Δ-CURVATURES AND Δ-GEODESIC PRINCIPAL DIRECTIONS OF A CONGRUENCE OF CURVES IN THE SUBSPACE OF A FINSLER SPACE [1])

C. M. PRASAD

The process of δ-differentiation [4] leads to the use of the DUPIN indicatrix in finding out the principal directions of a congruence. The process of Δ-differentiation [2] requires the use of the osculating indicatrix corresponding to a direction $\dot{x}^i$ in finding out the principal directions of a congruence of curves. In the present paper, by using the process of Δ-differentiation, the Δ-curvatures and the Δ-geodesic principal directions of a congruence of curves in the subspace of a FINSLER space have been obtained and some of their properties have been studied.

1. Introduction. Let $F(x, \dot{x})$ be the fundamental metric function of the FINSLER space F_n with local coordinates x^i $(i = 1, \ldots, n)$ and satisfy the conditions usually imposed upon a FINSLER metric [1]. Let F_m be a FINSLER subspace with local coordinates u^α $(\alpha = 1, 2, \ldots, m)$. Let $L : u^\alpha = u^\alpha(t)$ be a curve on F_m. The components $\dot{x}^i = dx^i/dt$ and $\dot{u}^\alpha = du^\alpha / dt$ of its unit tangent vectors with respect to the x^i and u^α coordinate system are related by $\dot{x}^i = B^i_\alpha \dot{u}^\alpha$ where $B^i_\alpha = \partial x^i/\partial u^\alpha$. At each point of L, a combination $(u^\alpha, \dot{u}^\alpha)$ or $(x^i, \dot{x}^i)$ determines a line-element of F_m. Quantities in our discussion will be considered for this line-element unless stated otherwise. The fundamental metric tensors $g_{ij}(x, \dot{x})$ and $g_{\alpha\beta}(u, \dot{u})$ of F_n and F_m are connected by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})\, B^i_\alpha\, B^j_\beta. \tag{1.1}$$

There exists a system of n-m linearly independent unit vectors $\overset{*}{n}{}^i_{(\nu)}(x, \dot{x})$, $\nu = m + 1, \ldots, n$, normal to F_m. These vectors are called secondary normal vectors [1] and are defined by the solutions

$$\overset{*}{n}_{(\nu)j}\, B^j_\alpha = g_{ij}(x, \dot{x})\, \overset{*}{n}{}^i_{(\nu)}\, B^j_\alpha = 0 \tag{1.2}$$

and are normalised by

$$F(x, \overset{*}{n}_{(\nu)}) = g_{ij}(x, \overset{*}{n}_{(\nu)})\, \overset{*}{n}{}^i_{(\nu)}\, \overset{*}{n}{}^j_{(\nu)} = 1. \tag{1.3}$$

These vectors also satisfy the relation

$$g_{ij}(x, \dot{x})\, \overset{*}{n}{}^i_{(\nu)}\, \overset{*}{n}{}^j_{(\mu)} = \delta^\nu_\mu\, \psi_{(\nu)}, \quad \text{(no summation on } \nu). \tag{1.4}$$

1) The author is extremely grateful to Professor K. B. LAL for his encouragement and guidance.

The covariant derivative of B^i_α with respect to u^β is given by [1],

(1.5) $$B^i_{\alpha;\beta} \stackrel{\text{def}}{=} I^i_{\alpha\beta} = \sum_\nu \overset{*}{\Omega}_{(\nu)\alpha\beta} \, \overset{*}{n}_{(\nu)}$$

where $\overset{*}{\Omega}_{\alpha\beta}\,(u,\dot{u})$ is defined as the secondary second fundamental tensor of the subspace. The covariant derivative of the secondary unit normal is given by

(1.6) $$\overset{*}{n}{}^i_{(\nu);\beta} = A^\delta_{(\nu)\beta}\, B^i_\delta + \sum_\mu N^{(\mu)}_{(\nu)}\, \overset{*}{n}{}^i_{(\mu)}$$

where

(1.7) $$A^\delta_{(\nu)\beta} = -\,\psi_{(\nu)}\, \overset{*}{\Omega}_{(\nu)\alpha\beta}\, g^{\alpha\delta} - \overset{*}{E}_{ijk}\,(x,\dot{x})\, B^k_\beta\, B^i_\alpha\, \overset{*}{n}{}^j_{(\nu)}\, g^{\alpha\delta}$$

and

(1.8) $$N^{(\mu)}_{(\nu)\beta}\, \psi_{(\mu)} = \overset{*}{n}{}^j_{(\nu);\beta}\, \overset{*}{n}_{(\mu)j}\,, \quad \overset{*}{E}_{ijk} = g_{ij;k}\,(x,\dot{x})\,.$$

Consider a system of congruences determined by the vector-field $\lambda^i_{(\sigma)}$. At a point of the subspace, it can be expressed as

(1.9) $$\lambda^i_{(\sigma)}\,(u,\dot{u}) = t^\alpha_{(\sigma)}\,(u,\dot{u})\, B^i_\alpha\,(u) + \sum_\nu C_{(\sigma\nu)}\,(u,\dot{u})\, \overset{*}{n}{}^i_{(\nu)}\,(u,\dot{u})\,.$$

It may be noted that at a point of F_m, the functions $\lambda^i_{(\sigma)}\,(u,\dot{u})$ are single-valued functions of $(u,\dot{u})$. Let $C : u^\alpha = u^\alpha\,(s)$ be a curve (not in the direction of $\dot{u}^\alpha$) of the subspace. The components, in the x^i and u^α coordinate system, of the tangent vector to C are such that

(1.10) $$x'^i = B^i_\alpha\, u'^\alpha\,,$$

where $x'^i = \dfrac{dx^i}{ds}$ and $u'^\alpha = \dfrac{du^\alpha}{ds}$.

These vectors satisfy the condition

(1.11) $$g_{ij}\,(x,\dot{x})\, x'^i x'^j = g_{\alpha\beta}\,(u,\dot{u})\, u'^\alpha u'^\beta = \varphi\,.$$

Let

$$x'^i / \sqrt{\varphi} = g^i \text{ and } u'^\alpha / \sqrt{\varphi} = g^\alpha$$

then equation (1.11) gives

(1.12) $$g_{ij}\,(x,\dot{x})\, g^i g^j = g_{\alpha\beta}\,(u,\dot{u})\, g^\alpha g^\beta = 1.$$

In the following section, by using the process of Δ-differentiation [2], we shall define the Δ-curvatures of the congruence $\lambda^i_{(\sigma)}$.

2. Δ-curvatures of a congruence

The Δ-differential of the vector $\lambda^i_{(\sigma)}$ (given by (1.9)) along C is given by

(2.1) $$\frac{\Delta\lambda^i_{(\sigma)}}{\Delta s} = \lambda^i_{(\sigma);\beta}\, y^\beta = \Big(W^\alpha_{(\sigma)}\, B^i_\alpha + \sum_\nu D_{(\sigma\nu)}\, \overset{*}{n}{}^i_{(\nu)} \Big),$$

where

$$W^{\alpha}_{(\sigma)} = q^{\alpha}_{(\sigma)\beta}\; y^{\beta} = \Big(t^{\alpha}_{(\sigma);\beta} + \sum_{\nu} C_{(\sigma\nu)}\, A^{\alpha}_{(\nu\beta)} \Big) y^{\beta} \tag{2.2}$$

and

$$D_{(\sigma\nu)} = P_{(\sigma\nu)\beta}\, y^{\beta} = \Big(t^{\alpha}_{(\sigma)}\, \overset{*}{\Omega}_{(\sigma)\alpha\beta} + \sum_{\tau} C_{(\sigma\tau)}\, N^{(\nu)}_{(\tau),\beta} + C_{(\sigma\nu);\beta} \Big) y^{\beta} . \tag{2.3}$$

In analogy to the definiton of the geodesic curvature of a subspace [4], we define the geodesic curvature of the congruence. $\lambda^{i}_{(\sigma)}$ as follows :

Definition (2.1). The quantity $K_{(\sigma)G}$ defined by

$$K^{2}_{(\sigma)G} = g_{\alpha\beta}\,(u, \dot{u})\, W^{\alpha}_{(\sigma)} W^{\beta}_{(\sigma)} = \widetilde{\Omega}_{(\sigma)\alpha\beta}\, y^{\alpha}\; y^{\beta} \tag{2.4}$$

where

$$\widetilde{\Omega}_{(\sigma)\alpha\beta}(u, \dot{u}) = g_{\gamma\delta}\,(u, \dot{u})\; q^{\gamma}_{(\sigma)\alpha}\, q^{\delta}_{(\sigma)\beta}\, , \tag{2.5}$$

is called Δ-geodesic curvature of the congruence $\lambda_{(\sigma)}$ along C in F_m .

Definition (2.2.). If the Δ-geodesic curvature of the congruence $\lambda_{(\sigma)}$ in the subspace vanishes at every point of the curve C, the curve is called a Δ-geodesic.

The differential equation of a Δ-geodesic of the congruence is given by

$$\widetilde{\Omega}_{(\sigma)\alpha\beta}\,(u, \dot{u})\, \frac{du_{\alpha}}{ds}\, \frac{du^{\beta}}{ds} = 0. \tag{2.6}$$

If $\dot{u}^{\alpha} = u'^{\alpha}$, the Δ-geodesic is called the *λ-geodesic* and its differential equation is given by

$$\widetilde{\Omega}_{(\sigma)\alpha\beta}\,(u, u')\; du^{\alpha}\; du^{\beta} = 0. \tag{2.7}$$

If the space F_m is a Riemannian space, (2.7) reduces to λ-geodesic [9] .

Since the Δ-geodesic curvature of the congruence $\lambda^{i}_{(\sigma)}$ in F_m vanishes along a Δ-geodesic (i.e. $w^{\alpha}_{(\sigma)} = 0$), the equation (2.1) reduces to

$$\left(\frac{\Delta\lambda^{i}_{(\sigma)}}{\Delta s}\right)_{G} = \sum_{\nu} D_{(\nu)}\; \overset{*}{n}{}^{i}_{(\nu)} \, . \tag{2.8}$$

In view of the definition of the secondary normal curvature of the subspace [1], we define the following :

Definition (2.3). The scalar $K_{n^{*}(\sigma)}$ defined by

$$K^{2}_{n^{*}(\sigma)} \overset{\text{def}}{=} g_{ij}\,(x, \dot{x}) \left(\frac{\Delta\lambda^{i}_{(\sigma)}}{\Delta s}\right)_{G} \cdot \left(\frac{\Delta\lambda^{j}_{(\sigma)}}{\Delta s}\right)_{G} \tag{2.9}$$

is called the Δ-secondary normal curvature of the congruence in the subspace and since $\overset{*}{n}_{(\sigma)}$ are unit vectors, it follows from (2.8) that the curvature of this Δ-geodesic (regarded as a curve of F_n) is represented by (2.9).

Simplifying the equation (2.9), wc have

$$K^2_{n^*(\sigma)} = \overset{\sim *}{\Omega}_{(\sigma)\alpha\beta}\ (u\,,\,\dot{u})\ g^\alpha g^\beta\ , \tag{2.10}$$

where

$$\overset{\sim *}{\Omega}_{(\sigma)\alpha\beta} = \sum_{\nu} \psi_{(\nu)}\, p_{(\sigma\nu)\alpha}\, p_{(\sigma\nu)\beta}. \tag{2.11}$$

Definition (2.4). The scalar $K_{(\sigma)}$ defined by

$$K^2_{(\sigma)} = g_{ij}\,(x\,,\,\dot{x}) \left(\frac{\Delta\lambda^i_{(\sigma)}}{\Delta s}\right)\left(\frac{\Delta\lambda^i_{(\sigma)}}{\Delta s}\right) \tag{2.12}$$

is called the Δ -absolute curvature of the congruence in the subspace.

Theorem (2.1). *If the conguruence $\lambda^i_{(\sigma)}$ does not lie in a variety spanned by the secondary normals, the Δ-geodesic curvature of the congruence is the magnitude of the Δ-derived vector of the tangential component of the congruence along C.*

Proof. If $\lambda^i_{(\sigma)}$ does not lie in a variety spanned by the vectors $\overset{*}{n}{}^i_{(\sigma)}$, $C_{(\sigma\nu)} = 0$ and $W^\alpha_{(\sigma)} = \dfrac{\Delta\, t^\alpha_{(\sigma)}}{\Delta\, s}$ and the equation (2.4) reduces to

$$K^2_{(\sigma)G} = g_{\alpha\beta}\ (\mathbf{u}\,,\,\dot{\mathbf{u}}) \left(\frac{\Delta\, t^\beta_{(\sigma)}}{\Delta\, s}\right)\left(\frac{\Delta\, t^\beta_{(\sigma)}}{\Delta\, s}\right), \tag{2.13}$$

which is the required result. Moreover, if $du^\alpha/dt = du^\alpha/ds$, the equation (2.13) yields

$$K^2_{(\sigma)G} = g_{\alpha\beta}\ (u\,,\,u')\ \frac{\delta\, t^\alpha_{(\sigma)}}{\delta\, s}\ \frac{\delta\, t^\alpha_{(\sigma)}}{\delta\, s}\ . \tag{2.14}$$

Theorem (2.2). *If the components of the congruence $\lambda^i_{(\sigma)}$ tangential to F_m are tangential to the curve C and $du^\alpha/ds = du^\alpha/dt$, the Δ -absolute curvature, the Δ -geodesic curvature and the Δ -secondary normal curvature are respectively the geodesic curvature with respect to F_n, the geodesic curvature with respect to F_m and the secondary normal curvature of the subspace.*

Proof. Since $\lambda^i_{(\sigma)} = x'^i = B^i_\alpha\ u'^\alpha$, the proof of the theorem follows from the definitions (2.4), (2.1) and (2.3).

From theorem (2.2) obviously the Δ -curvatures of the congruence may be considered as the generalised curvatures of the subspace and as such we may regard $\overset{\sim *}{\Omega}_{(\sigma)\alpha\beta}\,(u\,,\,\dot{u})\, du^\alpha\, du^\beta$ as the generalised secondary second fundamental form of the subspace.

Definition (2.5). A direction along which the Δ-secondary normal curvature of the congruence in F_m vanishes, is called the Δ-asymptotic direction and a curve whose direction at each point of it is Δ-asymptotic, is called a Δ-asymptotic line of the congruence in F_m. Its differential equation is given by

$$\tilde{\Omega}^{*}_{(\sigma)\alpha\beta}(u, \dot{u})\, du^{\alpha}\, du^{\beta} = 0 \qquad \text{for all } \sigma. \tag{2.15}$$

3. Δ-geodesic principal directions

Definition (3.1). For a preassigned $(u^{\alpha}, \dot{u}^{\alpha})$, a direction du^{α} / ds is said to be Δ-geodesic principal direction with repect to $\overset{*}{n}{}^{i}_{(\sigma)}$ provided that $K^{2}_{(\sigma)G}(u, \dot{u}, u')$ assumes an extreme value. The corresponding value of $K^{2}_{(\sigma)G}$ is called a Δ-geodesic principal curvature of the congruence in the subspace.

We now give the following propositions.

Theorem (3.1) *The Δ-geodesic principal directions and the Δ-geodesic principal curvatures (with respect to $\overset{*}{n}{}^{i}_{(\sigma)}$) of the congruence are respectively the eigenvectors and eigenvalues of the tensor $\tilde{\Omega}_{(\sigma)\alpha\beta}$.*

Theorem (3.2). *There exist m Δ-geodesic principal directions $\xi^{\alpha}_{(\mu)}$ $(\mu = 1, 2, \dots, m)$ of the congruence in F_m satisfying the conditions*

$$\tilde{\Omega}_{(\sigma)\alpha\beta}(u, \dot{u})\, \xi^{\alpha}_{(\mu)}\, \xi^{\beta}_{(\rho)} = 0 \tag{3.1}$$

and

$$g_{\alpha\beta}(u, \dot{u})\, \xi^{\alpha}_{(\mu)}\, \xi^{\beta}_{(\rho)} = 0 \qquad (\varrho \neq \mu). \tag{3.2}$$

The proofs of the above theorems are similar to those of the corresponding propositions in [2].

4. Some Invariants

Let $\xi^{\alpha}_{(\mu)}$ $(\mu = 1, 2, \dots, m)$ be the Δ-geodesic principal directions and ν^{α} be a vector field of F_m normalised by the condition

$$g_{\alpha\beta}(u, \dot{u})\, \nu^{\alpha}\, \nu^{\beta} = 1. \tag{4.1}$$

Since $\xi^{\alpha}_{(\mu)}$ are linearly independent in F_m, we write

$$\nu^{\alpha} = \sum_{\mu} l_{(\mu)}\, \xi^{\alpha}_{(\mu)}. \tag{4.2}$$

Substituting from (4.1) and using the relation (3.2), we get

$$\sum_{\mu} l^{2}_{(\mu)} = 1. \tag{4.3}$$

Th square of the Δ-geodesic curvature $K_{(\sigma)G}$ along ν^{α} is given by

$$K^2_{(\sigma)G}\ (u^\alpha, \dot u^\alpha, v^\alpha) = \widetilde{\Omega}_{(\sigma)\alpha\beta}\ (u, \dot u)\, v^\alpha v^\beta .$$

Substituting from (4.2) and putting

$$\widetilde{\Omega}_{(\sigma)\alpha\beta}\ (u, \dot u)\ \xi^\alpha_{(\rho)} \xi^\beta_{(\mu)} = \delta_{\mu\rho}\, \bar K_{(\sigma\mu)} \text{ (no summation on } \mu)$$

we obtain

$$K^2_{(\sigma)G} = \sum_\mu l^2_{(\mu)}\, \bar K_{(\sigma\mu)} \tag{4.4}$$

where $\bar K_{(\sigma\mu)}$ $(\mu = 1, 2, ..., m)$ are the Δ-geodesic pricipal curvatures of the congruence.

Theorem (4.1). *The sum of the squares of the Δ-geodesic curvatures of the congruence along m mutually orthogonal directions of F_m is an invariant and is equal to the sum of Δ-geodesic principal curvatures of the congruence.*

Proof. Consider m mutually orthogonal vectors $v^\alpha_{(\rho)}$ $(\varrho = 1, ..., m)$ each satisfying the normalising condition (4.1). We write

$$v^\alpha_{(\rho)} = \sum_\mu l_{(\rho\mu)}\, \xi^\alpha_{(\mu)} \qquad (\varrho = 1, ..., m) . \tag{4.5}$$

Since $g_{\alpha\beta}\ (u, \dot u)\ v^\alpha_{(\rho)}\, v^\beta_{(\mu)} = \delta_{\rho\mu}$, we get after simplification,

$$\sum_\varepsilon l_{(\rho\varepsilon)}\ l_{(\mu\varepsilon)} = \delta_{\rho\mu}. \tag{4.6}$$

Hence the matrix $\| l_{\rho\mu} \|$ is orthogonal and

$$\sum_\rho l^2_{(\rho\mu)} = 1 \qquad \text{for} \qquad \mu = 1, 2, ..., m . \tag{4.7}$$

By (4.4), the square of the Δ-geodesic curvature $K_{(\sigma)G}$ in the direction of $v^\alpha_{(\rho)}$ is given by

$$K^2_{(\sigma)G(\rho)} = \sum_\mu l^2_{(\rho\mu)}\, \bar K_{(\sigma\mu)} . \tag{4.8}$$

Hence using the equation (4.7), we get

$$\sum_\rho K^2_{(\sigma)G(\rho)} = \sum \bar K_{(\sigma\mu)}$$

which was to be proved.

5. Δ-absolute geodesics

From the equations (2.4), (2.10) and (2.12), we have

$$K^2_{(\sigma)} = \Phi_{(\sigma)\alpha\beta}\ (u, \dot u)\, g^\alpha g^\beta , \tag{5.1}$$

where

$$\Phi_{(\sigma)\alpha\beta}\,\{\widetilde{\Omega}_{(\sigma)\alpha\beta} + \widetilde{\Omega}^{*}_{(\sigma)\alpha\beta}\}$$

is a symmetric covariant tensor.

Defination (5.1). If the Δ-absolute curvature of the congtuence vanishes along a curve C, the curves is called the Δ-abslohte geodesic of the congruence in the subspace. Its differntial equation is given by

$$\text{(5.2)} \qquad \Phi_{(\sigma)\alpha\beta}\,(\mathrm{u}\,,\,\dot{\mathrm{u}})\,\frac{du^{\alpha}}{ds}\,\frac{du^{\beta}}{ds} = 0\ .$$

The principal directions associated with the tensor $\Phi_{(\sigma)\alpha\beta}$ may be called Δ -absolute principal directions and corresponding values of $K_{(\sigma)}$ are called the Δ -absolute principal curvatures of the congruence $\lambda^{i}_{(\sigma)}$. When $du/dt = dn/ds$, the equation (5.2) gives the equation of the absolute geodesic of the congruence in F_m.

From the equations (2.4), (2.9) and (2.12), we have

$$\text{(5.3)} \qquad K^{2}_{(\sigma)} = K^{2}_{(\sigma)G} + K^{2}_{n^{*}_{(\sigma)}}\ .$$

Now we prove the following

Theorem (5.1). *A necessary and sufficient condition that a curve of the subspace be Δ-absolute geodesic is that it is a Δ -geodesic as well as a Δ -asymptotic line of the congruence in the subspace.*

Proof. From the definition (5.1) and the equation (5.3), we have $K_{(\sigma)} = 0$ and then from the equation (5.3) obviously $K_{(\sigma)G} = 0$ and $K_{n^{*}_{(\sigma)}} = 0$. Conversely, if $K_{(\sigma)G} = K_{n^{*}_{(\sigma)}} = 0$, then $K_{(\sigma)} = 0$ which completes the proof.

In view of the equation (5.3) and the definitions (2.1)-(2.5) and (5.1), the following propositions are immediate.

Theorem (5.2). *The Δ-absolute curvature and Δ-geodesic curvature of the congruence are equal along the Δ-asymptotic line of the congruence in F_m.*

Theorem (5.3). *The Δ-absolute curvature and Δ-secondary normal curvatures are equal along a Δ-geodesic curve of the congruence in the subspace.*

Theorem (5.4). *If any two of the following hold, the third will also hold :*

(*i*) *the curve be a Δ-absolute geodesic,*

(*ii*) *it is a Δ-geodesic and*

(*iii*) *it is a Δ-asymptotic line of the congruence.*

REFERENCES

[1] ELIOPOULOS, H. A. : *Subspaces of a generalised metric space*, Canad. J. of Math., XI (3), (1959).

[2] LAL, K. B. AND PRASAD C. M. : *The generalised curvatures of a congruence of curves in the subspace of a Finsler space*, Rev. Fac. Sci. Univ. Istanbul, Serie A, 31, 49 - 55, (1966).

[3] NIRMALA, K. : *Curves and Invariants associated with a vector-field of a Riemannian V_m in relation to a curve C in a subspace V_n*, Proc. Nat. Inst. of Sciences, India, 29, A, No. 4, (1963).

[4] RUND, H. : The Differential Geometry of Finsler Spaces, SPRINGER VERLAG (1959).

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF GORAKHPUR,
GORAKHPUR, (U. P.), INDIA

(Manuscript received March 21, 1970)

ÖZET

Δ-türetme işlemi [1], bir kongrüansın esas doğrultularının belirtilmesi için DUPIN göstergesinin kullanılmasına imkân vermektedir. Δ-türetme işlemi [2] ise, bir kongrüansın esas doğrultularının bulunulmasında bir $\dot{x}^i$ doğrultusuna tekabül eden oskülatör göstergenin kullanılmasını gerektirmektedir. Bu araştırmada, Δ-türetme işlemini kullanarak bir FINSLER uzayının bir alt uzayında bir eğri kongrüansının Δ-eğrilikleri ve Δ-geodezik esas doğrultuları elde edilmiş ve bunlardan bâzı sonuçlar çıkarılmıştır.