A NOTE ON LOGARITHMIC PROXIMATE ORDERS [1])

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The logarithmic proximate orders of an entire functiou $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are defined as

$$\bar{\rho} = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r},\ \bar{\sigma}_f = \limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\bar{\rho}(r)}}$$

where $M(r) = \underset{z=r}{\text{l.u.b.}} |f(z)|$. Some results concerning these quantites are optained.

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Let $M(r) = \underset{|z|=r}{\text{l.u.b.}} |f(z)|$ and $\mu(r) = \max_{n\geq 1} \{ |a_n| r^n \}$ be the maximum term in the TAYLOR expansion of $f(z)$. It is well known that for functions of finite order, [1, 31]

$$\log M(r) \sim \log \mu(r). \tag{1.1}$$

Let the order ϱ of $f(z)$ be zero. We define the logarithmic order $\bar{\varrho}$ [3], by

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \bar{\varrho},\ 1 \leq \bar{\varrho} \leq \infty. \tag{1.2}$$

Let $1 < \bar{\varrho} < \infty$. A real valued function $\bar{\varrho}(r)$ is said to be a logarithmic proximate order [2] of $f(z)$ if,

(1.3) $\bar{\varrho}(r)$ is continuous, piecewise differentiable for $r > r_0$,

(1.4) $\bar{\varrho}(r) \to \bar{\varrho}$ as $r \to \infty$,

(1.5) $\bar{\varrho}'(r)\, r (\log r)(\log\log r) \to 0$ as $r \to \infty$, where $\bar{\varrho}'(r)$ is either the left or right hand side derivative at points where they are different,

$$\limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\bar{\varrho}(r)}} = 1. \tag{1.6}$$

Although $\bar{\varrho}(r)$ satisfies (1.6), it will be more convenient, in order to compare $\log M(r)$ with $\bar{\varrho}(r)$ etc., to take

$$\limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\bar{\varrho}(r)}} = \bar{\sigma}_f,\ 0 < \sigma_f < \infty. \tag{1.7}$$

1) The author is thankful to Professor R. S. L. SRIVASTAVA for his kind guidance.

We shall call $\bar{\sigma}_f$ the logarithmic type of $f(z)$ with respect to $\bar{\varrho}(r)$. In this note we obtain a formula for $\bar{\sigma}_f$ in terms of the coefficients $\{a_n\}$.

2. To formulate our result more precisely, we define the function $\varphi(t)$ as the unique solution of the equation

$$t = (\log r)^{\bar{\varrho}(r)-1}. \tag{2.1}$$

Next we prove :

Lemma. *For the function $\varphi(t)$ defined as above, we have*

$$\lim_{t\to\infty} \frac{d(\log\log\varphi(t))}{d(\log t)} = \frac{1}{\bar{\varrho}-1} \tag{2.2}$$

and

$$\lim_{t\to\infty} \frac{\log\varphi(Kt)}{\log\varphi(t)} = K^{1/(\bar{\varrho}-1)}, \quad K>0. \tag{2.3}$$

Proof. From (2.1) we have

$$\log t = [\bar{\varrho}(r)-1]\log\log r.$$

Differentiating with respect to $\log\log r'$, we get,

$$\frac{d(\log t)}{d(\log\log r)} = (\bar{\varrho}(r)-1) + \bar{\varrho}'(r)\, r(\log r)(\log\log r).$$

Hence, in view of (1.4) and (1.5)

$$\lim_{t\to\infty} \frac{d(\log t)}{d(\log\log r)} = \bar{\varrho}-1.$$

This is equivalent to

$$\lim_{t\to\infty} \frac{d(\log\log\varphi(t))}{d(\log t)} = 1/(\bar{\varrho}-1)$$

and (2.2) follows. Hence we have the asymptotic inequality

$$\left(\frac{1}{\bar{\varrho}-1}-\varepsilon\right) d(\log t) < d\left(\log\log\varphi(t)\right)\left(\frac{1}{\bar{\varrho}-1}+\varepsilon\right) d(\log t),\ \varepsilon>0.$$

Integrating from t to Kt, we get

$$\left(\frac{1}{\bar{\varrho}-1}-\varepsilon\right)\log K < \log\left[\frac{\log\varphi(Kt)}{\log\varphi(t)}\right] < \left(\frac{1}{\bar{\varrho}-1}+\varepsilon\right)\log K$$

or

$$\lim_{t\to\infty} \frac{\log\varphi(Kt)}{\log\varphi(t)} = K^{1/(\bar{\varrho}-1)}.$$

This proves the lemma.

Theorem. *For the logarithmic type $\bar{\sigma}_f$ defined by (1.7), we have*

$$\limsup_{n\to\infty} \frac{\varphi(n)}{\log|a_n|^{-\bar{\varrho}/n}} = \left(\frac{1}{\bar{\varrho}-1}\right)(\bar{\sigma}_f\,\bar{\varrho})^{1/(\bar{\varrho}-1)}. \tag{2.4}$$

Proof. For any $\sigma > \bar{\sigma}_f$, we have

$$\log M(r) < \sigma (\log r)^{\bar{\varrho}(r)}, \quad r > r_0(t).$$

Since, by CAUCHY's inequality, $|a_n| r^n < M(r)$ for all n and r, we have

$$\log |a_n| + n \log r < \sigma (\log r)^{\bar{\varrho}(r)}$$

or

$$\log |a_n| < \sigma (\log r)^{\bar{\varrho}(r)} - n \log r$$

Let

$$n = \left[\bar{\varrho}\, \sigma (\log r)^{\bar{\varrho}(r)-1} \right].$$

Then

$$\log |a_n| < \log r \left[\frac{n}{\bar{\varrho}} - n \right]$$

or

$$\left[\frac{\bar{\varrho}-1}{\bar{\varrho}} \right] n\, \varphi (n / \bar{\varrho}\, \sigma) < \log |a_n|^{-1}$$

or

$$\frac{\varphi(n)}{\log |a_n|^{-\bar{\varrho}/n}} < \left(\frac{1}{\bar{\varrho}-1} \right) \frac{\varphi(n)}{\varphi(n/\bar{\varrho}\sigma)}$$

or, by (2.3)

$$\frac{\varphi(n)}{\log |a_n|^{-\bar{\varrho}/n}} < \left(\frac{1}{\bar{\varrho}-1} \right) (\sigma\bar{\varrho})^{1/(\bar{\varrho}-1)}, \quad n > n_0 .$$

Hence

$$\limsup_{n\to\infty} \frac{\varphi(n)}{\log |a_n|^{-\bar{\varrho}/n}} \leq \left(\frac{1}{\bar{\varrho}-1} \right) (\sigma\bar{\varrho})^{1/(\bar{\varrho}-1)}, \quad \sigma_f < \sigma,$$

and, since σ is arbitrary,

$$\limsup_{n\to\infty} \frac{\varphi(n)}{\log |a_n|^{-\bar{\varrho}/n}} \leq \left(\frac{1}{\bar{\varrho}-1} \right) (\bar{\sigma}_f\, \bar{\varrho})^{1/(\bar{\varrho}-1)} .$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\varphi(n)}{\log |a_n|^{-\bar{\varrho}/n}} = \left(\frac{1}{\bar{\varrho}-1} \right) (\sigma_1\, \bar{\varrho})^{1/(\bar{\varrho}-1)} \quad \sigma_1 < \bar{\sigma}_f .$$

Let σ_2 be any number, $\sigma_1 < \sigma_2 < \bar{\sigma}_f$. Then we have

$$\frac{\varphi(n)}{\log |a_n|^{-\bar{\varrho}/n}} < \left(\frac{1}{\bar{\varrho}-1} \right) (\sigma_2\, \bar{\varrho})^{1/(\bar{\varrho}-1)}, \text{ for all } n > n_1$$

or,

$$\log |a_n| < -\left(\frac{\bar{\varrho}-1}{\bar{\varrho}} \right) \varphi(n)\ (n)\, (\sigma_2\, \bar{\varrho})^{1/(\bar{\varrho}-1)}$$

or,

$$\log |a_n| + n \log r < n \log r - \left(\frac{\bar{\varrho}-1}{\bar{\varrho}} \right) \frac{n\, \varphi(n)}{(\sigma_2\, \bar{\varrho})^{1/(\bar{\varrho}-1)}}$$

or,

$$\log |a_n| + n \log r < n \log r - \left(\frac{\bar{\varrho}-1}{\bar{\varrho}}\right) n\, \varphi\, (n/\sigma_2\, \bar{\varrho})\,(1-\varepsilon),\ \ \varepsilon > 0\,.$$

The above inequality holds for all r and all large values of n. Hence

$$\log \mu\ (r) < \max_{n \geq n_1} [n \log r - \left(\frac{\bar{\varrho}-1}{\bar{\varrho}}\right) n\, \varphi\, (n/\sigma_2\, \bar{\varrho})\,(1-\varepsilon)].$$

Using (2.2) it can be seen that for large values of n, the maxima on the right hand side of the above inequality is attained for $n = [\bar{\varrho}\sigma_2\,(\log r)^{\bar{\varrho}(r)-1}]$. Hence we have

$$\log \mu\,(r) < \sigma_2\, \bar{\varrho}\,(\log r)^{\bar{\varrho}(r)} - \left(\frac{\bar{\varrho}-1}{\bar{\varrho}}\right)(1-\varepsilon)\ (\log r)^{\bar{\varrho}(r)}\, \sigma_2\, \bar{\varrho}$$

$$= \sigma_2\,(\log r)^{\bar{\varrho}(r)}[1 + \varepsilon\,(\bar{\varrho}-1)]$$

or,

$$\frac{\log \mu\,(r)}{(\log r)^{\bar{\varrho}(r)}} < \sigma_2\,[1 + \varepsilon\,(\bar{\varrho}-1)],\, r > r_0\,.$$

In view of (1.1), this gives,

$$\lim_{r\to\infty} \sup \frac{\log M(r)}{(\log r)^{\bar{\varrho}(r)}} \leq \sigma_2 < \sigma_f\,.$$

This is a contradiction to (1.7). Hence we have

$$\lim_{n\to\infty} \sup \frac{\varphi\,(n)}{\log |a_n|^{-\bar{\varrho}/n}} = \left(\frac{1}{\bar{\varrho}-1}\right)(\bar{\varrho}\,\bar{\sigma}_f)^{1/(\bar{\varrho}-1)}.$$

This completes the proof of the Theorem.

Remark. Taking $\bar{\varrho}\,(r) = \bar{\varrho}$ in the theorem, we get the following interesting result :

$$\lim_{n\to\infty} \sup \frac{n^{1/(\bar{\varrho}-1)}}{\log |a_n|^{-\bar{\varrho}/n}} = \left(\frac{1}{\bar{\varrho}-1}\right)(\bar{\varrho}\,\bar{\sigma}_f)^{1/(\bar{\varrho}-1)}$$

where $\bar{\sigma}_f$ is the logarithmic type of $f(z)$ given by

$$\lim_{r\to\infty} \sup \frac{\log M(r)}{(\log r)^{\bar{\varrho}}} = \bar{\sigma}_f\,,\ 0 < \sigma_f < \infty\,.$$

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(Manuscript received July 23, 1970)

ÖZET

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ şeklinde verilen bir tam fonksiyon için logaritmik yaklaşım dereceleri

$$\bar{\rho} = \lim_{r\to\infty} \sup \frac{\log \log M(r)}{\log \log r}, \quad \bar{\sigma}_f = \lim_{r\to\infty} \sup \frac{\log M(r)}{(\log r)^{\bar{\rho}(r)}}$$

şeklinde tanımlanır. Burada $M(r) = \sup_{|z|=r} |f(z)|$ dir. Bu araştırmada, bu dereceleri ilgilendiren bâzı teoremler ispat edilmiştir.