 u+\wp_0 u-\wp_1 u-\wp_2 u+\wp_3 u)+2\lambda(\zeta_0 u-\zeta_1 u+\zeta_2 u-\zeta_3 u)-K
\end{aligned}\right\}(21)$$

and those of the point $(u+2\sigma, u-2\sigma)$ *are proportional to*

$$\left.\begin{aligned}
Z_0(u)&=(\alpha-\beta)(\wp_0 u-\wp_1 u+\wp_2 u+\wp_3 u-\wp_4 u)-2\mu(\zeta_1 u+\zeta_2 u-\zeta_3 u-\zeta_4 u)-L\\
Z_1(u)&=(\alpha-\beta)(\wp_1 u-\wp_2 u+\wp_3 u+\wp_4 u-\wp_0 u)-2\mu(\zeta_2 u+\zeta_3 u-\zeta_4 u-\zeta_0 u)-L\\
Z_2(u)&=(\alpha-\beta)(\wp_2 u-\wp_3 u+\wp_4 u+\wp_0 u-\wp_1 u)-2\mu(\zeta_3 u+\zeta_4 u-\zeta_0 u-\zeta_1 u)-L\\
Z_3(u)&=(\alpha-\beta)(\wp_3 u-\wp_4 u+\wp_0 u+\wp_1 u-\wp_2 u)-2\mu(\zeta_4 u+\zeta_0 u-\zeta_1 u-\zeta_2 u)-L\\
Z_4(u)&=(\alpha-\beta)(\wp_4 u-\wp_0 u+\wp_1 u+\wp_2 u-\wp_3 u)-2\mu(\zeta_0 u+\zeta_1 u-\zeta_2 u-\zeta_3 u)-L
\end{aligned}\right\}(21')$$

where

$$K = \alpha\ (\alpha - \beta) + 2\lambda\varphi, \quad L = \beta\ (\alpha - \beta) + 2\mu\theta.$$

Proof. By (16), the coordinates of $(u+\sigma, u-\sigma)$ are proportonal to the five sums

$$\sum_i (p_{jk}(u+\sigma)\, p_{lm}(u-\sigma) + p_{jk}(u-\sigma)\, p_{im}(u+\sigma)) \qquad (i = 0,\ldots,4).$$

But evidently $p_{ij}(u+\sigma) = p_{i-1,\, j-1}(u)$ and $p_{ij}(u-\sigma) = p_{i+1,\, j+1}(u)$, from the way (10) were obtained by the cyclic permutation, the suffixes here being again treated as residues (mod 5). Thus the first of these sums,

$$\begin{aligned} &p_{23}(u+\sigma)\, p_{14}(u-\sigma) + p_{14}(u+\sigma)\, p_{23}(u-\sigma) + p_{31}(u+\sigma)\, p_{24}(u-\sigma) \\ &+ p_{24}(u+\sigma)\, p_{31}(u-\sigma) + p_{12}(u+\sigma)\, p_{34}(u-\sigma) + p_{34}(u+\sigma)\, p_{12}(u-\sigma) \\ &\qquad = p_{12}(u)\, p_{20}(u) + p_{03}(u)\, p_{34}(u) + p_{20}(u)\, p_{30}(u) \\ &\qquad\quad + p_{13}(u)\, p_{42}(u) + p_{01}(u)\, p_{40}(u) + p_{23}^2(u) \end{aligned} \tag{22}$$

These six terms are functions of several different types, and require to be evaluated separately.

$p_{23}^2(u)$ has double poles at $u = \pm\sigma$, and double zeros at $u = \pm 2\sigma$. At $u = \sigma$,

$$\begin{aligned} p_{23}(u) &= (\alpha - \beta)\,((u-\sigma)^{-1} - (\theta + 2\varphi) + O(u-\sigma)) \\ &= (\alpha - \beta)\,(u-\sigma)^{-1} + \frac{1}{2}(\lambda - \mu) + O(u-\sigma) \end{aligned}$$

so that

$$p_{23}^2(u) = (\alpha-\beta)^2 (u-\sigma)^{-2} + (\lambda-\mu)(\alpha-\beta)(u-\sigma)^{-1} + O(1).$$

Hence

$$(\alpha-\beta)^{-1} p_{23}^2(u) = (\alpha-\beta)(\wp_1 u + \wp_4 u) + (\lambda-\mu)(\zeta_1 u - \zeta_4 u) - (\alpha^2 - \beta^2) - (\lambda-\mu)(\theta+\varphi), \tag{23}$$

the constant term being determined by the zeros. (It is easily verified that the derivative also vanishes at these).

$p_{42}(u)\, p_{13}(u) = p_{31}(u)\, p_{24}(u)$ was determined as $(\alpha-\beta)^2 (\wp_0 u - \beta)$ in the course of verifying the Grassmann relations. We therefore write

$$(\alpha-\beta)^{-1} p_{42}(u)\, p_{13}(u) = (\alpha-\beta)\,\wp_0 u - \beta\,(\alpha-\beta). \tag{24}$$

$p_{20}(u)\, p_{30}(u)$ has simple poles at $u = \pm\sigma$, and a double zero at $u = 0$. At $u = \sigma$, $p_{20}(u) = -\mu$, and $p_{30}(u)$ has residue $(\beta - \alpha)$. Thus

$$(\alpha-\beta)^{-1} p_{20}(u)\, p_{30}(u) = \mu\,(\zeta_1 u - \zeta_4 u) + 2\mu\theta \tag{25}$$

$p_{01}(u)\, p_{40}(u)$ is obtained from this last by the substitution of 2σ for σ, together with change of sign ; thus

$$(\alpha-\beta)^{-1} p_{01}(u)\, p_{40}(u) = \lambda\,(\zeta_3 u - \zeta_2 u) - 2\lambda\varphi. \tag{26}$$

$p_{34}(u)\, p_{03}(u)$ has a double pole at $u = 2\sigma$ and simple at $u = \sigma$, and a double zero at $u = 3\sigma$ and simple at $u = 4\sigma$. At $u = 2\sigma$ the two factors have residues $\pm(\alpha - \beta)$, and at $u = \sigma$ $p_{03}(u)$ has residue $(\alpha - \beta)$ and $p_{34}(u) = \lambda$. Thus

$$(\alpha-\beta)^{-1} p_{34}(u)\, p_{03}(u) = -(\alpha-\beta)\,\wp_2 u - \lambda\,(\zeta_1 u - \zeta_2 u) + \alpha\,(\alpha-\beta) + \lambda\,(\theta - \varphi), \tag{27}$$

the constant term being determined so as to make the function vanish at $u = 3\sigma$; it is easily verified that the derivative also vanishes, here, and that the function also vanishes at $u = 4\sigma$, using (6), (7).

Finally $p_{12}(u)\,p_{20}(u)$ is obtained from this last by substituting 4σ (or $-\sigma$) for σ, which interchanges the suffixes (14) (23) by pairs, leaves α, β unchanged, and changes the sign of $\lambda, \mu, \theta, \varphi$. Thus

$$(\alpha-\beta)^{-1}\,p_{12}(u)\,p_{20}(u) = -(\alpha-\beta)\,\mathfrak{p}_3 u + \lambda(\zeta_3 u - \zeta_4 u) + \alpha(\alpha-\beta) + \lambda(\theta-\varphi). \quad (28)$$

Adding up now the right hand members of (23),..., (28), we obtain the value of $Y_0(u)$ in (21) as that of $(\alpha-\beta)^{-1}$ times the right hand member of (22) ; the constant term being

$$-K = (\alpha-\beta)(\alpha-2\beta) + \lambda(\theta-3\varphi) + \mu(3\theta+\varphi) - 2\lambda\varphi = -\alpha(\alpha-\beta) - 2\lambda\varphi,$$

since by (7), (6)

$$\lambda(\theta-3\varphi) = \mu(3\theta+\varphi) = \frac{\lambda\mu}{\beta-\alpha} = -(\alpha-\beta)^2 .$$

$Z_0(u)$ is obtained from $Y_0(u)$ by substituting 2σ for σ, and changing the sign throughout ; and $Y_i(u)$, $Z_i(u)$ $(i=1,...,4)$ from these two by the usual cyclic permutation. Theorem 6 is thus proved.

We note that $K+L = -(\alpha^2-\beta^2)/5$, by (8). Thus trivially $2X_i(u) = Y_i(u) + Z_i(u)$ $(i=0,...,4)$, expressing the collinearity of the points (u, u), $(u+\sigma, u-\sigma)$, $(u+2\sigma, u-2\sigma)$, which are all on the generator $l(2u)$. As a simple extension of this we now prove

Theorem 7. *The coordinates of the point* $(u+v, u-v)$ *of* R^5 *are proportional to*

$$2\mathfrak{p}\,v \,.\, X_i(u) - W_i(u) \qquad (i=0,...,4),$$

where $X_i(u)$ $(i=0,...,4)$ *are as defined* in (19), *and*

$$W_i(u) = \beta Y_i(u) + \alpha Z_i(u) \qquad (i=0,...,4),$$

$Y_i(u)$, $Z_i(u)$ $(i=0,...,4)$ *being as defined in* (21), (21').

Proof. Evidently these coordinates are proportional to $Y_i(u) + f(v)\,Z_i(u)$, where $f(v)$ is a function of v (mod 2Ω), of order 2, zero at $v=\pm\sigma$, infinite at $v=\pm 2\sigma$, and with the value 1 at $v=0$. This means that $f(v) = \dfrac{\mathfrak{p}v-\alpha}{\mathfrak{p}v-\beta}$, and the coordinates in question are proportional to

$$(\mathfrak{p}v-\beta)\,Y_i(u) + (\mathfrak{p}v-\alpha)\,Z_i(u) \qquad (i=0,...,4),$$

which is the theorem.

6. Grassmann coordinates of the generators

The generators $l(w)$ of R^5 form of course a system of lines dual to the system $\pi(u)$ of planes ; and in dealing with these, in order to utilise as much as possible of the work already done, it is convenient to introduce temporarily a new coordinate system $(y_0,...,y_4)$, whose relation to the whole figure is dual to that of the coordinate system $(x_0,...,x_4)$, so that instead of the points P_{ii} $(i=0,...,4)$ being the vertices of the simplex of reference, the primes Σ_{ii} $(i=0,...,4)$ are its faces. As the equations of these primes, in the original coordinate system, are the vanishing the of sums of all but one of the coordinates, we write

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad (29)$$

Any permutation on $(x_0, \dots, x_4)$ clearly induces the same permutation on $(y_0, \dots, y_4)$; thus the whole figure is still invariant under the group $S_5 \cap S'_5$. We must bear in mind however, that as the lines l_0, l_1, l_2, l_3, l_4 are the generators $l(0)$, $l(2\sigma)$, $l(4\sigma)$, $l(\sigma)$, $l(3\sigma)$ respectively, the cyclic permutation of 0, 1, 2, 3, 4 applied to either system of coordinates cooresponds to the substitution of $w - 2\sigma$, not $w - \sigma$, for w, and consequently induces the cyclic permutation of 0, 2, 4, 1, 3 on the functions $\zeta_i w$ or $\wp_i w$. We now prove

Theorem 8. *The Grassmann coordinates of the generator $l(w)$ of R^5 are proportional to*

$$\left.\begin{aligned}
q_{23}(w) &= \zeta_1 w - \zeta_4 w - 2(\theta - \varphi) & q_{41}(w) &= \zeta_3 w - \zeta_2 w + 2(\theta + \varphi)\\
q_{34}(w) &= \zeta_3 w - \zeta_1 w - 2(\theta - \varphi) & q_{02}(w) &= \zeta_0 w - \zeta_4 w + 2(\theta + \varphi)\\
q_{40}(w) &= \zeta_0 w - \zeta_3 w - 2(\theta - \varphi) & q_{13}(w) &= \zeta_2 w - \zeta_4 w + 2(\theta + \varphi)\\
q_{01}(w) &= \zeta_2 w - \zeta_0 w - 2(\theta - \varphi) & q_{24}(w) &= \zeta_4 w - \zeta_3 w + 2(\theta + \varphi)\\
q_{12}(w) &= \zeta_4 w - \zeta_2 w - 2(\theta - \varphi) & q_{30}(w) &= \zeta_1 w - \zeta_0 w + 2(\theta + \varphi)
\end{aligned}\right\} \quad (30)$$

Proof. The prime $y_0 - y_1 + y_2 + y_3 - y_4 = 0$, or $x_1 + x_4 = 0$, contains the plane π_0, and the lines l_2, l_3, or $l(\pm\sigma)$; the prime $y_0 + y_1 - y_2 - y_3 + y_4 = 0$, or $x_2 + x_3 = 0$, contains π_0 and the lines l_1, l_4, or $l(\pm 2\sigma)$; and the prime $y_0 = 0$ contains π_0 and the line l_0, or $l(0)$. Thus the prime joining π_0 to the generator $l(w)$ is $f(w)\, y_0 - y_1 + y_2 + y_3 - y_4 = 0$, where $f(w)$ is an even elliptic function of w, of order 2, infinite at $w = 0$, and with the values 1 at $w = \pm\sigma$ and -1 at $w = \pm 2\sigma$, i. e. $f(w) = \dfrac{2\wp w - (\alpha + \beta)}{\alpha - \beta}$. Thus the equations of the five primes joining $l(w)$ to the five planes $\pi_0, \dots, \pi_4$ are

$$\begin{aligned}
&(2\wp_0 w - \alpha - \beta)\, y_0 + (\beta - \alpha)\, y_1 + (\alpha - \beta)\, y_2 + (\alpha - \beta)\, y_3 + (\beta - \alpha)\, y_4 = 0\\
&(\beta - \alpha)\, y_0 + (2\wp_2 w - \alpha - \beta)\, y_1 + (\beta - \alpha)\, y_2 + (\alpha - \beta)\, y_3 + (\alpha - \beta)\, y_4 = 0\\
&(\alpha - \beta)\, y_0 + (\beta - \alpha)\, y_1 + (2\wp_4 w - \alpha - \beta)\, y_2 + (\beta - \alpha)\, y_3 + (\alpha - \beta)\, y_4 = 0\\
&(\alpha - \beta)\, y_0 + (\alpha - \beta)\, y_1 + (\beta - \alpha)\, y_2 + (2\wp_1 w - \alpha - \beta)\, y_3 + (\beta - \alpha)\, y_4 = 0\\
&(\beta - \alpha)\, y_0 + (\alpha - \beta)\, y_1 + (\alpha - \beta)\, y_2 + (\beta - \alpha)\, y_3 + (2\wp_3 w - \alpha - \beta)\, y_4 = 0.
\end{aligned}$$

The matrix of coefficients in these equations however is what the matrix (9) becomes, on applying the cyclic permutation (1243) to both rows and columns. Exactly as in Theorem 4, the Grassmann coordinates r_{ij} of $l(w)$, relative to the coordinate system $(y_0, \dots, y_4)$, are proportonal to the square roots of the diagonal cubic minors of this matrix, which we have already found; and as the permutation (1243) is equivalent to doubling each of the symbols (mod 5), we can write, interpreting the suffixes as residue classes, $r_{ij} = p_{2i,\,2j}(w)$, i. e.

$$\begin{aligned}
&r_{23} = p_{41}(w),\ r_{34} = p_{13}(w),\ r_{40} = p_{30}(w),\ r_{01} = p_{02}(w),\ r_{12} = p_{24}(w)\\
&r_{41} = p_{32}(w),\ r_{02} = p_{04}(w),\ r_{13} = p_{21}(w),\ r_{24} = p_{43}(w),\ r_{30} = p_{10}(w).
\end{aligned} \quad (31)$$

Now the coordinate systems $(x_0, \dots, x_4)$, $(y_0, \dots, y_4)$ being related as in (29), the Grassmann coordinates q_{ij} of any line with respect to the system $(x_0, \dots, x_4)$ are linear combinations of its coordinates r_{ij} with respect to the system $(y_0, \dots, y_4)$, with a matrix of coefficients which is the cubic adjoint, i. e. the ten-by-ten matrix of cubic minors, of the matrix of coefficients in (29). These cubic minors are easly found, comparatively few needing to be actually calculated, on account of the symmetry of the matrix in the ordinary sense, as well as its cyclic symmetry. Putting in the values of r_{ij} from (31), and a constant of proportonality ϱ to which we shall give a convenient value later, we have

$$\varrho \begin{bmatrix} q_{23}(w) \\ q_{34}(w) \\ q_{40}(w) \\ q_{01}(w) \\ q_{12}(w) \\ q_{41}(w) \\ q_{02}(w) \\ q_{13}(w) \\ q_{24}(w) \\ q_{30}(w) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 1 & 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 & 0 & 2 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} p_{41}(w) \\ p_{13}(w) \\ p_{30}(w) \\ p_{02}(w) \\ p_{24}(w) \\ p_{32}(w) \\ p_{04}(w) \\ p_{21}(w) \\ p_{43}(w) \\ p_{10}(w) \end{bmatrix}$$

In the ten sums on the right, when we express $p_{ij}(u)$ in the second form given in (10), most of the terms cancel, and on putting $\varrho = 2(\alpha - \beta)$ we quite staightforwardly obtain (30).

We verify, as we did for the coordinates of $\pi(u)$, that for $w = 0, 2\sigma, 4\sigma, \sigma, 3\sigma$, (32) gives the coordinates of the lines l_0, l_1, l_2, l_3, l_4 respectively. For $w = 0$, $q_{40}(w)$, $q_{01}(w)$, $q_{02}(w)$, $q_{30}(w)$, which all contain a term in $\zeta_0 w$, have simple poles with residues $1, -1, 1, -1$ respectively, and the other $q_{ij}(w)$ are all finite, i. e. for the line l_0

$$q_{40} : q_{01} : q_{02} : q_{30} : \text{any other } q_{ij} = 1 : -1 : 1 : -1 : 0,$$

agreeing with the coordinates of l_0 as found from those of any two of

$$P_{00} : (1, 0, 0, 0, 0),\ P_{23} : (1, -1, 1, 1, -1),\ P_{41} : (1, 1, -1, -1, 1).$$

The coordinates of l_1, l_2, l_3, l_4 are similarly given by the poles of $q_{ij}(w)$ at $w = 2\sigma, 4\sigma, \sigma, 3\sigma$ respectively.

Writing the GRASSMANN coordinates of any line or plane, regarded as a vector in a ten dimensional space, in the order

$$(p_{23}, p_{34}, p_{40}, p_{01}, p_{12} ; p_{41}, p_{02}, p_{13}, p_{24}, p_{30}),$$

we define the twelve vectors

$$\left.\begin{array}{ll} \mathbf{p}_0 = (0, -1, 0, 0, 1 ;\ 0, 0, 1, -1, 0) & \mathbf{q}_0 = (0, 0, 1, -1, 0 ;\ 0, 1, 0, 0, -1) \\ \mathbf{p}_1 = (1, 0, -1, 0, 0 ;\ 0, 0, 0, 1, -1) & \mathbf{q}_1 = (0, 0, 0, 1, -1 ;\ -1, 0, 1, 0, 0) \\ \mathbf{p}_2 = (0, 1, 0, -1, 0 ;\ -1, 0, 0, 0, 1) & \mathbf{q}_2 = (-1\ 0, 0, 0, 1 ;\ 0, -1, 0, 1, 0) \\ \mathbf{p}_3 = (0, 0, 1, 0, -1 ;\ 1, -1, 0, 0, 0) & \mathbf{q}_3 = (1, -1, 0, 0, 0 ;\ 0, 0, -1, 0, 1) \\ \mathbf{p}_4 = (-1, 0, 0, 1, 0 ;\ 0, 1, -1, 0, 0) & \mathbf{q}_4 = (0, 1, -1, 0, 0 ;\ 1, 0, 0, -1, 0) \\ \mathbf{a}\ = (1, 1, 1, 1, 1 ;\ 0, 0, 0, 0, 0) & \mathbf{b}\ = (0, 0, 0, 0, 0 ;\ 1, 1, 1, 1, 1) \end{array}\right\} \quad (33)$$

By Theorem 4. a coordinate vector for the plane $\pi(u)$ can be taken to be

$$\mathbf{p}(u) = \mathbf{p}_0 \zeta_0 u + \mathbf{p}_1 \zeta_1 u + \mathbf{p}_2 \zeta_2 u + \mathbf{p}_3 \zeta_3 u + \mathbf{p}_4 \zeta_4 u - \mathbf{a}\,(\theta + \varphi) - \mathbf{b}\,(\theta - \varphi)$$

and by Theorem 8, one for the generator $l(w)$ is

$$\mathbf{q}(w) = \mathbf{q}_0\zeta_0 w + \mathbf{q}_1\zeta_2 w + \mathbf{q}_2\zeta_4 w + \mathbf{q}_3\zeta_1 w + \mathbf{q}_4\zeta_3 w - 2\mathbf{a}(\theta - \varphi) + 2\mathbf{b}(\theta + \varphi).$$

Now the condition for a plane with coordinate vector **p** and a line with coordinate vector **q** to intersect is the vanishing of the scalar product

$$\mathbf{p} \cdot \mathbf{q} = \sum p_{ij}\, q_{ij} = 0.$$

the summation being over the ten pairs ij. From (33) we have at once

$$\mathbf{p}_i \cdot \mathbf{q}_j = \mathbf{p}_i \cdot \mathbf{a} = \mathbf{p}_i \cdot \mathbf{b} = \mathbf{q}_j \cdot \mathbf{a} = \mathbf{q}_j \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} = 0\ (i, j = 0,\dots,4),$$

$$\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b},$$

whence, identically in (u, w)

$$\mathbf{p}(u) \cdot \mathbf{q}(w) = 0,$$

expressing that the plane $\pi(u)$ meets the generator $l(w)$ for all u and w. The vector $\mathbf{p}_i$ is the coordinate vector of the plane π_i, and $\mathbf{q}_i$ that of the generator l_i, and the relations $\mathbf{p}_i \cdot \mathbf{q}_j = 0$ $(i, j = 0,\dots,4)$ simply express that all five lines meet all five planes. Also, it is worth noting the obvious relations $\sum_{i=0}^{4} \mathbf{p}_i = \sum_{i=0}^{4} \mathbf{q}_i = 0$, which express that $\pi_0,\dots,\pi_4$ are five associated planes, and $l_0,\dots,l_4$ are five associated lines.

7. Cubic primals containing R^5.

The generators $\{l\}$ of R^5 are the intersection of five linearly independent linear complexes of lines, i. e. are the lines common to an ∞^4 linear system $|L|$ of linear complexes. In this system are of course a quadruple infinity of webs, or ∞^3 linear subsystems, and a sextuple infinity of nets, or ∞^2 linear subsystems, of linear complexes. The lines common to a general web of linear complexes are one system of generators of a SEGRE cubic primal, meeting five associated planes on the primal, and those common to a general net of linear complexes are the trisecants of a VERONESE surface. (By this term we shall denote the projection into four dimensions of the normal VERONESE surface in five dimensions; we shall omit the word projected usually associated with this surface, as we shall have no further occasion to refer to the normal surface). If a net N of linear complexes is contained in a web W, the VERONESE surface V_2^4 whose trisecants are the common lines of N lies on the SEGRE cubic V_3^3 generated by the common lines of W; for V_2^4 has a pencil of trisecants through each of its points; and the generators (of the relevant system) of V_3^3 are the trisecants of V_2^4 that belong to a linear complex not containing all of them, and hence include either one line or the whole of each pencil of trisecants, i. e. every point of V_2^4 lies on at least one generator of V_3^3.

Theorem 9. *The general cubic primal containing R^5 is a* SEGRE *primal V_3^3, and its intersection with the quintic primal W_3^5 generated by the planes $\{\pi\}$ of the curves $\{C^3\}$ on R^5 consists of R^5 counted twice, together with five associated planes of $\{\pi\}$. R^5 is the base surface of an ∞^4 linear system of such cubic primals, the intersection of two general members of which, residual to R^5, is a* VERONESE *surface V_2^4 trisecant to the generators of R^5.*

Proof. Let V_3^3 be a cubic primal containing R^5; its intersection with W_3^5 includes R^5 counted twice, since R^5 is the double locus on W_3^5, each of its points being the intersection of two generating planes $\pi(u)$, $\pi(u')$ of W_3^5. The residual intersection, of order 5, consists of five of the planes $\{\pi\}$; since if V_3^3 contains a point of $\pi(u)$ not on R^5, i.e. not on $C(u)$, it must contain the whole of $\pi(u)$. The complete linear system $|V_3^3|$ of all cubic primals through R^5 thus traces residually on W_3^5 a linear series of sets of five planes $\pi(u_1),\ldots,\pi(u_5)$; and as one such set consists of $\pi_0,\ldots,\pi_4$, every such set satisfies $\sum_{i=1}^{5} u_i \equiv 0$, i.e. consists of five associated planes. Conversely, every set of five associated planes in $\{\pi\}$ determines a SEGRE cubic containing it, generated by the lines meeting the five planes, among which are the generators of R^5, and hence containing R^5; thus $|V_3^3|$ traces on W_3^5 the complete series of sets of five associated planes in $\{\pi\}$, of dimension 4, i.e. $|V_3^3|$ is of dimension 4. (We have already seen that there are ∞^4 webs of linear complexes in $|L|$, the common lines of each web being the generators of a SEGRE cubic containing R^5.)

Now let V_3^3, V'^3_3 be two general members of $|V_3^3|$, and let W, W' be the corresponding webs of linear complexes. As W, W' are both contained in the ∞^4 linear system $|L|$ of linear complexes, their intersection is a net N, the common lines of all whose complexes are the trisecants of a VERONESE surface V_2^4, and include the generators (of the relevant systems) of both V_3^3, V'^3_3. Thus V_2^4 lies on both V_3^3 and V'^3_3, and the intersection of these two cubics consists just of the surfaces R^5, V_2^4, since their total order is 9. This completes the proof of Theorem 9.

Clearly, some cubics of $|V_3^3|$ will meet W_3^5 in sets of five planes not all distinct, as any linear series contains some sets not all distinct. These cubics are not strictly SEGRE cubics, as they do not contain fifteen planes; they are to be regarded as limiting cases of the SEGRE cubic, in which some of the planes coincide. Each is however generated by at least one system of ∞^2 lines, common to a web of linear complexes, one of these including the generators $\{l\}$ of R^5.

Similarly, some of the VERONESE surfaces V_2^4 which are the characteristic system of $|V_3^3|$ are degenerate. For a net N of linear complexes, though in general it does not include any that are special, consisting of all lines that meet a fixed directrix plane, may include one, two or three special complexes. (There are of course ∞^1 special complexes in $|L|$, whose directrix planes are the planes $\{\pi\}$.) The VERONESE surface corresponding to N breaks up accordingly into the directrix plane π of the special complex and a ruled cubic with directrix in π; the directrix planes π, π' of the two special complexes and a quadric meeting π, π' in lines; or the directrix planes π, π', π'' of the three special complexes and the unique plane meeting these three in lines, the join of their intersections by pairs. In particular on the SEGRE cubic primal $V^*{}_3^3$ from which we began, containing the planes $\pi_0,\ldots,\pi_4$, there are in the web $|V_2^4|$ of VERONESE surfaces traced residually by cubics through R^5, ten such completely reducibles surfaces, consisting of four planes π_{ij}, π_k, π_l, π_m, where $ijklm$ is any permutation of 01234.

The ruled quintics $|R|$ with generators in the line system (5) on $V^*{}_3^3$ are a linear system of dimension 5; the Grassmannian of this line system being the quintic DEL PEZZO surface in five dimensions, each of whose prime sections is the Grassmannian of an R^5. Some prime sections of the DEL PEZZO quintic are of course reducible; in particular it has completely reducible prime sections, consisting of five lines forming a skew pentagon, and these correspond to degenerate members of $|R|$, consisting of five planes π_{kl}, π_{lm}, π_{mi}, π_{ij}, π_{jk}, where again $ijklm$ is any permutation of 01234.

The characteristic system of $|R^5|$ consisting of sets of five associated lines, every set of five associated lines in the system (5) is the base of a pencil of surfaces in $|R|$. In particular the lines $l_0,\ldots,l_4$ are the base of a pencil $\{R^5(\lambda:\mu)\}$ of which one member is the particular surface R^5 we are studying, and two other members are the degenerate quintics $\pi_{23}\,\pi_{34}\,\pi_{40}\,\pi_{01}\,\pi_{12}$ and $\pi_{41}\,\pi_{02}\,\pi_{13}\,\pi_{24}\,\pi_{30}$.

The equation of the SEGRE cubic primal $V^*{}_3{}^3$ containing the fifteen planes with which we began is, in the original coordinate system before we suppressed the redundent coordinate x_5,

$$\sum_{i=0}^{5} x_i^3 = 0 ;$$

since every point of the plane $x_i + x_j = x_k + x_l = x_m + x_n = 0$ satisfies also $x_i^3 + x_j^3 = x_k^3 + x_l^3 = x_m^3 + x_n^3 = 0$. On suppressing x_5 this becomes

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = (x_0 + x_1 + x_2 + x_3 + x_4)^3 ,$$

i. e.

$$\Phi^* = \sum x_i^2 x_j + 2 \sum x_i x_j x_k = 0, \tag{34}$$

the first summation being over the 20 ordered pairs ij, and the second over the ten unordered triads ijk. Before finding the equations of other cubic primals containing R^5, it is convenient to prove the following

Lemma. Taking x_0, x_1, x_2 as coordinates in π_0 (with $x_3 = -x_2$, $x_4 = -x_1$) the equation of the curve $C(0)$ is

$$\lambda (x_1 + x_2)(x_0^2 - x_1^2) + \mu (x_1 - x_2)(x_0^2 - x_2^2) = 0, \tag{35}$$

and those of $C(\sigma), \ldots, C(4\sigma)$ are obtained from this by cyclic permutation of $x_0, \ldots, x_4$.

Proof. From (4), the points P_{00}, P_{01}, P_{02}, P_{30}, P_{40} are $(1, 0, 0)$, $(1, 1, -1)$, $(1, 1, 1)$, $(1, -1, -1)$, $(1, -1, 1)$, so that the lines $P_{00} P_{01} P_{40}$, $P_{00} P_{02} P_{30}$, $P_{01} P_{02}$, $P_{30} P_{40}$, $P_{01} P_{30}$, $P_{02} P_{40}$ are

$$x_1 + x_2 = 0,\ x_1 - x_2 = 0,\ x_0 - x_1 = 0,\ x_0 + x_1 = 0,\ x_0 + x_2 = 0,\ x_0 - x_2 = 0$$

respectively. On $C(0)$, u is a normal parameter for the point $(u, 0)$, so that the tangent at $(i\sigma, 0)$ meets the curve again at $(3i\sigma, 0)$, i. e. P_{00} is an inflexion, and the tangents at P_{01}, P_{02}, P_{30}, P_{40} are $x_0 + x_2 = 0$, $x_0 - x_1 = 0$, $x_0 + x_1 = 0$, $x_0 - x_2 = 0$, respectively. The cubics in the plane satisfying these contact conditions are a pencil, two reducible members of which are $(x_1 + x_2)(x_0^2 - x_1^2) = 0$ and $(x_1 - x_2)(x_0^2 - x_2^2) = 0$; and as from (20) the tangent to $C(0)$ at P_{00} satisfies $\lambda (x_1 + x_2) + \mu (x_1 - x_2) = 0$, $C(0)$ is the curve of this pencil given by (35), which establishes the lemma, the application of the cyclic permutation being obvious.

It may be observed that the two triangles $(x_1 + x_2)(x_0^2 - x_1^2) = 0$ and $(x_1 - x_2)(x_0^2 - x_2^2) = 0$ are the traces on π_0 of the two degenerate quintic surfaces $\pi_{23} \pi_{34} \pi_{40} \pi_{01} \pi_{12}$ and $\pi_{41} \pi_{02} \pi_{12}$ $\pi_{24} \pi_{30}$. In each case the two planes out of the five that have 0 for one of their suffixes meet π_0 only in points, and the other three meet it in lines, forming the degenerate cubic curve in question. We now prove the main result of this section :

Theorem 10. *The surface R^5, together with the degenerate* VERONESE *surface consisting of the planes π_{ij}, π_k, π_l, π_m form the complete intersection of the cubic primal $\Phi^* = 0$ with the cubic primal $\Phi_{ij} = 0$ where*

$$\left.\begin{aligned}
\Phi_{23} &= \lambda\,(x_1+x_2)\,(x_2+x_3)\,(x_3+x_4) + \mu\,(x_1+x_2+x_3+x_4)\,(x_3+x_0)\,(x_0+x_2)\\
\Phi_{34} &= \lambda\,(x_2+x_3)\,(x_3+x_4)\,(x_4+x_0) + \mu\,(x_2+x_3+x_4+x_0)\,(x_4+x_1)\,(x_1+x_3)\\
\Phi_{40} &= \lambda\,(x_3+x_4)\,(x_4+x_0)\,(x_0+x_1) + \mu\,(x_3+x_4+x_0+x_1)\,(x_0+x_2)\,(x_2+x_4)\\
\Phi_{01} &= \lambda\,(x_4+x_0)\,(x_0+x_1)\,(x_1+x_2) + \mu\,(x_4+x_0+x_1+x_2)\,(x_1+x_3)\,(x_3+x_0)\\
\Phi_{12} &= \lambda\,(x_0+x_1)\,(x_1+x_2)\,(x_2+x_3) + \mu\,(x_0+x_1+x_2+x_3)\,(x_2+x_4)\,(x_4+x_1)\\
\Phi_{41} &= \lambda\,(x_1+x_2+x_3+x_4)\,(x_4+x_0)\,(x_0+x_1) - \mu\,(x_2+x_4)\,(x_4+x_1)\,(x_1+x_3)\\
\Phi_{02} &= \lambda\,(x_2+x_3+x_4+x_0)\,(x_0+x_1)\,(x_1+x_2) - \mu\,(x_3+x_0)\,(x_0+x_2)\,(x_2+x_4)\\
\Phi_{13} &= \lambda\,(x_3+x_4+x_0+x_1)\,(x_1+x_2)\,(x_2+x_3) - \mu\,(x_4+x_1)\,(x_1+x_3)\,(x_3+x_0)\\
\Phi_{24} &= \lambda\,(x_4+x_0+x_1+x_2)\,(x_2+x_3)\,(x_3+x_4) - \mu\,(x_0+x_2)\,(x_2+x_4)\,(x_4+x_1)\\
\Phi_{30} &= \lambda\,(x_0+x_1+x_2+x_3)\,(x_3+x_4)\,(x_4+x_0) - \mu\,(x_1+x_3)\,(x_3+x_0)\,(x_0+x_2)
\end{aligned}\right\} \quad (36)$$

Proof. $\Phi_{ij} = 0$ contains the planes π_{ij}, π_k, π_l, π_m for all λ, μ; for instance Φ_{23} is a linear combination (with quadratic coefficients) of $x_1 + x_2$, $x_3 + x_4$ which vanish on π_{23}; of $x_1 + x_4$, $x_2 + x_3$ which vanish on π_0; of $x_0 + x_2$, $x_3 + x_4$ which vanish on π_1; and of $x_0 + x_3$, $x_1 + x_2$ which vanish on π_4; the proofs for the other cubics Φ_{ij} are exactly the same, and obtained by applying the usual permutations to the coordinates.

Further, in Φ_{01}, Φ_{02}, Φ_{30}, Φ_{40}, the only four of the ten cubics which do not vanish identically on π_0, if we substitute $-x_2$, $-x_1$ for x_3, x_4 respectively, we obtain in each case either plus or minus the left hand member of (35). Thus each of the primals $\Phi_{ij} = 0$ either contains the plane π_0, or cuts it in the curve $C(0)$; similarly each of these ten primals either contains the plane π_k or cuts it in the curve $C(k\sigma)$, by the usual cyclic permutation. Thus each of these primals contains at least five points on every generator of R^5, one in each of the planes $\pi_0, \ldots, \pi_4$ and hence (being a cubic) contains R^5. Theorem 10 is thus proved.

It is worth remarking that whereas both terms in Φ_{ij} vanish on the four planes π_{ij}, π_k, π_l, π_m, the term in each of these cubics which has the coefficient λ vanishes also on the five planes π_{23}, π_{34}, π_{40}, π_{01}, π_{12}, and that which has the coefficient μ vanishes also on π_{41}, π_{02}, π_{13}, π_{24}, π_{30}. Thus allowing λ, μ to vary, $\Phi_{ij} = 0$ represents a pencil of cubic primals, tracing on $\Phi^* = 0$, residually to the fixed degenerate VERONESE surface, the pencil $\{R^5\ (\lambda\,;\mu)\}$ of surfaces in $|\,R^5\,|$, with the base lines $l_0, \ldots, l_4$, including the R^5 we are studying, and the two degenerate surfaces consisting of these pentads of planes. Each of the twenty terms, cofficients of λ, μ in Φ_{ij}, represents three of the fifteen primes Σ_{ii}, Σ_{ij}, of Section 2, each cutting $\Phi^* = 0$ in three planes; and these nine planes are in each case just the four composing the degenerate V_2^4 and the five composing the degenerate R^5.

We confirm also the uniqueness anticipated at the end of section 2 in the following

Corrollary. Given the configuration of Section 2, the period lattice 2Ω, and chosen fifth part σ of an element of 2Ω, the surface R^5 such that the planes $\pi_0, \ldots, \pi_4$ shall be those of the curves $C(0), \ldots, C(4\sigma)$ respectively, is uniquely determined; for the configuration determines the coordinate system, and relative to this coordinate system the equations of R^5 depend only on the constants $\lambda = \wp'(\sigma\,|\,2\Omega)$, $\mu = \wp'(2\sigma\,|\,2\Omega)$.

8. Cubics through R^5 in relation to W_3^5.

We have now to consider two five dimensional vector spaces over the complex numbers: $\{\Phi^3\}$, consisting of all cubic forms Φ in the coordinates that vanish on R^5; and $\{p(u)\}$, consisting of all elliptic functions

$$p(u) = \sum_{i=0}^{4} A_i\,\zeta_i\,u + C \qquad \left(\sum_{i=0}^{4} A_i = 0\right) \tag{37}$$

with at most simple poles at $u = i\sigma$ $(i = 0,...,4)$; the condition in parentheses in (37) being classically necessary and sufficient for $p(u)$ there defined to be an elliptic function. Our objective of course is to obtain a linear mapping of these two vector spaces on each other, which shall express the fact that the projective model of the linear system traced on $W_3^{\,6}$, residually to R^5 counted twice, by the cubics $\Phi = 0$, for all Φ in $\{\Phi^3\}$, is the Grassmannian curve of $\{\pi\}$, parametrised in Theorem 4 in terms of $\{p(u)\}$.

In dealing with $\{p(u)\}$, or any similar vector space of elliptic functions with assigned poles, it is convenient to speak of any individual element of the vector space as having au s - ple root at a point which is an r - pie pole of the vector space (i. e.of its general element), if in fact it has an $(r - s)$ - pie pole there, or an $(s - r)$ - ple zero, or a non zero finite value for $s = r$; at a point that is not a pole of the vector space, an s-ple root will mean the same thing as an s-ple zero. (This is analogous to the way in which, in dealing with the vector space of polynomials $f(x)$ of degree n, we regard one which is actually of degree $n - s$ as having an s — ple root at infinity, as well as its ordinary roots or zeros elsewhere). With this convention, every element of $\{p(u)\}$ has five roots, with due allowance for coincidence in multiple roots ; and the points of intersection of the Grassmannian curve of $\{\pi\}$ with the prime $\sum q_{ij}\, p_{ij} = 0$, or the planes common to $\{\pi\}$ and the linear complex with this equation, arc given by the roots of $\sum q_{ij}\, p_{ij}\ (u)$. In particular, if $ijklm$ is any cyclic permutation either of 01234 or of 02413 , as $p_{kl}\,(u)$ has simple poles only at $u = j\sigma,\ m\sigma$, simple zeros at $u = k\sigma,\ l\sigma$, and a non zero finite value at $u = i\sigma$, its roots arc simple at $u = i\sigma$ and double at $u = k\sigma, l\sigma$; and it represents a set of five associated planes consisting of π_i counted once and π_k , π_l, each twice. In the same wav, a constant element of $\{p(u)\}$, having simple roots at the five poles of the vector space, corresponds to the set of planes $\pi_0,...,\pi_4$.

The planes $\{\pi\}$ being common to five linearly independent linear complexes, there are of course five linearly independent linear identities between the ten functions $p_{ij}(u)$. These are first of all those expressing that $\pi\,(u)$ meets l_0 ,...,l_4 , of which only four are linearly independent (the five lines being associated), namely $\mathbf{q}_i \cdot \mathbf{p}(u) = 0$, where $\mathbf{q}_0$,...,$\mathbf{q}_4$ are defined in (33), i. e.

$$\left.\begin{aligned} p_{40}(u) - p_{01}(u) + p_{02}(u) - p_{30}(u) &= 0 \\ p_{01}(u) - p_{12}(u) + p_{13}(u) - p_{41}(u) &= 0 \\ p_{12}(u) - p_{23}(u) + p_{21}(u) - p_{02}(u) &= 0 \\ p_{23}(u) - p_{34}(u) + p_{30}(u) - p_{13}(u) &= 0 \\ p_{34}(u) - p_{40}(u) + p_{41}(u) - p_{24}(u) &= 0 \end{aligned}\right\} \qquad (38)$$

and since further, from the second form of (10), and from (7),

$$\left.\begin{aligned} p_{23}(u) + p_{34}(u) + p_{10}(u) + p_{01}(u) + p_{12}(u) &= 5(\beta - \alpha)\ (\theta + \varphi) = 2\lambda - \mu \\ p_{41}(u) + p_{02}(u) + p_{13}(u) + p_{24}(u) + p_{30}(u) &= 5(\beta - \alpha)\ (\theta - \varphi) = \lambda + 2\mu \end{aligned}\right\} \qquad (39)$$

we can take the equation of the remaining linear complex, which defines $\{\pi\}$ among the ∞^2 planes meeting l_0 ,...,l_4, in the form

$$(\lambda+2\mu)\,(p_{23} + p_{34} + p_{40} + p_{01} + p_{12}) = (2\lambda - \mu)\,(p_{41} + p_{02} + p_{13} + p_{24} + p_{30}). \qquad (40)$$

We remark also that from these relations, either of the pentads $p_{23}(u)$, $p_{34}(u)$, $p_{10}(u)$, $p_{01}(u)$, $p_{12}(u)$ and $p_{41}(u)$, $p_{02}(u)\,p_{13}(u)$, $p_{21}(u)$, $p_{30}(u)$ is linearly independent, and can be used as a base for $\{p(u)\}$; we have in fact

$$(2\lambda - \mu)\begin{pmatrix} p_{41}(u) \\ p_{02}(u) \\ p_{13}(u) \\ p_{24}(u) \\ p_{30}(n) \end{pmatrix} = \begin{pmatrix} \lambda & \lambda & \mu-\lambda & \mu-\lambda & \lambda \\ \lambda & \lambda & \lambda & \mu-\lambda & \mu-\lambda \\ \mu-\lambda & \lambda & \lambda & \lambda & \mu-\lambda \\ \mu-\lambda & \mu-\lambda & \lambda & \lambda & \lambda \\ \lambda & \mu-\lambda & \mu-\lambda & \lambda & \lambda \end{pmatrix}\begin{pmatrix} p_{23}(u) \\ p_{34}(u) \\ p_{40}(u) \\ p_{01}(u) \\ p_{12}(u) \end{pmatrix} \qquad (4i)$$

and inversely

$$(\lambda + 2\mu)\begin{pmatrix} p_{23}(u) \\ p_{34}(u) \\ p_{40}(u) \\ p_{01}(u) \\ p_{12}(u) \end{pmatrix} = \begin{pmatrix} \mu & \mu & -\lambda-\mu & -\lambda-\mu & \mu \\ \mu & \mu & \mu & -\lambda-\mu & -\lambda-\mu \\ -\lambda-\mu & \mu & \mu & \mu & -\lambda-\mu \\ -\lambda-\mu & -\lambda-\mu & \mu & \mu & \mu \\ \mu & -\lambda-\mu & -\lambda-\mu & \mu & \mu \end{pmatrix} \begin{pmatrix} p_{41}(u) \\ p_{02}(u) \\ p_{13}(u) \\ p_{24}(u) \\ p_{30}(u) \end{pmatrix} \tag{42}$$

Turning now to $\{\Phi^3\}$, we have of course five linearly independent linear identities between the ten elements $\bar{\Phi}_{ij}$, six between these and Φ^*, since $\{\Phi^3\}$, like $\{p(u)\}$, is five dimensional. It is easily seen from (36) that

$$\begin{aligned} \Phi_{13} - \Phi_{24} = {} & \lambda (x_2 + x_3)((x_0 + x_1)(x_1 + x_2) - (x_3 + x_4)(x_4 + x_0)) \\ & + \mu (x_1 + x_4)((x_0 + x_2)(x_2 + x_4) - (x_1 + x_3)(x_3 + x_0)) \end{aligned}$$

and

$$\begin{aligned} \Phi_{34} - \Phi_{12} = {} & \lambda (x_2 + x_3)((x_3 + x_4)(x_4 + x_0) - (x_0 + x_1)(x_1 + x_2)) \\ & + \mu (x_4 + x_1)((x_1 + x_3)(x_3 + x_0) - (x_0 + x_2)(x_2 + x_4)), \end{aligned}$$

so that applying the cyclic permutation

$$\left.\begin{aligned} \Phi_{13} - \Phi_{24} + \Phi_{34} - \Phi_{12} = 0 \\ \Phi_{24} - \Phi_{30} + \Phi_{40} - \Phi_{23} = 0 \\ \Phi_{30} - \Phi_{41} + \Phi_{01} - \Phi_{34} = 0 \\ \Phi_{41} - \Phi_{02} + \Phi_{12} - \Phi_{40} = 0 \\ \Phi_{02} - \Phi_{13} + \Phi_{23} - \Phi_{01} = 0 \end{aligned}\right\} \tag{43}$$

of which only four are linearly independent ; and also from (36), (34),

$$\left.\begin{aligned} \Phi_{41} + \Phi_{02} + \Phi_{13} + \Phi_{24} + \Phi_{30} = (2\lambda - \mu)\Phi^* \\ \Phi_{23} + \Phi_{34} + \Phi_{40} + \Phi_{01} + \Phi_{12} = (\lambda + 2\mu)\Phi^*. \end{aligned}\right\} \tag{44}$$

Comparison of (43), (44) with (38), (39) suggests, for any constant ϱ, an obvious linear mapping $\xi : \{\Phi^3\} \to \{p(u)\}$, in which

$$\left.\begin{aligned} &\xi(\Phi_{41}) = \varrho\, p_{23}(u) & &\xi(\Phi_{23}) = \varrho\, p_{41}(u) \\ &\xi(\Phi_{02}) = \varrho\, p_{34}(u) & &\xi(\Phi_{34}) = \varrho\, p_{02}(u) \\ &\xi(\Phi_{13}) = \varrho\, p_{40}(u) & &\xi(\Phi_{40}) = \varrho\, p_{13}(u) \qquad \xi(\Phi^*) = \varrho \\ &\xi(\Phi_{24}) = \varrho\, p_{01}(u) & &\xi(\Phi_{01}) = \varrho\, p_{24}(u) \\ &\xi(\Phi_{30}) = \varrho\, p_{12}(u) & &\xi(\Phi_{12}) = \varrho\, p_{30}(u) \end{aligned}\right\} \tag{45}$$

We now prove

Theorem 11. *The mapping $\xi : \{\Phi^3\} \to \{p(u)\}$ defined in (45) has the property that if Φ is any element of $\{\Phi^3\}$, the cubic primal $\Phi = 0$ cuts W_3^5, residually to R^5 counted twice, in the planes $\pi(u)$ corresponding to the five roots u of the element $\xi(\Phi)$ of $\{p(u)\}$.*

Proof. That there is some non singular linear mapping $\psi : \{\Phi^3\} \to \{p(u)\}$ having this property is obvious, since, as was remarked above, the projective model of the linear system traced by the primals $\Phi = 0$, for all Φ in $\{\Phi^3\}$, on W_3^5, is the Grassmannian curve of $\{\pi\}$. Clearly also, ψ is determined to within a coefficient of homogeneity, i.e. the mappings ψ having this property are all the scalar multiples of any one of them.

We begin by showing that the planes of $\{\pi\}$ that lie on the primal $\Phi_{23} = 0$ corrsepoud to the roots of $p_{41}(u)$, i.e. are π_0 counted once and π_4, π_1 each counted twice, so that $\psi(\Phi_{23})$ is a constant multiple of $p_{41}(u)$. Now if l is any line lying on $\Phi_{23} = 0$, not a generator of R^5, but of the system containing the generators of R^5, the five planes of $\{\pi\}$ that meet l are those that lie on $\Phi_{23} = 0$. We find the coordinates q_{ij} of a suitable line l as follows :

Putting $x_1 + x_4 = 0$ in $\Phi_{23} = 0$, it becomes

$$(x_2 + x_3)(\lambda(x_1 + x_2)(x_3 + x_4) + \mu(x_3 + x_0)(x_0 + x_2)) = 0,$$

indicating that this prime cuts $\Phi_{23} = 0$ in π_0 together with a quadric surface ; it cuts R^5 in the curve $C(0)$ together with the generators l_2, l_3. One system of generators of the quadric is

$$x_1 + x_4 = (x_0 + x_2) - k(x_3 + x_4) = \lambda(x_1 + x_2) + k\mu(x_3 + x_0) = 0 \tag{46}$$

with variable parameter k ; and of these, the lines $k = \infty$, $k = 0$ are l_2, l_3 by (5). Thus for any other value of k, the line (46) is on $\Phi_{23} = 0$, and belongs to the system containing the generators of R^5, but is not itself a generator. Putting $k = 1$ for smplicity, we obtain the line l ;

$$x_1 + x_4 = x_0 + x_2 - x_3 - x_4 = \mu x_0 + \lambda x_1 + \lambda x_2 + \mu x_3 = 0,$$

whose Grassmann coordinates are the cubic minors in the coefficient matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ \mu & \lambda & \lambda & \mu & 0 \end{pmatrix}$$

of its three equations. These minors are

$$(q_{23}, q_{34}, q_{40}, q_{01}, q_{12} ; q_{41}, q_{02}, q_{13}, q_{24}, q_{30})$$
$$= (\lambda - \mu, \mu - \lambda, \lambda + \mu, \lambda + \mu, 2\mu ; 0, -(\lambda + \mu), \mu - \lambda, 2\mu, 0) ;$$

and substituting these values in the condition $\sum q_{ij} p_{ij}(u) = 0$ for l to meet $\pi(u)$, it is at once verified that the terms containing $\zeta_i u$ $(i = 0,\ldots,4)$ reduce to $2\lambda(\alpha - \beta)(\zeta_3 u - \zeta_2 u)$; the constant term is in the first instance found as $2(\alpha - \beta)((\lambda - \mu)(\theta - \varphi) - (\lambda + 2\mu)(\theta + \varphi))$; but since, from (7), $(\lambda + 2\mu)(\theta + \varphi) = (2\lambda - \mu)(\theta - \varphi)$, it reduces to $-2\lambda(\alpha - \beta)(\theta - \varphi)$, so that $\sum q_{ij} p_{ij}(u) = 2\lambda p_{41}(u)$.

Thus $\psi(\Phi_{23})$ is a constant multiple of $p_{41}(u)$; and in exactly the same way $\psi(\Phi_{34})$, $\psi(\Phi_{40})$, $\psi(\Phi_{01})$, $\psi(\Phi_{12})$ are some constant multiples of $p_{02}(u)$, $p_{13}(u)$, $p_{24}(u)$, $p_{30}(u)$ respectively. But also, $\psi(\Phi^*)$ is a constant element of $\{p(u)\}$, since its roots are $i\sigma$ $(i = 0,\ldots,4)$; thus the constant multipliers are all equal, i. e. $\psi = \xi$, for some value of the constant ϱ in (45). This completes the proof of Theorem 11.

Corrollary. The values, on substituting the coordinates of any point in the plane $\pi(u)$, of all elements of $\{\Phi^3\}$, are proportional to the values at the point u of the corresponding elements of $\{p(u)\}$, under the mapping ξ.

9. Equation of W_3^5.

W_3^5, generated by the planes $\{\pi\}$, is a quintic primal, and its generators are the system of planes dual to those of R^5 as system of lines. Before actually finding the equation of this primal, it is convenient to take a look, in general terms, at the kind of equation we may expect.

Each R^5 of the pencil $\{R^5\ (\lambda:\mu)\}$ on $V^*{}_3{}^3$, obtained by varying the ratio $\lambda:\mu$ in the cubic equations $\Phi_{ij}=0$, determines of course a W_3^5, which we denote by $W_3^5(\lambda:\mu)$, and which varies with the parameter $\lambda:\mu$ in a system $\{W_3^5(\lambda:\mu)\}$. Any plane π meeting $l_0,\ldots,l_4$ (other than $\pi_0,\ldots,\pi_4$) is a generator of one member of the system $\{W_3^5(\lambda:\mu)\}$; for the lines meeting $\pi_0,\ldots,\pi_4$, π are the generators of a unique P^5, which is one of the pencil $\{R^5(\lambda:\mu)\}$, since this pencil consists of all R^5's on $V^*{}_3{}^3$ that have $l_0,\ldots,l_4$ as generators; and π is a generator of the corresponding $W_3^5(\lambda:\mu)$. Since through a general point P of space there pass two planes π, π' meeting $l_0,\ldots,l_4$, two members of $\{W_3^5(\lambda:\mu)\}$ also pass through P, one with π and one with π' as generators. Thus the equation of $W_3^5(\lambda:\mu)$ must be homogeneously quadratic in (λ,μ), as well as quintic in the coordinates. Only if P is on $V^*{}_3{}^3$, π and π' arc two planes $\pi(u)$, $\pi(u')$ of one $W_3^5(\lambda:\mu)$, and P is the point (u, u') on $R^5\ (\lambda:\mu)$; in this case the two members of $\{W_3^5(\lambda:\mu)\}$ through P coincide.

To the degenerate members of the pencil $\{R^5(\lambda:\mu)\}$, which occur for $\lambda=0$, $\mu=0$, and consist of the pentads of planes π_{23}, π_{34}, π_{40}, π_{01}, π_{12} and π_{41}, π_{02}, π_{13}, π_{24}, π_{30}, coorrespond degenerate members of $\{W_3^5(\lambda:\mu)\}$, consisting of the pentads of primes Σ_{41}, Σ_{02}, Σ_{13}, Σ_{24}, Σ_{30} and Σ_{23}, Σ_{34}, Σ_{40}, Σ_{01}, Σ_{12} respectively. For by (4), (4'), (5), Σ_{41}: $x_2+x_3=0$ contains π_{12}, π_{34}, meets π_{23} in the line $l_{41}=P_{40}\,P_{01}$, and meets π_{40}, π_{01} in lines throngh P_{40}, P_{01} respectively; thus any plane through l_{41} in Σ_{41}, like π_0 (which is in fact one of this pencil) meets π_{12}, π_{23}, π_{34} in lines and π_{01}, π_{40} in the points P_{01}, P_{40}, and hence meets every generator of the degenerate $R^5\ \pi_{23}\,\pi_{34}\,\pi_{40}\,\pi_{10}\,\pi_{12}$, as these generators are a pencil in each of the five planes, those in π_{01}, π_{40} having their vertices at P_{01}, P_{40}. The planes meeting all the generators of the degenerate R^5, and cutting the degenerate R^5 in degenerate plane cubic curves, are thus a pencil in each of the five primes $\Sigma_{41}\ \Sigma_{02}\ \Sigma_{13}\ \Sigma_{24}\ \Sigma_{30}$, which accordingly constitute the coorrespnding $W_3{}^5$.

Bearing in mind that the equation of $W_3{}^5$ must be invariant, not only under the cyclic permutation of x_0, x_1, x_2, x_3, x_4, but also under that of x_1, x_2, x_4, x_3 accompanied by the substitution of $(\mu,-\lambda)$ for (λ,μ), we see that there are two a priori possbilitics for the form of the equation :

$$\text{(a)}\quad \lambda^2 F+\lambda\,\mu G+\mu^2 H=0,\qquad \text{(b)}\quad \lambda^2 F+\lambda\,\mu G-\mu^2 H=0,$$

where

$$\left.\begin{aligned} F&=(x_2+x_3)(x_3+x_4)(x_4+x_0)(x_0+x_1)(x_1+x_2)=0\\ H&=(x_4+x_1)(x_0+x_2)(x_1+x_3)(x_2+x_4)(x_3+x_0)=0\end{aligned}\right\}\qquad (47)$$

are the equations of the two singular $W_3{}^5$'s, and G is invariant under the cyclic permutation of x_0, x_1, x_2, x_3, x_4, and in case (a) is changed in sign, but in case (b) is left unchanged, by that of x_1, x_2, x_4, x_3. We shall find that the actual case is (b). We prove in fact

Theorem 12. *The equation of $W_3{}^5$, generated by the planes $\{\pi\}$ of the curves $\{C^3\}$ on P^5, is*

$$\lambda^2 F-\lambda\,\mu G-\mu^2 H=0,\qquad (48)$$

where, F, H are as defined in (47), and G is the symmetric quintic form in the coordinates

$$G=\sum_{20} x_i{}^3 x_j+2\sum_{30} x_i{}^3 x_j\,x_k+4\sum_{30} x_i{}^2 x_j{}^2 x_k+7\sum_{20} x_i{}^2 x_j\,x_k\,x_l+12\,x_0\,x_1\,x_2\,x_3\,x_4\,,\qquad (49)$$

the summation being over all distinct monomials obtained by permuting the coordinates, and the number under each summation sign indicating the number of terms in the symmetric sum.

Proof. R^5 is the locus of double points of W_3^5, each point of R^5 being on two generators of W_3^5; and W_3^5 is the only quintic primal having R^5 as locus of double points, since any such primal must meet each plane $\pi\,(u)$ at least in the curve $C(u)$ counted twice, i.e. at least in a sextic curve, and hence must contain the whole of $\pi\,(u)$. We therefore show that the quintic primal (48) has R^5 as locus of double points, by verifying the identities

$$\left.\begin{aligned}
\Phi_{23}\Phi_{41} - \lambda\mu\,\Phi^{*2} &= (x_1 + x_2 + x_3 + x_4)\,(\lambda^2 F - \lambda\mu\, G - \mu^2 H)\\
\Phi_{34}\Phi_{02} - \lambda\mu\,\Phi^{*2} &= (x_0 + x_2 + x_3 + x_4)\,(\lambda^2 F - \lambda\mu\, G - \mu^2 H)\\
\Phi_{40}\Phi_{13} - \lambda\mu\,\Phi^{*2} &= (x_0 + x_1 + x_3 + x_4)\,(\lambda^2 F - \lambda\mu\, G - \mu^2 H)\\
\Phi_{01}\Phi_{24} - \lambda\mu\,\Phi^{*2} &= (x_0 + x_1 + x_2 + x_4)\,(\lambda^2 F - \lambda\mu\, G - \mu^2 H)\\
\Phi_{12}\Phi_{30} - \lambda\mu\,\Phi^{*2} &= (x_0 + x_1 + x_2 + x_3)\,(\lambda^2 F - \lambda\mu\, G - \mu^2 H)
\end{aligned}\right\} \qquad (50)$$

or indeed any one of them ; since the left hand member, equated to zero, is the equation of a sextic primal with R^5 as locus of double points, and the identity shows that this primal breaks up into a prime and the quintic (48), which can thus only be W_3^5. In the product $\Phi_{23}\,\Phi_{41}$ the coefficients of λ^2, μ^2 are immediately seen to be $(x_1+x_2+x_3+x_4)\,F$, $-(x_1+x_2+x_3+x_4)\,H$; the coefficient of $\lambda\mu$ is symmetrical in x_1, x_2, x_3, x_4 ; and the verification that it is in fact equal to $\Phi^{*2} - (x_1+x_2+x_3+x_4)\,G$ is tedious but perfectly straightforward. Theorem 12 is thus proved.

It will be seen that the identities (50) and the mapping defined in (45) provide an explicit expression for the known quadrocubic Cremona transformation in four dimensions [4], in which the homaloids are on the one hand quadrics through an elliptic quintic curve ${}^1C^5$, and on the other hand cubics through an R^5. We can take $(p_{23}, p_{34}, p_{40}, p_{01}, p_{12}\,;\,p_{41}, p_{02}, p_{13}, p_{24}, p_{30})$ as linear forms in the coordinates in the ambient four dimensional space of the Grassmannian ${}^1C^5$ of $\{\neg\}$, satisfying the linear identities (38) with the argument u omitted, and (40) ; for instance we can take either of the two pentads $(p_{23}, p_{34}, p_{40}, p_{01}, p_{12})$ or $(p_{41}, p_{02}, p_{13}, p_{24}, p_{30})$ as the coordinates, the relation between the two coordinate systems being given by (41), (42), again of course with the argument u omitted ; and we define the further linear form

$$\begin{aligned}
p^* &= (p_{23} + p_{34} + p_{40} + p_{01} + p_{12})/(2\lambda - \mu)\\
&= (p_{41} + p_{02} + p_{13} + p_{24} + p_{30})/(\lambda + 2\mu).
\end{aligned}$$

Then from (45) we have

$$\varrho\begin{pmatrix} p_{23}\\ p_{34}\\ p_{40}\\ p_{01}\\ p_{12}\end{pmatrix} = \begin{pmatrix}\Phi_{41}\\ \Phi_{02}\\ \Phi_{13}\\ \Phi_{24}\\ \Phi_{30}\end{pmatrix},\ \varrho\begin{pmatrix} p_{41}\\ p_{02}\\ p_{13}\\ p_{24}\\ p_{30}\end{pmatrix} = \begin{pmatrix}\Phi_{23}\\ \Phi_{34}\\ \Phi_{40}\\ \Phi_{01}\\ \Phi_{12}\end{pmatrix},\ \varrho p^* = \Phi^* \qquad (51)$$

as the equations of the mapping one way ; and since from (50)

$$4(\lambda^2 F - \lambda\mu G - \mu^2 H)\begin{pmatrix} x_0\\ x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & -1\\ 1 & -3 & 1 & 1 & 1 & -1\\ 1 & 1 & -3 & 1 & 1 & -1\\ 1 & 1 & 1 & -3 & 1 & -1\\ 1 & 1 & 1 & 1 & -3 & -1\end{pmatrix}\begin{pmatrix}\Phi_{23}\Phi_{41}\\ \Phi_{34}\Phi_{02}\\ \Phi_{40}\Phi_{13}\\ \Phi_{01}\Phi_{24}\\ \Phi_{12}\Phi_{30}\\ \lambda\mu\Phi^{*2}\end{pmatrix}$$

the equations of the inverse mapping are

$$\varrho' \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & -1 \\ 1 & -3 & 1 & 1 & 1 & -1 \\ 1 & 1 & -3 & 1 & 1 & -1 \\ 1 & 1 & 1 & -3 & 1 & -1 \\ 1 & 1 & 1 & 1 & -3 & -1 \end{pmatrix} \begin{pmatrix} p_{23}\,p_{41} \\ p_{34}\,p_{02} \\ p_{40}\,p_{13} \\ p_{01}\,p_{24} \\ p_{12}\,p_{30} \\ \lambda\mu p^{*2} \end{pmatrix} \tag{52}$$

where ϱ', like ϱ, is an arbitrary coefficient of homogeneity. The identities

$$p_{23}(u)\, p_{41}(u) = p_{34}(u)\, p_{02}(u) = p_{40}(u)\, p_{13}(u) = p_{01}(u)\, p_{24}(u) = p_{12}(u)\, p_{30}(u) = \mu\lambda$$

which are obvious form the first form of (10), mean that the five quadrics on the right of (52) all contain the Grassmannian curve, of which (10) is the parametrisation.

The envelope $G^2 + 4FH = 0$ of $\{W_3{}^5(\lambda : \mu)\}$, of total order 10, consists of two parts. The locus of double points of $W_3{}^5$ being R^5, that of the whole system $\{W_3{}^5(\lambda : \mu)\}$ is $V^*{}_3{}^3$, $\{R^5(\lambda : \mu)\}$ being a pencil on $V^*{}_3{}^3$; this counts twice in the envelope. The residual part is the quartic primal $M_3{}^4$, locus of a point P such that the two planes π, π' through P that meet $l_0,\ldots,l_4$ coincide ; since for any point P not on $V^*{}_3{}^3$, if and only if π, π' coincide, the two members of $\{W_3{}^3(\lambda : \mu)\}$ through P, one containing each of these planes, will likewise coincide.

The quartic primal $M_3{}^4$ is well known [1]. It has the fifteen lines l_i, l_{ij} in (5) as double lines, and touches each of the ten primes Σ_{ij} along a quadric surface Q_{ij}, of which the six that arc in that prime of the fifteen lines (5) are generators (three of each system). $M_3{}^4$ is in fact the only quartic primal touching these ten primes, or indeed any five of them that are linearly independent, along the respective quadric surfaces Q_{ij} ; for taking five such primes as the faces of a simplex of reference, the section of the quartic by each of these (the corresponding quadric Q_{ij} counted twice) determines all the terms in its equation that do not contain a particular coordinate; and as the equation is quartic, no term can contain all five coordinates.

This last remark enables us to verify that the equation of $M_3{}^4$ (in the coordinate system used throughout) is

$$\Theta = \sum_{10} x_i^2\, x_j^2 + 2 \sum_{30} x_i^2\, x_j\, x_k + 2 \sum_{5} x_i\, x_j\, x_k\, x_l = 0. \tag{53}$$

For on substituting $-x_0$ for x_4 in Θ (i. e. taking $(x_0,\ldots,x_3)$ as coordinates in the prime $\Sigma_{13} : x_4 + x_0 = 0$) many terms cancel, and the expression reduces to $(x_0{}^2 + x_2\, x_3 + x_3\, x_1 + x_1\, x_2)^2$. Moreover, the six of the lines (5) that lie in this prime are

$$l_1 :\ x_0 = x_2 = -x_3 \quad l_{13} :\ x_0 = x_3 = -x_1 \quad l_3 :\ x_0 = x_1 = -x_2$$

$$l_{02} :\ x_0 = x_3 = -x_2 \quad l_{40} :\ x_0 = x_1 = -x_3 \quad l_{24} :\ x_0 = x_2 = -x_1,$$

all of which clearly lie on the quadric $x_0{}^2 + x_2\, x_3 + x_3\, x_1 + x_1\, x_2 = 0$, so that this quadric is the quadric Q_{13} . Thus $\Theta = 0$ touches Σ_{13} along the quadric Q_{13}, and similarly it touches each prime Σ_{ij} along the quadric Q_{ij} ; i. e. $\Theta = 0$ is the equation of $M_3{}^4$.

There is thus clearly the identity

$$G^2 + 4FH = \Phi^{*2}\, \Theta\,, \tag{54}$$

except that, a priori, there might be a numerical coefficient on one side or the other ; but that this coefficient is unity can be seen by giving the value 1 to all the coordinates at once, when, by merely counting the terms in the symmetric forms, we see that

$$F = H = 32 \ , \quad G = 352, G^2 + 4FH = 128{,}000 \ ; \Phi^* = 40, \Theta = 80, \quad \Phi^{*2}\,\Theta = 128{,}000.$$

The quintic primal $G = 0$ cuts each of the ten primes Σ_{ij} in the quadric surface Q_{ij}, together with the three of the fifteen planes π_i, π_{ij} that are in Σ_{ij} (incidentally, these three planes cut Q_{ij} in the six out of the fifteen lines that are in Σ_{ij}). $G = 0$ cuts $V^*{}_3{}^3$ in the fifteen planes π_i, π_{ij} ; and it cuts $M_3{}^4$ in the ten quadrics Q_{ij}. The intersection of $V^*{}_3{}^3$ with $M_3{}^4$ is a surface of order 12, having the fifteen lines l_i, l_{ij} as double lines, and on which the focal curves of the surfaces $\{(R^5 (\lambda : \mu)\}$ form a pencil, with base points at P_{00} ,..., P_{44} . The focal curve on R^5 is in fact clearly its section by $M_3{}^4$, residual to the five lines l_0 ,...,l_4 counted twice.

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ÖZET

Dört boyutlu uzaydaki beşinci dereceden normal eliptik R^5 regle yüzeyleri için özel bir koordinat sistemi o tarzda tanımlanmaktadır ki, yüzey bu koordinat sisteminin koordinatlarına 20 permütasyonundan oluşan bir grubun dönüşümleri altında invaryam kalmaktadır. Bu koordinat sistemi sâyesinde, R^5 yüzeyini kübik bir eğri boyunca kesen genel düzlemin ve yüzeyin genel doğuranının GRASSMANN koordinatlarının beşinci dereceden eliptik fonksiyonları cinsinden ifadeleri elde edilmekte ve yüzeyin iki değişkenin eliptik fonksiyonları cinsinden iki farklı parametrelenmesi bulunmaktadır. Böylece, yüzey için bir kübik denklem takımı ve yüzey üzerindeki kübik eğrilerin düzlemleri tarafından doğurulan hiperyüzey için beşinci dereceden bir denklem elde edilmektedir : bu denklemler, σ, eliptik fonksiyonların peryodunun ilkel parçasının beşte biri olmak üzere,

$$\lambda = \wp' (\sigma) \text{ ve } \mu = \wp' (2\sigma)$$

parametrelerine homogen bir tarzda bağlıdırlar. Üstelik, R^5 yüzeyinden geçen kübikler ile elde edilen CREMONA dönüşümü ve normal eliptik beşinci dereceden bir eğriden geçen kuadriklerle elde edilen ve yukardaki dönüşümün tersi olan dönüşümlerin açık ifadeleri bulunmaktadır.