ON THE ZEROS OF ENTIRE FUNCTIONS [1]

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Let f be an entire function and $n(r,f)$ the number of zeros of f in $D=\{z\in C: |z|\leq r\}$, where $n(r,f)$ is assumed non zero. The *convergence exponent* α and the *lower convergence exponent* β of f are defined as the upper and lower limits of the ratio $\log n(r,f)/\log r$ when r tends to infinity. The object of this paper is to establish relationships between convergence exponents and lower convergence exponents of two or more entire functions and then a result about entire functions of infinite order.

1. Let E be the set of mappings $f: C \to C$ (C is the complex field) such that the image under f of an element $z \in C$ is $f(z)=\sum_{n\in N} a_n z^n$ with $\lim_{n\to+\infty} \inf |a_n|^{-1/n} = +\infty$; N is the set of natural numbers $0, 1, 2, \ldots$; $\{a_n : n \in N\}$ is a sequence in C and $z = x + iy$; $x, y \in R$ (R is the field of reals). Since $\lim_{n\to+\infty} \inf |a_n|^{-1/n} = +\infty$, i.e. the power series defining f converges for each complex z, f is an entire function.

If $f \in E$ is an entire function such that it has at least one zero in the disc $D=\{z\in C: |z|\leq r\}$, and if $n(r, f)$ is the number of zeros of f in D, then the convergence exponent of the zeros of f or briefly the convergence exponent α and lower convergence exponent β of f are defined as

$$(1.1) \qquad \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log n(r,f)}{\log r} = \begin{matrix}\alpha \\ \beta\end{matrix} .$$

In this paper we first establish relationships between convergence exponents and lower convergence exponents of two or more entire functions and then a result about entire functions of infinite order.

Theorem 1. *Let $f, f_1, f_2 \in E$ be three entire functions, each having at least one zero in D, of convergence exponents $\alpha, \alpha_1, \alpha_2$ and lower convergence exponents β, β_1, β_2. If $n(r,f)$, $n(r,f_1)$, $n(r,f_2)$ denote, respectively, the number of zeros of f, f_1, f_2 in D, and if, as $r\to+\infty$,*

$$(1.2) \qquad \log n(r,f) \sim \left(\log n(r,f_1) \log n(r,f_2)\right)^{1/2},$$

then

$$(1.3) \qquad (\beta_1 \beta_2)^{1/2} \leq \beta \leq \alpha \leq (\alpha_1 \alpha_2)^{1/2},$$

and

$$(1.4) \qquad \beta^2 \leq \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \leq \alpha^2.$$

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Proof. Making use of (1.1) for f_1 and f_1 we get, for any $\varepsilon > 0$ and sufficiently large r [1]),

$$(1.5) \qquad (\beta_1 - \varepsilon) < \frac{\log n(r, f_1)}{\log r} < (\alpha_1 + \varepsilon),$$

and

$$(1.6) \qquad (\beta_2 - \varepsilon) < \frac{\log n(r, f_2)}{\log r} < (\alpha_2 + \varepsilon).$$

Multiplying (1.5) and (1.6) we get, for any $\varepsilon > 0$ and sufficiently large r,

$$(\beta_1 - \varepsilon)(\beta_2 - \varepsilon) < \frac{\log n(r, f_1) \log n(r, f_2)}{(\log r)^2} < (\alpha_1 + \varepsilon)(\alpha_2 + \varepsilon),$$

or

$$(\beta_1 - \varepsilon)(\beta_2 - \varepsilon) < \left(\frac{\log n((r, f)}{\log r}\right)^2 < (\alpha_1 + \varepsilon)(\alpha_2 + \varepsilon),$$

in view of condition (1.2). On proceeding to limits and making use of (1.1) for f we, therefore, get

$$(\beta_1 \beta_2)^{1/2} \leq \beta \leq \alpha \leq (\alpha_1 \alpha_2)^{1/2}.$$

Thus (1. 3) is established.

In order to prove (1.4) we make use of the well known fact that if $< x_n >$ and $< y_n >$ are two sequences in R^* (R^* is the set of extended reals), then

$$(1.7) \qquad \lim_{n \to +\infty} \inf n\, (x_n \,.\, y_n) \leq \begin{matrix} \lim\limits_{n \to +\infty} \inf x_n \,.\, \lim\limits_{n \to +\infty} \sup y_n \\ \\ \lim\limits_{n \to +\infty} \inf y_n \,.\, \lim\limits_{n \to +\infty} \sup x_n \end{matrix} \leq \lim_{n \to +\infty} \sup\, (x_n \,.\, y_n).$$

Here for large values of r, the sequences

$$\left\{ \frac{\log n(r, f_1)}{\log r} \right\} \text{ and } \left\{ \frac{\log n(r, f_2)}{\log r} \right\}$$

satisfy the conditions for (1.7) and so

$$\lim_{r \to +\infty} \inf \left(\frac{\log n(r, f_1)}{\log r} \,.\, \frac{\log n(r, f_2)}{\log r} \right) \begin{matrix} \leq \lim\limits_{r \to +\infty} \inf \dfrac{\log n(r, f_1)}{\log r} \,.\, \lim\limits_{r \to +\infty} \sup \dfrac{\log n(r, f_2)}{\log r} \\ \\ \lim\limits_{r \to +\infty} \inf \dfrac{\log n(r, f_2)}{\log r} \,.\, \lim\limits_{r \to +\infty} \sup \dfrac{\log n(r, f_1)}{\log r} \end{matrix}$$

[1]) ε need not be the same at each occurence.

$$\leq \lim_{r \to +\infty} \sup \left(\frac{\log n(r, f_1)}{\log r} \cdot \frac{\log n(r, f_2)}{\log r} \right),$$

or

$$\lim_{r \to +\infty} \inf \left(\frac{\log n(r, f)}{\log r} \right)^2 \leq \frac{\beta_1}{\beta_2} \frac{\alpha_2}{\alpha_1} \leq \lim_{r \to +\infty} \sup \left(\frac{\log n(r, f_1)}{\log r} \right)^2,$$

in view of (1.1) for f_1 and f_2, and condition (1.2). Now using (1.1) for f_1 we get

$$\beta^2 \leq \frac{\beta_1}{\alpha_1} \frac{\alpha_2}{\beta_2} \leq \alpha^2,$$

which proves (1.4).

Corollary 1. *Let $f_k \in E$ $(k = 1, 2, \ldots, m)$ be m entire functions, each having at least one zero in D, of convergence exponents α_k and lower convergence exponents β_k, and let, for each $k = 1, 2, \ldots, m$, $n(r, f_k)$ denote the number of zeros of f_k in D. Also let $f \in E$ be some other entire function, having at least one zero in D, of convergence exponent α and lower convergence exponent β. If $n(r, f)$ denote the number of zeros of f in D, and if, as $r \to +\infty$,*

$$\log n(r, f) \sim \left(\log n(r, f_1) \ldots \log n(r, f_m)\right)^{1/m},$$

then

$$(\beta_1 \ldots \beta_m)^{1/m} \leq \beta \leq \alpha \leq (\alpha_1 \ldots \alpha_m)^{1/m}.$$

This is an immediate generalization of (1.3).

Corollary 2. *Under the hypothesis of Corollary 1, if each of the entire functions be of regular growth and nonintegral order, then,*

$$\varrho \leq (\varrho_1 \ldots\ldots \varrho_m)^{1/m},$$

where ϱ_k is the order of f_k for each $k = 1, 2, \ldots, m$, and ϱ is the order of f

This follows from Corollary 1, since for an entire function of regular growth and nonintegral order the convergence exponent equals the order [1, 24].

Theorem 2. *Let $f, f_1, f_2 \in E$ be three entire functions, each having at least one zero in D, of convergence exponents $\alpha, \alpha_1, \alpha_2$, and lower convergence exponents β, β_1, β_2. If $n(r, f)$, $n(r, f_1)$, $n(r) f_2)$ denote, respectively, the number of zeros of f, f_1, f_2 in D and if, as $r \to +\infty$,*

$$\log n(r, f) \sim \log \left(n(r, f_1) \cdot n(r, f_2)\right), \tag{1.8}$$

then

$$\beta \leq \frac{\beta_1 + \alpha_2}{\beta_2 + \alpha_1} \leq \alpha. \tag{1.9}$$

The result in (1.9) follows from (1.8) and the well known fact that if $\{x_n\}$ and $\{y_n\}$ are two sequences in R^*, then

$$\lim_{n\to+\infty}\inf(x_n+y_n)\leq \begin{matrix}\lim\limits_{n\to+\infty}\inf x_n+\lim\limits_{n\to+\infty}\sup y_n\\ \\ \lim\limits_{n\to+\infty}\inf y_n+\lim\limits_{n\to+\infty}\sup x_n\end{matrix}\leq \lim_{n\to+\infty}\sup(x_n+y_n).$$

Theorem 3. *Let $f_1, f_2 \in E$ be two entire functions, each having at least one zero in* **D**, *of convergence exponents α_1, α_2 and lower convergence exponents β_1, β_2, and let $n(r,f_1)$, $n(r,f_2)$ denote, respectively, the number of zeros of f_1, f_2 in* **D**. *Also let*

$$\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\chi(r; f_1, f_2)=\begin{matrix}c\\ d\end{matrix}, \tag{1.10}$$

where $\chi(r; f_1, f_2)=n(r, f_1)-n(r, f_2)$, and $c, d\in R^*$. If c and d are finite, then $\alpha_1=\alpha_2$, and $\beta_1=\beta_2$. Futhermore if the limit in (1.10) *exists then*

$$\int_{r_0}^{r} x\,d\chi = 0(r),\ r > r_0(\varepsilon, f_1, f_2) > 0, \tag{1.11}$$

as $r\to+\infty$.

Proof. If c and d are finite, (1.10) implies that

$$n(r,f_1)-n(r,f_2)=0(1)$$

when $r\to+\infty$, and so, as $r\to+\infty$,

$$n(r,f_1)\sim n(r, f_2). \tag{1.12}$$

From (1.12) it follows that

$$\begin{matrix}\alpha_1\\ \beta_1\end{matrix}=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log n(r,f_1)}{\log r}=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log n(r,f_2)}{\log r}=\begin{matrix}\alpha_2\\ \beta_2\end{matrix},$$

which proves the first part of the theorem.

Now, if the limit in (1.10) exists, then $c=d$, and so, for any $\varepsilon>0$ and $r>r_0(\varepsilon, f_1, f_2)$, we have,

$$c-\varepsilon<\chi(r; f_1, f_2)<c+\varepsilon.$$

Hence

$$(c-\varepsilon)\left(1-0(1)\right)<\frac{1}{r}\int_{r_0}^{r}\chi(x; f_1, f_2)\,dx<(c+\varepsilon)\left(1-0(1)\right).$$

Taking limit as $r\to+\infty$, we get

$$\lim_{r\to+\infty}\frac{1}{r}\int_{r_0}^{r}\chi(x; f_1, f_2)\,dx=c,$$

whence

$$\lim_{r\to+\infty}\left(\chi(r\,;f_1,f_2)-\frac{1}{r}\int_{r_0}^{r} xd\{\chi(x\,;f_1,f_2)\}\right)=c.$$

But $\lim_{r\to+\infty}\chi(r\,;f_1,f_2)=c$, and hence (1.11) holds. This completes the proof of the theorem.

2. In the end we give a result regarding entire functions of infinite order. Let $f\in E$ be an entire function of order $\varrho(0\leq\varrho\leq+\infty)$ and lower order λ, $\mu(r,f)$ be the maximum term, for $|z|=r$, in the power series defining f, and $\nu(r,f)$ be the rank of $\mu(r,f)$. It is known [2,80] that

$$\lim_{r\to+\infty}\inf\frac{\log\mu(r,f)}{\nu(r,f)}\leq\frac{1}{\varrho}\leq\frac{1}{\lambda}\leq\lim_{r\to+\infty}\sup\frac{\log\mu(r,f)}{\nu(r,f)}. \tag{2.1}$$

It follows from (2.1) that if f is of infinite order, then

$$\lim_{r\to+\infty}\inf\frac{\log\mu(r,f)}{\nu(r,f)}=0. \tag{2.2}$$

Since for every entire function $\mu(r,f)\leq M(r,f)$, where $M(r,f)=\sup\limits_{0\leq\theta\leq2\pi}|f(re^{i\theta})|$, a result better than (2.2) viz.,

$$\lim_{r\to+\infty}\inf\frac{\log M(r,f)}{\nu(r,f)}=0, \tag{2.3}$$

for every entire function of infinite order, has been proved by SHAH [6, 112]. We show that (2.3) hold with $M_2(r,f)$ in place of $M(r,f)$, where $M_2(r,f)=\left(\frac{1}{2\pi}\int_0^{r\pi}|f(re^{i\theta})|^2\,d\theta\right)^{1/2}$ is the second root of the quadratic mean of f

Theorem 4. *If $f\in E$ is an entire function of infinite order, then*

$$\lim_{r\to+\infty}\inf\frac{\log M_2(r,f)}{\nu(r,f)}=0.$$

Proof. We know [1, 13] that

$$M_2(r,f)\leq M(r,f)\leq\left(\frac{R+r}{R-r}\right)^{1/2}.\,M_2(R,f),\ 0<r<R.$$

Putting $R=kr$, $(k>1)$ and taking logarithms, we get

$$\log M_2(r,f)\leq\log M(r,f)\leq\log M_2(kr,f)+\tfrac{1}{2}\log\left(\frac{k+1}{k-1}\right).$$

Hence

$$\lim_{r\to+\infty}\inf\frac{\log M_2(r,f)}{\nu(r,f)}\leq\lim_{r\to+\infty}\inf\frac{\log M(r,f)}{\nu(r,f)}$$

$$\leq \liminf_{r \to +\infty} \left(\frac{\log M_2(kr,f) + \frac{1}{2} \log \frac{k+1}{k-1}}{\nu(r,f)} \cdot \frac{\nu(kr,f)}{\nu(r,f)} \right)$$

$$= \liminf_{r \to +\infty} \frac{\log M_2(r,f)}{\nu(r,f)},$$

and the result follows from this in view of (2.3).

It is known [5, 215] that

$$M_2(r,f) \leq 2^{1/2} \mu(r,f) \nu(r,f).$$

Hence

$$\log M_2(r,f) < \tfrac{1}{2} \log 2 + \log \mu(r,f) + \log \nu(r,f).$$

Dividing throughout by $\nu(r,f)$ and taking inferior limits of both sides we get (2.4) in view of (2.2).

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ÖZET

f bir tam fonksiyon ve $n(r,f)$ bu fonksiyonun $D = \{z \in C : |z| \leq r\}$ diskindeki sıfırlarının sayıs olsun. $n(r,f)$ sıfırdan farklı farzedilmektedir. f fonksiyonunun *yakınsaklık üssü* ve *alt yakınsaklık üssü* $\log n(r,f)/\log r$ kesrinin, r nin sonsuza gitmesi halindeki üst ve alt limiti olarak tanımlanmıştır. Bu araştırmanın gâyesi iki veya daha fazla tam fonksiyonun yakınsaklık ve alt yakınsaklık üsleri arasında bağıntılar elde etmek ve sonsuz mertebeden bir tam fonksiyon hakkında bir sonuç ispat etmektir.