ON POLYNOMIAL POWER SERIES WITH HOLOMORPHIC COEFFICIENTS

ALIMARDON A. ATAMURATOV AND X.Q. KAMOLOV

ABSTRACT. In this paper, we studied multiple polynomial power series with holomorphic coefficients and determined domain of convergence.

1. Introduction

Power series plays an important role on the theory of functions of several complex variables. There are several special types of series, which are widely used in the theory of analytical functions. One of them is Jacobi-Hartogs series, which were successfully used by A.Sadullaev and E.M.Chirka on problems of extension of analytical functions with singularities and also by S. Imomqulov and J. Khujamov on extension of analytic functions along the boundary sections ([**3**],[**4**]).
In this paper we study properties of multidimensional Jacobi-Hartogs series.
Let us consider the following power series:

$$f(z_1,...,z_n)=\sum_{|k|=0}^{\infty} c_k(z)\cdot\varphi_1^{k_1}(z_1)\cdot...\cdot\varphi_n^{k_n}(z_n), \tag{1}$$

where $k=(k_1,...,k_n)$, $|k|=k_1+k_2+\cdot...\cdot+k_n$, $c_k(z)=c_{k_1,..,k_n}(z)$, $\varphi_1(z_1),...,\varphi_n(z_n)$ are polynomials in $\mathbb{C}^n$, and $t=\deg c_{k_1,...,k_n}\leq C\deg(\varphi_1\cdot...\cdot\varphi_n)$, $\deg\varphi_j=s$. We call (1) as polynomial multiseries.
We define the sequences of partial sums by the standard diagonal order:

$$Q_j=\sum_{|k|=0}^{q} c_k(z)\cdot\varphi_1^{k_1}\cdot\varphi_2^{k_2}\cdot...\cdot\varphi_n^{k_n}\,, j=t+sq. \tag{2}$$

Definition 1.1. If the sequence $Q_j(z)$ converges as $j\to\infty$, then the polynomial series (1) is called convergent at the point z and certain limits $Q(z)$ are called the sum of power series.

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It is well known that for usual power series with given constant coefficients, statement of Abelian theorem [**1**] is true, i.e. if the power series

$$\sum_{|k|=0}^{\infty} c_k (z-a)^k$$

is convergent at a point $z^0 \in \mathbb{C}^n$, then it is absolutely convergent in the domain

$$U = \left\{ |z_j - a_j| < \left| z_j^0 - a_j \right|, \quad j = 1, 2, ..., n \right\}.$$

The next example shows that for the series (1), in general, Abelian theorem is not true.
Example. In $\mathbb{C}^2$, the polynomial power series

$$\sum_{|k|=0}^{\infty} (1 + k! z_1 \cdot z_2)(z_1 + \frac{1}{2})^{k_1} (z_2 + \frac{1}{2})^{k_2}$$

is convergent at the point $z = (z_1, z_2) = (0, 0)$. But it is not convergent at all points of the set

$$\Pi = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \left| z_1 + \frac{1}{2} \right| < \frac{1}{2}, \left| z_2 + \frac{1}{2} \right| < \frac{1}{2} \right\}.$$

Indeed, if we put $z_1 = -\frac{1}{4}, z_2 = -\frac{1}{4}$ then we can see that

$$\sum_{|k|=0}^{\infty} \left(1 + \frac{k!}{16} \right) \frac{1}{4^{k_1}} \cdot \frac{1}{4^{k_2}} = \sum_{|k|=0}^{\infty} \frac{1}{4^k} + \sum_{|k|=0}^{\infty} \frac{k!}{4^{k+2}} = \infty.$$

Thus from convergence of (1) at a point $z^0 \in \mathbb{C}^n$ we cannot state the convergence of the series in the set $\Pi = \{z \in \mathbb{C}^n : |\varphi_j(z_j)| \leq \left|\varphi_j(z_j^0)\right|, j = 1, 2, ..., n\}$. Main reason of it is the variance of coefficients of power series. We cannot say anything about the convergence of the series, although we know behavior of series on several points. But, it turns out to be estimating coefficients on some thick sets gives us to obtain the set of convergence of series (1). Main result of the paper is the following theorem.

Theorem 1.2. *If for nonpluripolar compact set $K \subset \mathbb{C}^n$ there exist positive numbers $r_j > 0$, $j = 1, 2, ..., n$, such that for the coefficients of power series (1) we have*

$$\overline{\lim_{|k|\to\infty}} \sqrt[|k|]{\|c_k(z)\|_K r_1^{k_1} \cdot r_2^{k_2} \cdot ... \cdot r_n^{k_n}} = 1, \tag{3}$$

then the power series (1) is convergent in the set

$$\Pi(r_1, ..., r_n) = \{ z \in \mathbb{C}^n \ : \ |\varphi_1(z_1)| < r_1 \ , ..., |\varphi_n(z_n)| < r_n \}. \tag{4}$$

Note that $\|c_k\|_K = \sup\limits_K |c_k(z)|$, and in generally the set $\Pi(r_1, ..., r_n)$ is not connected and contains at most $s^n, s = \deg \varphi_j(z)$, connected components.
Before we prove the theorem recall some preliminary results from pluripotential theory.

2. Preliminary results

Pluricomplex Green functions (see [**2**]). For the compact subset $K \subset \mathbb{C}^n$ we define class of functions $L = \{u(z) \in psh(\mathbb{C}^n) : u(z) \leq C_u + \ln(1 + \|z\|)\}, C_u$ is a constant depending only on functions u. Then for the function $V(z,K) = \sup\{u(z) : u(z) \in L, \ u|_K \leq 0\}$, its upper regularization $V^*(z,K) = \overline{\lim\limits_{w \to z}} V(w,K)$ is called as extremely plurisubharmonic function (or Green function) of the compact set K.

We consider here some useful properties of Green functions:
1. For the existence of $V^*(z,K) \in psh(\mathbb{C}^n)$ it is sufficient and necessary that set K is nonpluripolar. If K is pluripolar, then $V^*(z,K) \equiv +\infty$. Another case is when $V^*(z,K) \in L$ and the set $\{z \in \mathbb{C}^n : V(z,K) < V^*(z,K)\}$ is pluripolar.
2. *Bernstein-Walsh inequality.* (in details see [2]) If $P_m(z)$ is a polynomial of degree m, then it holds that

$$\frac{1}{m} \ln |P_m(z)| \leq \frac{1}{m} \ln \|P_m\|_K + V(z,K), \ z \in \mathbb{C}^n.$$

Let us prove this inequality. Let $P_m(z)$ be a polynomial of degree m, then function $\ln |P_m(z)| \in psh(\mathbb{C}^n)$ and

$$u(z) = \frac{1}{m} \ln |P_m(z)| - \frac{1}{m} \ln \|P_m\|_K = \frac{1}{m} \ln \frac{|P_m(z)|}{\|P_m\|_K}.$$

Consequently for the function $u(z)$ we get that $u(z)|_K = \frac{1}{m} \ln \frac{|P_m(z)|}{\|P_m\|_K} |_K \leq 0$. Now we will show that $u(z) \in L$. In fact as $P_m(z)$ is polynomial of degree m it follows that the function $\left| \frac{P_m(z)}{|z|^m} \right|$ is bounded as $z \to \infty$. So we have

$$u(z) = \frac{1}{m} \ln |P_m(z)| - \frac{1}{m} \ln \|P_m\|_K = \frac{1}{m} \ln |z|^m + \frac{1}{m} \ln \left| \frac{P_m(z)}{|z|^m} \right| +$$

$$+ \ln \|P_m\|_K^{-\frac{1}{m}} \leq \ln |z| + \ln \left(\left| \frac{P_m(z)}{|z|^m \|P_m\|_K} \right| \right)^{\frac{1}{m}} \leq C_u + \ln(1 + |z|),$$

and $u(z) \in L$. From here we get that $u(z) \leq V(z,K)$.

3. The proof of the main result

Following Bernstein-Walsh inequality, for the coefficients of the series (1) we can write the next inequality:

$$\frac{1}{t} \ln |c_k(z)| \leq \frac{1}{t} \ln \|c_k\|_K + V(z,K) \ , \ z \in \mathbb{C}^n.$$

For arbitrary small numbers $\delta_j > 0, \quad j = 1, 2, ..., n$ we estimate the terms of series (1) in the set

$$\Pi \supset \Pi' = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |\varphi_j(z_j)| < r_j - \delta_j \quad j = 1, ..., n\},$$

$$\left|c_{k_1 k_2 \ldots k_n}(z) \cdot \varphi_1^{k_1}(z_1) \cdot ... \cdot \varphi_n^{k_n}(z_n)\right| \leq |c_k(z)| \cdot (r_1 - \delta_1)^{k_1} \cdot ... \cdot (r_n - \delta_n)^{k_n} \leq$$

$$\leq e^{tV(z,K)} \|c_k\|_K \cdot (r_1 - \delta_1)^{k_1} \cdot ... \cdot (r_n - \delta_n)^{k_n}. \tag{5}$$

From conditions of theorem, for each $\varepsilon > 0$, there exists number k_0 such that for all $|k| > k_0$ it holds that

$$\|c_{k_1 k_2 \ldots k_n}\|_K \cdot r_1^{k_1} \cdot ... \cdot r_n^{k_n} \leq (1 + \varepsilon)^k.$$

Consequently, we obtain

$$\|c_{k_1 k_2 \ldots k_n}\|_K \leq \frac{(1+\varepsilon)^k}{r_1^{k_1} \cdot ... \cdot r_n^{k_n}}. \tag{6}$$

Putting together (6) and (5) we get that

$$\left|c_{k_1 k_2 \ldots k_n}(z) \cdot \varphi_1^{k_1}(z_1) \cdot ... \cdot \varphi_n^{k_n}(z_n)\right| \leq$$

$$\leq \exp(tV(z, K)) \cdot (1 + \varepsilon)^k \left(\frac{r_1 - \delta_1}{r_1}\right)^{k_1} ... \left(\frac{r_n - \delta_n}{r_n}\right)^{k_n} \leq$$

$$\leq M \cdot \left(\left(1 - \frac{\delta_1}{r_1}\right)(1 + \varepsilon)\right)^{k_1} \cdot ... \cdot \left(\left(1 - \frac{\delta_n}{r_n}\right)(1 + \varepsilon)\right)^{k_n}.$$

Therefore, if we denote $q_j = \left(1 - \frac{\delta_j}{r_j}\right)(1 + \varepsilon), (j = 1, 2, ..., n)$ then as $\varepsilon > 0$ is arbitrary we can supply $0 < q_j < 1$. Thus for the terms of the power series (1) we obtain the estimation

$$\left|c_k \varphi^k\right| < M \cdot q_1^{k_1} \cdot ... \cdot q_n^{k_n} = M\, q^k.$$

Therefore as power series $\sum\limits_{|k|=0}^{\infty} M\, q^k$ is convergent in accordance of the Weierstrass theorem we conclude that series (1) converges uniformly in the set Π'. Additionally, from here we can conclude that the sum of the power series (1) is analytic in the set (4).
Using arguments from above we can establish the next interesting fact.

Theorem 3.1. *If for any nonpluripolar set $K \subset \mathbb{C}^n$, there exists numbers $r_j > 0$ $j = 1, 2, ..., n$ such that*

$$\overline{\lim_{|k| \to \infty}} \sqrt[|k|]{\|c_k(z)\|_K r_1^{k_1} \cdot r_2^{k_2} \cdot ... \cdot r_n^{k_n}} = 1\,,$$

then this equality takes place for any nonpluripolar compact subset E.

Proof. For the compact K we apply Bernstein-Walsh inequality

$$\frac{1}{t} \ln |c_k(z)| \leq \frac{1}{t} \ln \|c_k\|_K + V(z, K),$$

and write it in a form

$$|c_k(z)| \leq e^{tV(z,K)} \cdot \|c_k\|_K. \tag{7}$$

We take supnorm on (7) by nonpluripolar compact set E and get

$$\|c_k(z)\|_E \leq \left\|e^{tV(z,K)}\right\|_E \cdot \|c_k\|_K \quad . \tag{8}$$

Now we apply Bernstein-Walsh inequality for compact E,

$$|c_k(z)| \leq e^{tV(z,E)} \cdot \|c_k\|_E \quad . \tag{9}$$

If we take supnorm by compact set K on inequality (9) then we get

$$\|c_k(z)\|_K \leq \left\|e^{tV(z,E)}\right\|_K \cdot \|c_k\|_E \quad . \tag{10}$$

Putting together (8) and (10) we get the estimation

$$\frac{\|c_k(z)\|_K}{\left\|e^{tV(z,E)}\right\|_K} \leq \|c_k\|_E \leq \left\|e^{tV(z,K)}\right\|_E \cdot \|c_k\|_K \quad . \tag{11}$$

Multiplying (11) by $r_1^{k_1} \cdot r_2^{k_2} \cdot ... \cdot r_n^{k_n}$ and by the help of the fact that Green function is bounded from above in compact sets we obtain the relation

$$1 = \overline{\lim_{|k|\to\infty}} \sqrt[|k|]{\frac{\|c_k(z)\|_K}{\left\|e^{tV(z,E)}\right\|_K} r_1^{k_1} r_2^{k_2} \cdot ... \cdot r_n^{k_n}} \leq \overline{\lim_{|k|\to\infty}} \sqrt[|k|]{\|c_k\|_E r_1^{k_1} r_2^{k_2} \cdot ... \cdot r_n^{k_n}} \leq$$

$$\leq \overline{\lim_{|k|\to\infty}} \sqrt[|k|]{\left\|e^{tV(z,K)}\right\|_E \cdot \|c_k\|_K r_1^{k_1} r_2^{k_2} \cdot ... \cdot r_n^{k_n}} = 1\,.$$

Therefore we get $\overline{\lim\limits_{|k|\to\infty}} \sqrt[|k|]{\|c_k\|_E r_1^{k_1} r_2^{k_2} \cdot ... \cdot r_n^{k_n}} = 1$. □

Denote the set of convergence of power series (1) by S and its open kernel by S^0 i.e. S^0 is domain of convergence. Let $\Pi(r_1, ..., r_n)$ be maximally polyhedron contained in S^0 then the radius-vector of it, $r = (r_1, ..., r_n)$, is called dual radii of S^0.
If r is radius-vector of dual radii then for each $R = (R_1, ..., R_n)\, R_j > r_j$ $(j = 1, 2, ..., n)$ it follows that $\Pi(R_1, ..., R_n) \not\subset S^0$.
If we put

$$\chi\left(r_1, r_2, ..., r_n\right) = \overline{\lim_{|k|\to\infty}} \sqrt[|k|]{\|c_k(z)\|_K r_1^{k_1} \cdot r_2^{k_2} \cdot ... \cdot r_n^{k_n}}\,,$$

then

$$S^0 = \bigcup_{\chi(r)=1} \Pi^r = \bigcup_{\chi(r)=1} \{(z_1, ..., z_n) \in \mathbb{C}^n : |\varphi_1(z_1)| < r_1, ..., |\varphi_n(z_n)| < r_n\}\,.$$

References

[**1**] Shabat B.V., *Indroduction to complex analysis*, Part2. Moscow, Nauka, 1985.
[**2**] Sadullaev A.S., *Plurisubharmonic functions*, Problem of Mathematics, VINITI- Springer, 8 (1985), 65–113.
[**3**] Sadullaev A.S., Chirka E.M., *On continuation of functions with polar singularities*, Math. Sb., 132(174) (1987).
[**4**] Imomkulov S.A, Khujamov J.U., *On holomorphic continuation of functions along boundary sections*, Mathematica Bohemica (Czech Republic), 130(3) (2005).

Alimardon A. ATAMURATOV, Urgench State University named after Al-Khorezmi, 220100, Urgench city, Khamid Alimdjan street 14., Uzbekistan, *e-mail*: `alimardon01@mail.ru`

X.Q. KAMOLOV, Urgench State University named after Al-Khorezmi, 220100, Urgench city, Khamid Alimdjan street 14., Uzbekistan, *e-mail*: `xkamolov@mail.ru`