# ON THE $\psi$ - TYPE, LOWER $\psi$ - TYPE AND $\psi$ - $\lambda$ - TYPE OF THE FUNCTIONS REPRESENTED BY THE SERIES

$$\sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$$

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This paper introduces a series which unifies various aspects of the two theories of entire functions defined by Taylor series and Dirichlet series, which have so far been treated separately by different workers in the two fields. Applications given at the end of each theorem are intended to emphasize this fact [1]).

**1.** Consider the function

$$F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)} \tag{1.1}$$

where

$$\lim_{n\to\infty} \sup n/\lambda_n = n < \infty, \tag{1.2}$$

$\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lim_{n\to\infty} \lambda_n = \infty$ ; $\{a_n\}$ $(n = 1, 2, 3 \ldots)$ is a sequence of real numbers and $\psi(x)$ is an increasing continuous function of the real variable $x$, defined either for all $x$ on the real line or for all $x$ in the interval $\eta < x < \infty$ where $-\infty < \eta < +\infty$, satisfying the following conditions :

(i) $\psi(x)$ tends to infinity as $x \to \infty$ ,

(ii) $\psi(x)$ assumes every value from $-\infty$ to $+\infty$,

(iii) $\psi(x)$ has an inverse, that is, if $y = \psi(x)$, then there exists a function $\psi^{-1}$ such that $\psi^{-1}(y) = x$,

(iv) $\psi(x) = \psi(x - k) = \Phi(x) = O(1)$ for every fixed $k > 0$.

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Let $x_c$ and $x_a$ be the abscissa of $\psi$-convergence and the abscissa of absolute $\psi$-convergence respectively. If $x_c = \infty$ and $x_a = \infty$, then the series in (1.1) is convergent for all values of $x > \eta$ and the sum function $F(x)$ is defined and continuous for all values of $x > \eta$.

2. If, in the series (1.1), we put $\psi(\sigma) = \sigma$, we get a DIRICHLET series viz.,

$$F(\sigma) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \sigma} \qquad (0 < \sigma < \infty)$$

in the real variable $\sigma$.

If, in the series (1.1), we substitute $\psi(r) = \log r$ and $\lambda_n = n$, one may get a TAYLOR series, viz.,

$$F(r) = \sum_{n=1}^{\infty} a_n \cdot r^n \qquad (0 < r < \infty)$$

in the real variable $r$.

If, $\lambda_n$ and $\psi(x)$ are respectively replaced by $n$ and $x$, the series takes the form of a TAYLOR-D-series,

$$F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{nx}.$$

In this paper we have attempted to unify various aspects of the two theories of entire functions defined by TAYLOR series and DIRICHLET series which have so far been treated separately by different workers in the two fields. Applications given at the end of each theorem are intended to emphasize this fact.

3. Let

$$\text{(3.1)} \qquad \lim_{x \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log F(x)}{\psi(X)} = \begin{matrix} \varrho \\ \lambda \end{matrix}, \qquad (0 \leq \lambda \leq \varrho < \infty)$$

We shall refer to the constants $\varrho$ and $\lambda$ as defined in (3.1) by $\psi$-order [1, 18] and lower $\psi$-order [1, 18] respectively of the function $F(x)$, which shall be said to be of regular $\psi$-growth when $\varrho = \lambda$. Justification for this lies in the fact that $\varrho$ and $\lambda$ depend on the function $\psi(x)$. Again let

$$\text{(3.2)} \qquad \lim_{x \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log F(x)}{e^{\varrho} \psi(x)} = \begin{matrix} T \\ t \end{matrix}, \qquad (0 \leq t \leq T < \infty)$$

where $\varrho$ $(0 < \varrho < \infty)$ is the $\psi$-order of the function $f(x)$. We define $T$ to be the $\psi$-type [1, 21] and $t$ the lower $\psi$-type [1, 21] of the function $F(x)$ of $\psi$-order $\varrho$, and in the case when the limit in (3.2) exists i.e. $T = t\ (< \infty)$ and $\lambda = \varrho$, we say that $F(x)$ is of perfectly regular $\psi$-growth.

In a previous paper [1, 15-26] we have obtained an expression for the $\psi$-type in terms of the coefficients. Now, we obtain the same for the lower $\psi$-type.

**Theorem 1.** *Let*

$$F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \psi(x)}$$

*be a function of* $\psi$*-order* $\varrho\,(0 < \varrho < \infty)$ *such that*

$$\lim_{x\to\infty} \inf \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} = t .$$

*If* $\lambda_{n+1} \sim \lambda_n$, *then*

$$t \geqq \lim_{n\to\infty} \inf \frac{\lambda_n}{e\varrho} (a_n)^{\varrho/\lambda_n} \tag{3.3}$$

*and further, if* $\dfrac{\log (a_n/a_{n+1})}{\lambda_{n+1} - \lambda_n}$ *forms a non-decreasing function of n for* $n > n_0$ *then*

$$t = \lim_{n\to\infty} \inf \frac{\lambda_n}{e\varrho} (a_n)^{\varrho/\lambda_n} . \tag{3.4}$$

**Proof.** Let

$$\lim_{n\to\infty} \inf \frac{\lambda_n}{e\varrho} (a_n)^{\varrho/\lambda_n} = a.$$

Suppose first $0 < a < \infty$, then, for any $\varepsilon > 0$, $n > N = N(\varepsilon)$, we have

$$\lambda_n (a_n)^{\varrho/\lambda_n} > (a - \varepsilon)\, e\, \varrho .$$

We know that

$$F(x) > a_n e^{\lambda_n \cdot \psi(x)} \text{ for all } x > x_0 .$$

Therefore

$$\frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} > \frac{\log (a_n) + \lambda_n \cdot \psi(x)}{e^{\varrho \cdot \psi(x)}} .$$

That is

$$\frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} > \frac{1}{e^{\varrho \cdot \psi(x)}} \left[ \lambda_n \psi(x) + \frac{\lambda_n}{\varrho} \log (a - \varepsilon)\, e\, \varrho - \frac{\lambda_n}{\varrho} \log \lambda_n \right] .$$

Let

$$(\lambda_n/\varrho \cdot a)^{1/\varrho} \leqq e^{\psi(x)} < (\lambda_{n+1}/\varrho \cdot a)^{1/\varrho} .$$

Then

$$\frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} > \frac{a\,\varrho}{\lambda_{n+1}} \left[ \frac{\lambda_n}{\varrho} \log (1/a\,\varrho) + \frac{\lambda_n}{\varrho} \log \{(a - \varepsilon)\, e\, \varrho\} + 0(1) \right]$$

$$\sim a \log \left[ \frac{(a-\varepsilon)}{a} \cdot e \right], \quad \text{since} \quad \lambda_n \sim \lambda_{n+1} .$$

Hence $t \geqq a$, which obviously holds when $a = 0$. If $a = \infty$, the above argument with a arbitrary large number $k$ in place of $(a - \varepsilon)$ gives $t = \infty$ and hence (3.3).

If $\mu(x, F)$ denotes the maximum term of $F(x)$, then

$$\frac{\log \mu(x, F)}{e^{\varrho \cdot \psi(x)}} = \frac{1}{e^{\varrho \cdot \psi(x)}} \left[ \log a_n + \lambda_n \cdot \psi(x) \right]$$

for

$$\frac{\log (a_{n-1}/a_n)}{\lambda_n - \lambda_{n-1}} \leq \psi(x) < \frac{\log (a_n/a_{n+1})}{\lambda_{n+1} - \lambda_n} \cdot$$

Suppose first $t < \infty$, then

$$\log a_n + \lambda_n \cdot \psi(x) \geq (t - \varepsilon) \cdot e^{\varrho \cdot \psi(x)} \tag{3.5}$$

for all $x > x_0$ and for all $n$ such that

$$\frac{\log (a_{n-1}/a_n)}{\lambda_n - \lambda_{n-1}} \leq \psi(x) < \frac{\log (a_n/a_{n+1})}{\lambda_{n+1} - \lambda_n}$$

Let

$$X = \lambda_n (a_n)^{\varrho/\lambda_n}$$

then for $n > n_0$,

$$\log X > \log \lambda_n + \frac{\varrho}{\lambda_n} \left[ (t - \varepsilon) e^{\varrho \cdot \psi(x)} - \lambda_n \cdot \psi(x) \right]$$

or

$$X > \frac{\lambda_n}{e^{\varrho \cdot \psi(x)}} \exp \left[ \frac{\varrho \cdot (t - \varepsilon) \; e^{\varrho \cdot \psi(x)}}{\lambda_n} \right]$$

$$\geq \frac{\lambda_n}{e^{\varrho \cdot \psi(x)}} \left[ \frac{e \cdot \varrho (t - \varepsilon) \cdot e^{\varrho \cdot \psi(x)}}{\lambda_n} \right]$$

since $e^x \geq ex$, for all values of $x > 0$.

Further, if

$g(n) = \dfrac{\log (a_n/a_{n+1})}{\lambda_{n+1} - \lambda_n} - g(n-1) = \ldots = g(n-m)$ and if $1 \leq p \leq m$, $(n - m) > n_0$,
we obtain from (3.5)

$$\lambda_{n-p} \cdot (a_{n-p})^{\varrho/\lambda_{n-p}} \geq e \, \varrho \, t.$$

Therefore

$$\liminf_{n \to \infty} \frac{\lambda_n}{e \, \varrho} (a_n)^{\varrho/\lambda_n} \geq t \tag{3.6}$$

which on the same arguments shows that if $t = \infty$, then $a = \infty$. Hence (3.4) follows from (3.3) and (3.6).

**Application.**

*(i) Let* $f(z) = \sum a_n z^n$, $(z = x + iy)$, *be an entire function of order* $\varrho$ *and lower order* $\lambda$ $(0 < \lambda \leq \varrho < \infty)$. *Then*

$$t \geq \lim_{n\to\infty} \inf \frac{n}{e\varrho} |a_n|^{\varrho/n} \qquad (0 \leq t < \infty)$$

*and further if* $\log \left| \frac{a_n}{a_{n+1}} \right|$ *and consequently* $|a_n/a_{n+1}|$ *forms a nondecreasing function of n for* $n > n_0$, *then* [2, 26]

$$t = \lim_{n\to\infty} \inf \frac{n}{e\varrho} |a_n|^{\varrho/n}.$$

(*ii*) In the case of DIRICHLET series the result [4, 29-31] is the same as in (3.3) and in (3.4) under the conditions mentioned in the theorem.

**Theorem 2.** *Let* $F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$ *be the function of ψ-order ϱ and lower ψ-order* $\lambda$ $(0 \leq \lambda < \varrho < \infty)$. *Then*

(3.7) (*i*) $$\lim_{x\to\infty} \inf \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} = \lim_{x\to\infty} \inf \frac{\log \mu(x, F)}{e^{\varrho \cdot \psi(x)}} = 0$$

i.e. *the lower ψ-type of the function* $F(x)$ *of irregular ψ-growth and of finite ψ-order is zero.*

(*ii*) $$\lim_{x\to\infty} \inf \frac{\lambda_{\psi(x,F)}}{e^{\varrho \cdot \psi(x)}} = 0.$$

**Proof.** From (3.1) we obtain

(3.8) $\log F(x) > e^{(\lambda-\varepsilon)\cdot\psi(x)}$, for any $\varepsilon > 0$ and $x > x_0 = x_0(\varepsilon)$ and

(3.9) $\log F(x) < e^{(\lambda+\varepsilon)\cdot\psi(x)}$ for a sequence of values of $x \to \infty$.

Dividing (3.8) and (3.9) by $e^{\lambda \cdot \psi(x)}$ and then proceeding to limits, the argument shows that

$$\lim_{x\to\infty} \inf \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} = 0.$$

Since $\log F(x) \sim \log \mu(x, F)$, for $\varrho < \infty$, hence the result in (3.7).

Also, it is known [1, *a*] that

$$\lim_{x\to\infty} \inf \frac{\log \lambda_{\psi(x,F)}}{\psi(x)} = \lambda.$$

Therefore proceeding on the same lines as above, we obtain the second part of the theorem.

**Application.** *The lower type of entire functions represented by* DIRICHLET *series and* TAYLOR *series of irregular growth of finite order, is zero ; and the values of*

$$\lim_{\sigma\to\infty} \inf \frac{\lambda_{\psi(\sigma)}}{e^{\varrho\sigma}} \quad \text{and} \quad \lim_{r\to\infty} \inf \frac{\nu(r)}{r^{\varrho}}$$

*vanish in the two series respectively.* [4, 250], [5, 345].

**4.** The fact that

$$\lim_{x\to\infty} \inf \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} = 0 \quad \text{when} \quad (0 \leq \lambda < \varrho < \infty)$$

opens the question of comparing the function $\log F(x)$ with the function $e^{\lambda \cdot \psi(x)}$ when $(0 < \lambda < \varrho < \infty)$. Evidently, since $\lambda < \varrho$,

$$\lim_{x\to\infty} \sup \frac{\log F(x)}{e^{\lambda \cdot \psi(x)}} = \infty,$$

yet $\lim\limits_{x\to\infty} \inf \dfrac{\log F(x)}{e^{\lambda \cdot \psi(x)}}$ may still be a finite constant. We shall refer to this constant as $\psi$-$\lambda$-type of the function $F(x)$ and denote this by $t_\lambda$. Thus, for function $F(x)$ of $\psi$-order $\varrho$ and lower $\psi$-order $\lambda$, such that $(0 < \lambda < \varrho < \infty)$, we define the $\psi$-$\lambda$-type $t_\lambda$ by

$$\lim_{x\to\infty} \inf \frac{\log F(x)}{e^{\lambda \cdot \psi(x)}} = t_\lambda. \tag{4.1}$$

**Application.** From (4.1), we obtain the definitions for $\lambda$-type in case of DIRICHLET series and TAYLOR series respectively as obtained by R. S. L. SRIVASTAVA and P. SINGH [4, 250] and [6, 345] viz.,

(*i*)
$$\lim_{\sigma\to\infty} \inf \frac{\log M(\sigma)}{e^{\lambda \cdot \sigma}} = t_\lambda.$$

(*ii*)
$$\lim_{r\to\infty} \inf \frac{\log M(r)}{r^\lambda} = t_\lambda.$$

Here we find $t_\lambda$ in terms of coefficients.

**Theorem 3.** *Let*

$$F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$$

*be a function of lower $\psi$-order $\lambda$ $(0 < \lambda < \infty)$ such that*

$$\lim_{x\to\infty} \inf \frac{\log F(x)}{e^{\lambda \cdot \psi(x)}} = t_\lambda.$$

*If $\lambda_{n+1} \sim \lambda_n$, then*

$$t_\lambda \geq \lim_{n\to\infty} \inf \frac{\lambda_n}{e\lambda} (a_n)^{\lambda/\lambda_n} \tag{4.2}$$

*and further, if $\log a_n/a_{n+1})/(\lambda_{n+1} - \lambda_n)$ forms a non-decreasing function of $n$ for $n > n_0$, then*

$$t_\lambda = \lim_{n\to\infty} \inf \frac{\lambda_n}{e\lambda} (a_n)^{\lambda/\lambda_n}. \tag{4.3}$$

The above theorem can be proved on the same lines as theorem 1, hence we omit the proof.

**Theorem 4.** *Let* $F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$ *be a function of* $\psi$*-order* $\varrho$*, lower* $\psi$*-order* $\lambda$*,* $(0 < \lambda < \varrho < \infty)$ *and lower* $\psi$*-*$\lambda$*-type* $t_\lambda$*. If*

$$\lim_{n\to\infty} \lambda_n/n = D, D > 0.$$

*then*

(4.4) $$\lim_{n\to\infty} \inf \lambda_n (a_{n+1}/a_n)^{\lambda/D} \leq e\lambda t_\lambda,$$

*and*

(4.5) $$\lim_{n\to\infty} \inf \lambda_n (a_{n+1}/a_n)^{\varrho/D} = 0.$$

**Proof.** Let $\lim_{n\to\infty} \inf \lambda_n (a_{n+1}/a_n)^{\lambda/D} = c$, $(0 < c < \infty)$, then, for any $\varepsilon > 0$

(4.6) $$\lambda_n (a_{n+1}/a_n)^{\lambda/D} > c - \varepsilon \text{ for } n > n_0 = n_0(\varepsilon).$$

Substituting $n_0$, $n_0 - 1, \ldots, (n-1)$ in (4.6) and then multiplying all the $(n - n_0)$ inequalities, we have

$$(\lambda_n)^{n-n_0} \cdot (a_n/a_{n_0})^{\lambda/D} > (c-\varepsilon)^{n-n_0},$$

since $\lambda_{n+1} > \lambda_n$. Taking $(n - n_0)$th root of both sides and then proceeding to limits, we get $e^{\lambda t_\lambda} \geq c$, which is also true when $c = 0$. of $c = \infty$, then $t_\lambda = \infty$. If, in (4.6) we take $\varrho$ in place of $\lambda$ and proceed on the same lines as above, we obtain (4.5), in view of (3.7).

**5.** Let $F(x) = \sum^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$ be the function of $\psi$-order $\varrho\,(0 < \varrho < \infty)$, $\psi$-type $T$ and lower $\psi$-type $t$, then we have

(5.1) $$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(x, F)}{e^{\varrho \cdot \psi(x)}} = \frac{T}{t}, \quad (0 \leq t \leq T < \infty).$$

We now define $\psi$-growth number $\nu$ and lower $\psi$-growth number $\delta$ for the function $F(x)$ as

(5.2) $$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_{\nu(x,F)}}{e^{\varrho \cdot \psi(x)}} = \frac{\nu}{\delta}, \quad (0 \leq \delta \leq \nu < \infty)$$

In the next theorem we obtain a number of results involving $\nu$, $\delta$, $T$, $t$, and $\varrho$ etc.

**Theorem 5.** *If the symbols have the meanings, as defined in* (5.1) *and* (5.2) *then*

(5.3) (*i*) $$\nu \geq \varrho T \geq \varrho t \geq \delta$$

(*ii*) $$\nu \geq \varrho T \geq \frac{\nu e^{\delta/\nu}}{e} \geq \delta$$

(*iii*) $$\nu \geq \delta (1 + \log \nu/\delta) \geq \varrho t \geq \delta$$

(*iv*) $$\nu + \delta \leq e \varrho T$$

(*v*) *Equality cannot hold simultaneously in* (*iv*) *and* $\delta \leq \varrho T$.

**Proof.** From (5.2), we have

$$\lambda_{\nu(x,F)} > (\delta - \varepsilon).\, e^{\varrho \cdot \psi(x)}$$

for any $\varepsilon > 0$ and $x > x_0$.

Also, from (1.3) [1, 16] for $x > x_0$, and constant $\psi(k) > 0$, we have,

$$\log \mu[\psi^{-1}\{\psi(x) + (k)\}, F] = 0(1) + \left[\int_{x_0}^{x} + \int_{x}^{\psi^{-1}\{\psi(x)+\psi(k)\}} \lambda_{\nu(t,F)} \cdot \psi'(t)\, dt\right]$$

$$\geq 0(1) + (\delta - \varepsilon)\int_{x_0}^{x} e^{\varrho \cdot \psi(t)} \cdot \psi'(t)\, dt + \lambda_{\nu(x,F)} \cdot \psi(k)$$

$$= 0(1) + \frac{\delta - \varepsilon}{\varrho}\,[e^{\varrho \cdot \psi(x)} - e^{\varrho \cdot \psi(x_0)}] + \lambda_{\nu(x,F)} \cdot \psi(k).$$

Therefore,

$$\frac{\log \mu[\psi^{-1}\{\psi(x) + \psi(k)\}, F]}{e^{\varrho \cdot \{\psi(x) + \psi(k)\}}} \geq 0(1) + \frac{\delta - \varepsilon}{\varrho} \cdot \frac{e^{\varrho \cdot \psi(x)}}{e^{\varrho \cdot \{\psi(x) + \psi(k)\}}} + \frac{\lambda_{\nu(x,F)} \cdot \psi(k)}{e^{\varrho \cdot \{\psi(x) + \psi(k)\}}}$$

$$= 0(1) + \frac{\delta - \varepsilon}{\varrho} \cdot \frac{1}{e^{\varrho \cdot \psi(k)}} + \frac{\psi(k)}{e^{\varrho \cdot \psi(k)}} \cdot \frac{\lambda_{\nu(x,F)}}{e^{\varrho \cdot \psi(x)}}.$$

Hence, on proceeding to limits we obtain

(5.4) $$T \geq \frac{\delta + \gamma \cdot \varrho\, \psi(k)}{\varrho \cdot e^{\varrho \cdot \psi(k)}} \quad \text{and} \quad t \geq \frac{\delta\{1 + \varrho \cdot \psi(k)\}}{\varrho \cdot e^{\varrho \cdot \psi(k)}}.$$

The right hand side of the first inequality in (5.4) attains its maximum value $\frac{\nu\, e^{\delta/\nu}}{e\, \varrho}$ when $\psi(k) = \frac{\nu - \delta}{\nu\, \varrho}$, hence

(5.5) $$\varrho\, T \geq \frac{\nu\, e^{\delta/\nu}}{e}.$$

Similarily, the right hand side of the second inequality in (5.4) is maximum, when $\psi(k) = 0$. Thus

(5.6) $$\varrho\, t \geq \delta.$$

Again, as before, we have

$$\log \mu[\psi^{-1}\{\psi(x) + \psi(k)\}, F] \leq 0(1) + \frac{\nu + \varepsilon}{\varrho}\,[e^{\varrho \cdot \psi(x)} - e^{\varrho \cdot \psi(x_0)}] + \lambda_{\nu}[\psi^{-1}\{\psi(x) + \psi(k)\}, F) \cdot \psi(k).$$

Therefore

$$\frac{\log \mu[\psi^{-1}\{\psi(x) + \psi(k)\}, F]}{e^{\varrho \cdot \{\psi(x) + \psi(k)\}}} \leq 0(1) + \frac{\nu + \varepsilon}{\varrho \cdot e^{\varrho \cdot \psi(k)}} + \frac{\lambda_{\nu}[\psi^{-1}\{\psi(x) + \psi(k)\}, F]\, \psi(k)}{e^{\varrho \cdot \{\psi(x) + \psi(k)\}}}.$$

Hence, on proceeding to limits, we have

(5.7) $$T < \frac{\nu\,[1 + \psi(k) \cdot \varrho \cdot e^{\varrho \cdot \psi(k)}]}{\varrho \cdot e^{\varrho \cdot \psi(k)}} \quad \text{and} \quad \geq t \quad \frac{\nu + \delta \cdot \psi(k) \cdot \varrho \cdot e^{\varrho \cdot \psi(k)}}{\varrho \cdot e^{\varrho \cdot \psi(k)}}.$$

The right hand side of the first inequality in (5.7) is minimum when $\psi(k) = 0$, hence

(5.8) $$\varrho\, T \geq \nu.$$

Similarily, the right hand side of the second inequality in (5.7) attains its minimum value

$\frac{\delta}{\varrho}\,(1 + \log^{\nu/\delta})$ when $\psi(k) = \frac{1}{\varrho} \log \frac{\nu}{\delta}$. Thus

(5.9) $$\varrho\, t \geq \delta \left(1 + \log \frac{\nu}{\delta}\right).$$

Now, the first part of the theorem follows from (5.6) and (5.8).

The second and third parts of the theorem are the direct consequences of the first part, for

$$\frac{e^{\delta/\nu}}{e} \geq \frac{\delta}{\nu} \quad \text{and} \quad \left(1 + \log \frac{\nu}{\delta}\right) \geq \frac{\nu}{\delta}.$$

The fourth part of the theorem follows from (5.5), because

$$\varrho\, T \quad \geq \frac{\nu}{e}\, e^{\delta/\nu}$$

or

$$e\,\varrho\, T \quad \geq \nu \left(1 + \frac{\delta}{\nu} + \dots\right)$$

and hence the result.

To prove the fifth part of the theorem, let $\delta = \varrho\, T$, then from (5.4), we have

$$T \geq \frac{\varrho\, T + \nu\, \varrho \cdot \psi(k)}{\varrho \cdot e^{\varrho \cdot \psi(k)}}.$$

or

$$\nu \geq \frac{T\,(e^{\varrho \cdot \psi(k)} - 1)}{\psi(k)}.$$

Put $\psi(k) = 1/\varrho \cdot \log(1 + \eta)$ where $\eta \to 0$. So

$$\nu \geq \frac{\varrho\, T\, \eta}{\eta + 0\,(\eta^2)} \geq T.$$

Also, $\delta \geq \nu$, hence $\nu = \varrho\, T$.

So

$$\nu + \delta = 2\,\varrho\, T < e\,\varrho\, T.$$

Next suppose that

$$\nu + \delta = e\,\varrho\, T.$$

then $\delta$ will be less than $\varrho T$, for if it were equal to $\varrho T$, then by the above $\nu + \delta$ will have to be less than $e \varrho T$.

**Application.** The results proved in the case of TAYLOR series by SHAH [3, 220], S. K. SINGH [7, 6] and others and in the case of DIRICHLET series by K. N. SRIVASTAVA [6, 134-146], P. K. KAMTHAN [8, 28] and others, follow from the theorem proved above.

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## ÖZET


Tam fonksiyonların TAYLOR veya DIRICHLET serileri ile tanımlanmalarına göre elde edilen iki teoriyi birleştiren bir seri ele alınarak aynı konunun birçok yazar tarafından ayrı ayrı yapılan incelemelerini birleştirmek mümkün olmuştur. Teoremlerin sonunda yapılan uygulamalar bu durumu kesin bir şekilde teyid edecek mahiyettedir.