STRESSES IN A NON HOMOGENEOUS COMPOSITE SPHERICAL PRESSURE VESSEL [1])

YUDHISTHIR DE

Stresses have been computed in a composite spherical pressure vessel made of two layers of different POISSON's ratio and varying rigidity moduli and have been compared with the stresses when the shell is made up of a single layer. Concentration of the stress on the inter-face in the composite case has been obtained.

1. Introduction. Although in the classical theory of linear elasticity the solutions of problems are numerous for materials whose elastic coefficients are same at all points within the body in question, there are materials where these vary considerably from point to point.

The present paper is concerned with the determination of stresses in a closed form in a non-homogeneous isotropic composite pressure vessel made up of two layers of different POISSON's ratio. The non-homogeneity is characterised by the variation of shear modulus which is assumed to be proportional to an arbitrary function of the radial distance from the centre of the shell.

The solutions are obtained in terms of modified BESSEL Functions and it has been shown that for particular values of POISSON's ratio the displacement as also the stresses can be expressed in closed forms. Effect of non-homogeneity has been studied on the radial stress distribution where the pressure vessel is made up of two layers or a single layer. The shell has been presented in tabular form.

With a view towards a description of the type of non-homogeneity underlying this investigation, let (r, θ, φ) denote the spherical polar co-ordinaters and let the shell under consideration occupy the region $a \leq r \leq b$ where a is the inner radius and b is the outer radius.

If μ denote the shear modulus of the material at a point, we assume that

$$\mu = \mu_0 \, e^{-kr^m}, \qquad k \neq 0 \tag{1.1}$$

in which we assume that k has the dimension L^{-m} where m is a rational number and μ_0 is the value of μ in the homogeneous medium.

2. Governing equations and their solutions. The problem described in the preceding section is characterised by a radial displacement field corresponding with the problem of a shell held strained by internal pressure. With reference to spherical polar co-ordinates (r, θ, φ), we thus have

$$u_r = u(r), \quad u_\theta = u_\varphi = 0 \tag{2.1}$$

1) I express my heartful gratitude to Dr. J. G. CHAKRAVORTY, Department of Applied Mathematics, Science College, Calcutta, for his helpful guidance in preparing this paper.

where u_r, u_θ, u_φ are the polar components of displacement. Equations (2.1) in conjunction with the displacement-stress relations of the linear theory of the isotropic elastic media now yield the stresses [LOVE 1944]

$$\left\{\begin{aligned} \sigma_\pi &= \frac{2(1-\nu)\mu}{1-2\nu}\frac{du}{dr}+\frac{4\,\mu\,\nu}{1-2\nu}\cdot\frac{u}{r}\,, \\ \sigma_\theta = \sigma_\varphi &= \frac{2\,\mu\,\nu}{1-2\nu}\frac{du}{dr}+\frac{2\,\mu}{1-2\nu}\cdot\frac{u}{r}\,, \end{aligned}\right. \tag{2.2}$$

whereas all the remaining components of stress vanish, ν ($\neq 1/2$) being the POISSON's ratio.

The shear modulus μ is for the time being, assumed to be an arbitrary continuously differentiable function of r. The general equations of equilibrium

$$\frac{\partial\,\sigma_r}{\partial\,r}+\frac{1}{r}\,(2\,\sigma_r-\sigma_\theta-\sigma_\varphi)=0\,, \tag{2.3}$$

in view of (2.2) reduce to the single equation

$$\frac{d^2 u}{d\,r^2}+(2-k\,m\,r^m)\,\frac{1}{r}\,\frac{du}{d\,r}-\left(1+\frac{k\,m\,\nu\,r^m}{1-\nu}\right)\frac{2\,u}{r^2}=0\,, \tag{2.4}$$

The solution of this equation is obtained from

$$\frac{d^2\,v}{d\,r^2}+(1-k\,m\,r^m)\,\frac{1}{r}\,\frac{d\,v}{d\,r}-\left\{q/4-\frac{(1-2\nu)\,k\,m\,r^m}{2\,(1-\nu)}\right\}\frac{v}{r^2}=0\,, \tag{2.5}$$

where

$$u = r^{-1/2}\,v\,. \tag{2.6}$$

Setting

$$V = e^{x/2}\,\psi(x), \qquad x = k\,r^m\,, \tag{2.7}$$

where equation (2.5) reduces to

$$\frac{d^2\,\psi}{d\,x^2}+\frac{1}{x}\,\frac{d\,\psi}{d\,x}-\left(1/4+\frac{q}{4m^2\,x^2}\right)\psi=0, \tag{2.8}$$

in which we have assumed,

$$m=\frac{5\,\nu-1}{1-\nu}\,, \qquad m\neq 3\,. \tag{2.9}$$

The solution of (2.8) is obtained in terms of modified BESSEL functions I and K of the first and second kind respectively and is

$$\psi = A_1\,I_{3/2m}\,(x/2)+B_1\,K_{3/2m}\,(x/2)\,, \tag{2.10}$$

A_1 and B_1 being constants.

In view of (2.7) and (2.10) we have from (2.6)

$$u = r^{-1/2}\,e^{x/2}\,[A_1\,I_{3/2m}\,(x/2)+B_1\,K_{3/2m}\,(x/2)]. \tag{2.11}$$

If $3/2m$ is half an integer then (2.11) or in other words the solution of (2.4) can be expressed in a closed form. For example if $m = 1$ corresponding to $\nu = 1/3$ (copper $\nu = .378$, brass $\nu = .327$) we have [McLachlan (1955)]

(2.12) $$u = A\,(k\,r^{-1} - 2\,r^{-2})\,e^{kr} + B\,(k\,r^{-1} + 2\,r^{-2}).$$

For $m = 3/5$, x corresponding to $\nu = 2/7$ (for wrought iron $\nu = .275$, for steel $\nu = .268$) we have written l for k in (1.1)

(2.13) $$u = C\,(l^2\,r^{-4/5} - 6\,l\,r^{-7/5} + 12\,r^{-2})\,e^{l\,r^{3/5}} + D\,(l^2\,r^{-4/5} + 6\,l\,r^{-7/5} + 12\,r^{-2}).$$

Other solutions in closed form are possible for $m = 3/7$, $\nu = 5/19$ and so on. But these become more complicated and futhermore the value of ν becomes progressively less realistic.

3. **Formulation of the problem.** Henceforth, we shall call the material for which $m = 1$, $\nu = 1/3$ as I and the material for which $m = 3/5$ and $\nu = 2/7$ as II. We consider a spherical pressure vessel made up of two layers, the outer layer being made of material II and the inner layer of material I and subjected to an internal pressure p. The common surface of the two materials is at $r = c$.

We compute the stresses by (2.2) with (2.12) for the material I and with (2.13) for the material II and use suffixes 1 and 2 respectively.

For material I:

(3.1) $$(\sigma_r)_1 = 4\,\mu_0\,r^{-3}\,\{A\,(2 - 2\,k\,r + k^2\,r^2) - 2\,B\,e^{-kr}\},$$

(3.2) $$(\sigma_0)_1 = (\sigma_\varphi)_1 = 2\,\mu_0 r^{-3}\,\{A\,(k\,r^2 - 2) + 2\,B(1 + k\,r)\,e^{-r}\}.$$

For material II:

$$(\sigma_r)_2 = 2\,\mu_0\,[C\,(l^3\,r^{-6/5} - 6\,l^2\,r^{-9/5} + 18\,l\,r^{-12/5} - 24\,r^{-3})$$

(3.3) $$- D\,(6\,l\,r^{-12/5} + 24\,r^{-3})\,e^{-l r^{3/5}}],$$

$$(\sigma_0)_2 = (\sigma_\varphi)_2 = 2/5\,\mu_0\,[C\,(2\,l^3\,r^{-6/5} - 3\,l^2\,r^{-9/5} - 18\,l\,r^{-12/5} + 60\,r^{-3})$$

(3.4) $$+ D\,(9\,l^2\,r^{-9/5} + 42\,l\,r^{-12/5} + 60\,r^{-3})\,e^{-l r^{3/5}}].$$

Boundary conditions :

Case 1. When we consider a composite pressure vessel

(i) $(\varrho_r)_1 = -p$ on $r = a$; (ii) $(\varrho_r)_2 = 0$ on $r = b$;

(iii) $(\varrho_r)_1 = (\varrho_r)_2$ on $r = c$; (iv) $(u)_1 = (u)_2$ on $r = c$.

These give

(3.5) $$\left.\begin{aligned} A &= \frac{\lambda}{2c^3\,k^3}\left[4\,\xi_2\,e^{-ck} + \xi_1\,(kc + 2)\right], \\ B &= \frac{\lambda}{2c^3\,k^3}\left[2\,\xi_2\,f(c) - \xi_1\,(kc - 2)\,e^{ck}\right., \\ C &= \lambda\,N(b)\,e^{-lb^{3/5}} \quad , \quad D = \lambda\,M(b), \end{aligned}\right\}$$

where

$$\lambda = -\frac{p}{2\mu_o} a^3 k^3 c^3 \Big[4\xi_2 \{ f(a)\, e^{-ck} - f(c)\, e^{-nk} \}$$

$$+ \xi_1 \{ (kc+2)\, f(a) - 2\,(kc-2) e^{(c-a)k} \} \Big]^{-1},$$

$$\xi_1 = M(c)\, N(b)\, e^{-l\, b^{3/5}} - LM(b)\, N(c)\, e^{-l\, c^{3/5}},$$

$$\xi_2 = J(c)\, N(b)\, e^{l\, c^{3/5} - b^{3/5}} + M(b)\, H(c),$$

$$M(x) = l^3 x^{9/5} - 6\, l^2 x^{6/5} + 18 l\, x^{3/5} - 24,$$

$$N(x) = 6l\, x^{3/5} + 24,$$

$$H(c) = l^2 c^{6/5} + 6l\, c^{3/5} + 12,$$

$$J(c) = l^2 c^{6/5} - 6l c^{3/5} + 12,$$

$$f(x) = 2 - 2\,kx + k^2 x^2.$$

Case 2. When we consider the pressure vessel to be made up of a single layer of material I, the boundary conditions are :

(*i*) $(\sigma_r)_1 = -p$ on $r = a$; (*ii*) $(\sigma_r)_1 = 0$ on $r = b$.

These give

$$A = -\frac{pa^3 e^{-bk}}{4\,\mu_o} \Big[f(a)\, e^{-bk} - f(b)\, e^{-ak} \Big]^{-1}$$

$$B = -\frac{pa^3 f(b)}{8\,\mu_o} \Big[f(a)\, e^{-bk} - f(b)\, e^{-ak} \Big]^{-1}$$

where $f(x)$ is as defined in (3.6)

Case 3. When we consider the pressure vessel to be made up of a single layer of material II, the boundary conditions are:

(*i*) $(\sigma_r)_2 = -p$ on $r = a$; (*ii*) $(\sigma_r)_2 = 0$ on $r = b$.

These give

$$C = -\frac{pa^3}{2\,\mu_o\,\eta} N(b)\, e^{-l b^{3/2}},$$

$$D = -\frac{pa^3}{2\,\mu_o\,\eta} M(b),$$

where

$$\eta = M(a)\, N(b)\, e^{-l\, b^{3/5}} - M(b)\, N(a)\, e^{-l\, a^{3/5}}$$

and other expressions defined as in (3.6).

4. Numerical discussion. We assume $b/a = 2$ and $c/a = 1.5$. Furthermore for numerical evaluation of stresses we assume

$$k\,a = n = l\, a^{3/5} = 1.$$

In table I we give the values of $P = -\sigma_r/P$ for different values of δ $(= r/a)$. P_1, P_2 and P_3 are the values of P in the three different cases viz. the composite shell, the single layer shell of material *I* and the single layer shell of material *II*. It is observed from the table that the magnitude of stresses at every point in the shell of material *I* is less than its corresponding values in the shell of material *II* and the composite shell. The magnitude of the stress remains greater in material *II* than its corresponding values in the composite case up to $\delta = 1.4$ and then the situation becomes just the opposite.

Table I

(*Radial stress distribution in the three different cases*).

δ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
P_1	1.00000	0.67414	0.46459	0.32576	0.23145	0.16596	0.11218	0.07196	0.04151	0.01810	0.00000
P_2	1.00000	0.66201	0.44494	0.30137	0.20406	0.13674	0.08935	0.05550	0.03103	0.01316	0.00000
P_3	1.00000	0.68858	0.47913	0.33825	0.23752	0.16486	0.11142	0.07148	0.04123	0.01799	0.00000

In table II we give the values of $Q = \{(\sigma_0)_2 - (\sigma_0)_1\}/_p$ for different values of n and observe that Q increases together with n.

Table II

(*Shear stress concentration on the interface*)

n	1	2	3	4	5
Q	.057888	.116719	.155063	.172395	.181952

REFERENCES

[1] LOVE, A. E. H. : The Mathematical Theory of Elasticty, DOVER PUBLICATION, New York, 142, (1944).

[2] MCLACHLAN, N. W. : Bessel Functions for Engineers, CLARENDEN PRESS, Oxford, 202-204, (1955).

R. K. MISSION RESIDENTIAL COLLEGE
P. O. NARENDRAPUR
DIST. 24 - PARGANAS
WEST BENGAL, INDIA

(*Manuscript received November 13, 1970*)

ÖZET

Farklı POISSON oranları ve değişen rigidlik modüllerini hâiz iki tabakadan oluşan basınç altında bulunan karma bir küresel cisimdeki gerilimler hesaplanmış elde edilen sonuçlar tek bir tabakadan oluşan bir cisim için bulunan değerlerle karşılaştırılmış ve iki tabakayı ayıran yüzey üzerindeki gerilim durumları bulunmuştur.