WEAKLY REGULAR SETS

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Abstract. We consider the concept of weakly regular set in complex plane and in complex half-plane. It is proved that such sets are interpolation sets.

1. Introduction

Concept of regular set in complex plane was introduced by B. Ya. Levin [**1**]. Let $\rho(r)$ be a proximate order, $\lim_{r\to\infty}\rho(r)=\rho>\ 0$. Let $A=\{a_n, n=1,2,\dots\}$ be a set in $\mathbb{C}$. We assume that the points cannot come arbitrarily close to each other. More precisely, we assume that one of the following conditions (C) or (C') holds:

(C) *The points a_n lie inside of finite numbers of angles with a common vertex at the origin but with no other points in common, which are such that if one arranges the points of the set A within any one of these angles in the order of increasing moduli, then for all points which lie inside same angle it is true that*

$$|a_{n+1}|-|a_n|\geq r_n=d|a_n|^{1-\rho(|a_n|)}$$

for some $d>0$.

(C') *There exists a number $d>0$ such that the disks of radii*

$$r_n=d|a_n|^{1-\frac{\rho(|a_n|)}{2}}$$

with centers at the points a_n do not intersect.

A regular point set A satisfying one of the conditions (C) or (C') is called *a regular in sense of Levin*, or more briefly an *R-set in sense of Levin*, while the disks $|z-a_n|\leq r_n$ are called *the exceptional disks of the R-set* (C_R*-disks*).

The sets which satisfy the condition (C) play important role in the theory of entire functions, in particular, for constructing canonical products of the sets [**2**]-[**5**]. In this paper we generalize this concept by introducing the notions of regular sets and weakly regular sets in complex plane and half-plane. We will show that these sets are interpolation sets.

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In [**6, 7**] the regular sets $A = \{a_n, n = 1, 2, \dots\}$ in the upper half-plane $\mathbb{C}_+ = \{z : \Im z > 0\}$ with property "*For all points it is true that*

$$|a_{n+1}| - |a_n| \geq r_n = d \sin \arg(a_n) |a_n|^{1-\rho(|a_n|)}$$

for some $d > 0$*,* " were considered. Such sets were used also for constructing canonical products of the sets in the upper half-plane $\mathbb{C}_+$.

2. Weakly regular sets in complex plane

In this paper, following Titchmarsh, we will use the following definitions and notations. If there is a value which is not depending on the basic variables it is called as a constant. For denoting absolute positive constants, not necessarily same, we use letters A, M, K. Statements like "$|v(z)| < M\gamma(r)$ hence $3|v(z)| < M\gamma(r)$" should not cause any misunderstanding. Denote the class of entire functions f of order $\rho > 0$ by $[\rho, \infty]$ i.e.

$$\limsup_{r \to \infty} \frac{\log^+ \log^+ |f(re^{i\theta})|}{\log r} \leq \rho \, .$$

Definition 2.1. A sequence $A = \{a_n, n = 1, 2, \dots\}$ is called an interpolation sequence in the class $[\rho, \infty]$ if for any numerical sequence $\{b_n, n = 1, 2, \dots\}$ satisfying the condition

$$\limsup_{n \to \infty} \frac{\log^+ \log^+ |b_n|}{\log |a_n|} \leq \rho, \tag{1}$$

there exists a function $f \in [\rho, \infty]$ solving the interpolation problem

$$f(a_n) = b_n, \quad n = 1, 2, \dots . \tag{2}$$

Let $\rho(r)$ be a proximate order, $\lim_{r\to\infty} \rho(r) = \rho > 0$. Denote the class of entire functions f of at most normal type for $\rho(r)$ by $[\rho(r), \infty)$ i.e.

$$\log^+ |f(re^{i\theta})| \leq C_f V(r) \, ,$$

where $V(r) = r^{\rho(r)}$, and $C_f > 0$ is a finite constant.

Definition 2.2. A sequence $A = \{a_n, n = 1, 2, \dots\}$ is called an interpolation sequence in the class $[\rho(r), \infty)$ if for any numerical sequence $\{b_n, n = 1, 2, \dots\}$ satisfying the condition

$$\limsup_{n \to \infty} \frac{\log^+ |b_n|}{V(|a_n|)} < \infty, \tag{3}$$

there exists a function $f \in [\rho(r), \infty)$ solving the interpolation problem (2).

Let $C(a, r)$ be an open disc of radius r about a point a. From the set A we define the measure $n(G) = \sum_{a_n \in G} 1$ and the family of functions

$$\Phi_z(\alpha) = \frac{\max\{n(C(z, \alpha|z|)) - 1; 0\}}{V(|z|)} \, .$$

The following theorems were obtained in [**8**].

Theorem 2.3. *A sequence* $A = \{a_n, n = 1, 2, \dots\}$ *is an interpolation sequence in the class* $[\rho, \infty]$ *if and only if there exists a proximate order* $\rho(r)$, $\lim\limits_{r\to\infty} \rho(r) \le \rho$, *such that*

$$\Phi_z(\alpha) \le (\ln 1/\alpha)^{-1} . \tag{4}$$

Theorem 2.4. *A sequence* $A = \{a_n, n = 1, 2, \dots\}$ *is an interpolation sequence in the class* $[\rho(r), \infty)$ *if and only if*

$$\sup_{z\in\mathbb{C}} \int_0^{1/2} \frac{\Phi_z(\alpha)}{\alpha}\, d\alpha < \infty . \tag{5}$$

Now we will introduce the following definitions.

Definition 2.5. A sequence $A = \{a_n, n = 1, 2, \dots\}$ is called a weakly regular sequence of an order $\rho > 0$ or more briefly an $WR(\rho)$-set, if there exists a proximate order $\rho(r)$, $\lim_{r\to\infty} \rho(r) = \rho$ such that one of the conditions (C) or (C') is true and

$$n(C(0, r)) \le KV(r), \quad K > 0 . \tag{6}$$

Definition 2.6. A sequence $A = \{a_n, n = 1, 2, \dots\}$ is called a weakly regular sequence of a proximate order $\rho(r)$ or more briefly a $WR(\rho(r))$-set, if it satisfies (6) and one of the conditions (C) or (C′) holds.

Let us give the following two theorems.

Theorem 2.7. *Let a sequence* $A = \{a_n, n = 1, 2, \dots\}$ *be a weakly regular sequence of a proximate order* $\rho(r)$. *Then* A *is an interpolation sequence in the class* $[\rho(r), \infty)$.

Theorem 2.8. *Let a sequence* $A = \{a_n, n = 1, 2, \dots\}$ *be a weakly regular sequence of an order* ρ. *Then* A *is an interpolation sequence in the class* $[\rho, \infty]$.

We will use the following lemma from [**1**].

Lemma 2.9. *Let* ρ *be a proximate order,* $\lim_{r\to\infty} \rho(r) = \rho > 0$. *Then asymptotic inequality*

$$(1 - \varepsilon)k^\rho V(r) < V(kr) < (1 + \varepsilon)k^\rho V(r)$$

holds uniformly with respect to k, $0 < a \le k \le b$, *as* $r \to \infty$.

We now obtain some consequences of conditions (C) and (C').

Lemma 2.10. *If a sequence* $A = \{a_n, n = 1, 2, \dots\}$ *satisfies the condition* (C) *then there exists a number* $d_1 > 0$ *such that the disks of radii* $d_1|a_n|/V(|a_n|$ *with centers* a_n *do not intersect.*

We will prove the analogous lemma below (see Lemma 3.9). First we give a definition.

Definition 2.11. Let a sequence $A = \{a_n, n = 1, 2, \dots\}$ be a weakly regular sequence of a proximate order $\rho(r)$ (of an order ρ). Then exceptional disks are called $C_R(\rho(r))$-disks ($C_R(\rho)$-disks).

Lemma 2.12. *Let a sequence $A = \{a_n, n = 1, 2, \dots\}$ be a weakly regular sequence of a proximate order $\rho(r)$. If the condition (C) holds then*

$$\Phi_z(\alpha) \le K\alpha\,, \tag{7}$$

and if the condition (C') is true then

$$\Phi_z(\alpha) \le K\alpha^2\,, \tag{8}$$

fore some $K > 0$.

Proof. Let us assume that the condition (C) holds and the point z does not belong to any of $C_R([\rho(r), \infty))$-disks of an exceptional set. Let us take the disk $C(z, \alpha|z|)$ with center at the point z of the radius $\alpha|z|$. If the points $a_n = r_n e^{i\theta_n} \in C(z, \alpha|z|)$ denoted by $[\alpha_n, \beta_n]$ circular projection of a segment $[a_n, a_n + e^{i\theta_n} d|a_n|^{1-\rho(|a_n|)}]$ is on the ray $\arg\xi = \arg z$. Since the point z does not belong to the exceptional disk corresponding to the point a_n, $[\alpha_n, \beta_n]$ belong to the disk $C(z, 2\alpha|z|)$. The condition (C) implies that all such segments do not intersect and therefore

$$\sum_{a_n \in C(z,\alpha|z|)} d|a_n|^{1-\rho(|a_n|)} \le 4\alpha|z|\,.$$

From this inequality and lemma 2.9 we get

$$n(C(z, \alpha|z|)) \le M\alpha V(|z|), \tag{9}$$

for $\alpha \le 1/2$. The inequality (9) holds for all points z which do not belong to exceptional disks. If the point z belongs to an exceptional disk then right part of the inequality (9) can increase no more than by unit. Therefore, for all $z \in \mathbb{C}$, $\Phi(z\alpha) \le M\alpha$, $\alpha \le 1/2$.
Estimation of (8) under the condition (C') can be obtained by comparing the areas of the disks [**1**]. □

The proof of Theorem 2.7 follows from Lemma 2.12 and (5).
The proof of Theorem 2.8 follows from Theorem 2.7.

3. Weakly regular sets in half-plane

Let $\mathbb{C}_+ = \{z : \Im z > 0\}$ be the upper half-plane. Denote the class of analytic functions f of order $\rho > 0$ in $\mathbb{C}_+$ by $[\rho, \infty]^+$ [**9**].

Definition 3.1. A sequence $A = \{a_n, n = 1, 2, \dots\}$ is called an interpolation sequence in the class $[\rho, \infty]^+$ if for any numerical sequence $\{b_n, n = 1, 2, \dots\}$ satisfying the condition (1) there exists a function $f \in [\rho, \infty]^+$ solving the interpolation problem (2).

Let $\rho(r)$ be a proximate order, $\lim_{r\to\infty} \rho(r) = \rho > 0$. Denote the class of analytic functions f of half-formal order $\rho(r)$ in sense of Grishin [**10**] by $[\rho(r), \infty)^+$.

Definition 3.2. A sequence $A = \{a_n, n = 1, 2, \dots\}$ is called an interpolation sequence in the class $[\rho(r), \infty)^+$ if for any numerical sequence $\{b_n, n = 1, 2, \dots\}$ satisfying the condition condition (3) there exists a function $f \in [\rho(r), \infty)^+$ solving the interpolation problem (2).

From the set A we define the measure $n^+(G) = \sum_{a_n \in G} \sin(\arg a_n)$ and the family of functions

$$\Phi_z^+(\alpha) = \frac{\max\{n^+(C(z, \alpha|z|)) - \sin\arg a_n; 0\}}{V(|z|)},$$

where a_n is the point closest to z (if there are several such points, then we choose the one with the largest $\sin\arg a_n$).

The following theorem was obtained in [**8**].

Theorem 3.3. *A sequence $A = \{a_n, n = 1, 2, \dots\}$ is an interpolation sequence in the class $[\rho, \infty]^+$ if and only if there exists a proximate order $\rho(r)$, $\lim_{r\to\infty} \rho(r) \le \rho$ such that*

$$\Phi_z^+(\alpha) \le 2\alpha, \quad \alpha \ge (\sin(\arg z))/2\,, \tag{10}$$

$$\Phi_z^+(\alpha) \le \frac{\sin(\arg z)}{\ln(e\sin(\arg z))/(2\alpha))}, \quad \alpha < (\sin(\arg z))/2\,. \tag{11}$$

Necessary and sufficient criteria of solvability of interpolation problem in the class $[\rho(r), \infty)^+$ were obtained in [**11**].

Theorem 3.4. *A sequence $A = \{a_n, n = 1, 2, \dots\}$ is an interpolation sequence in the class $[\rho(r), \infty)^+$ if and only if*

$$\sup_{z\in\mathbb{C}_+} \sin(\arg z) \int_0^{1/2} \frac{\Phi_z^+(\alpha)\, d\alpha}{\alpha(\alpha + \sin(\arg z))^2} < \infty\,. \tag{12}$$

We will introduce the following definitions.

Definition 3.5. A sequence $A = \{a_n, n = 1, 2, \dots\}$, $A \in \mathbb{C}_+$, is called a weakly regular sequence in $\mathbb{C}_+$ at a proximate order $\rho(r)$, or more briefly an $WR^+(\rho(r))$-set, if one of the conditions (C_+) or ($\mathrm{C}_+{}'$) holds:

(C_+) 1) Among points of a set of A there are no multiple points and there are no points with identical modulus;
2) $A \cap C(0, 2) = \emptyset$;
3) the condition

$$n^+(C(0, r)) \le KV(r), \quad K > 0$$

holds;
4) there exists a number $d > 0$ such that for all points a_n and a_k of A satisfying the inequality $|a_n| \ge |a_k|$ we have

$$|a_n| \ge |a_k| + d\Im a_k / V(|a_k|). \tag{13}$$

(C'_+) 1) Among points of a set of A there are no multiple points and there are no points with identical modulus;
2) $A \cap C(0, 2) = \emptyset$;

3) the condition

$$n^+(C(0,r)) \leq KV(r), \quad K > 0\,,$$

holds;
4) there exists a number $d > 0$ such that the disks of radii

$$r_n = d(\sin(\arg a_n))^{1/2}|a_n|^{1-\frac{\rho(|a_n|)}{2}}$$

with centers at the points a_n do not intersect.

Definition 3.6. A sequence $A = \{a_n, n = 1, 2, \dots\}$, $A \in \mathbb{C}_+$, is called a weakly regular sequence in $\mathbb{C}_+$ of an order ρ, or more briefly a $WR^+(\rho)$-set, if there exists a proximate order $\rho(r)$, $\lim_{r\to\infty} \rho(r) = \rho$, such that one of the conditions (C_+) or ($\mathrm{C}_+{}'$) holds.

Let us give the following two theorems.

Theorem 3.7. *Let a sequence $A = \{a_n, n = 1, 2, \dots\}$ be a weakly regular sequence in $\mathbb{C}_+$ of a proximate order $\rho(r)$. Then A is an interpolation sequence in the class $[\rho(r), \infty)^+$.*

Theorem 3.8. *A sequence $A = \{a_n, n = 1, 2, \dots\}$, $A \in \mathbb{C}_+$, is called a weakly regular sequence in $\mathbb{C}_+$ of an order ρ. Then A is an interpolation sequence in the class $[\rho, \infty]^+$.*

We now obtain some consequences of conditions (C_+) and ($\mathrm{C}_+{}'$).

Lemma 3.9. *If a sequence $A = \{a_n, n = 1, 2, \dots\}$, $A \in \mathbb{C}_+$, satisfies condition* (C_+) *then there exists a number $d_1 > 0$ such that the disks of radii $d_1\Im a_n/V(|a_n|)$ with centers at the points a_n do not intersect.*

Proof. There exists a number $M_1 > 1$ such that for all r_1 and r_2, $1 \leq r_1 \leq r_2 \leq 2r_1(1 + d)$, we have

$$r_2/V(r_2) \leq M_1 r_1/V(r_1)\,. \tag{14}$$

Denote $a_n = r_n e^{i\theta_n}$, $n \in \mathbb{N}$. Let $r_j > r_i$ and $\sin\theta_j \leq 4\sin\theta_i$. If $r_j \geq 2(r_i + d\Im a_i/V(r_i))$ then the disks $C(a_i, d\Im a_i/V(r_i))$ and $C(a_j, r_j/2)$ do not intersect. Because in this case for all points $z \in C(a_j, r_j/2)$ the inequality is carried out

$$|z - a_i| \geq |a_j - a_i| - |z - a_j| \geq r_j - r_i - r_j/2 = r_j/2 - r_i \geq d\Im a_i/V(r_i).$$

Thus $z \notin C(a_i, d\Im a_i/V(r_i))$. Since $V(r_j) \geq 1$ then the disks $C(a_i, d\Im a_i/V(r_i))$ and $C(a_j, d\Im a_j/(2V(r_j)))$ do not intersect.
If $r_j < 2(r_i + d\Im a_i/V(r_i))$ and $\sin\theta_j \leq 4\sin\theta_i$ we have $r_j < 2r_i(1 + d)$. From (14), $d\Im a_j/(8M_1V(r_j)) \leq d\sin\theta_j\, r_i/(8V(r_i)) \leq d\Im a_i/(2V(r_i))$. From (13), we obtain that the disks $C(a_j, d\Im a_j/(8M_1V(r_j))$ and $C(a_i, d\Im a_i/(2V(r_i))$ do not intersect.
Let $\sin\theta_j > 4\sin\theta_i$ and $r_j < 2(r_i + d\Im a_i/V(r_i))$. Then the disks
$C(a_j, d_1\Im a_j/V(r_j))$ and $C(a_i, d_1\Im a_i/V(r_i))$, where $d_1 = d/2(9 + d)$, do not intersect.
Really, let us find d_1 such that

$$r_j - d_1\frac{\Im a_j}{V(r_j)} > r_i + d_1\frac{\Im a_i}{V(r_i)}.$$

We have

$$r_j - r_i \geq d\frac{r_i \Im a_i}{V(r_i)}.$$

Then

$$4d_1\frac{r_j}{V(r_j)} + d_1\frac{r_i}{V(r_i)} < d\frac{r_i}{V(r_i)},$$

$$d_1\left(8\left(r_i + \frac{d\Im a_i}{V(r_i)}\right) + r_i\right) < dr_i.$$

From this we get $d_1 < d/(9+d)$.
It is necessary to take

$$d_1 = \min\left\{\frac{d}{2(9+d)}; \frac{d}{8M_1}\right\}.$$

□

Definition 3.10. Let a sequence $A = \{a_n, n = 1, 2, \dots\}$, $A \in \mathbb{C}_+$, be a weakly regular sequence in $\mathbb{C}_+$ of a proximate order $\rho(r)$ (of an order ρ). Then exceptional disks are called $C_R^+(\rho(r))$-disks ($C_R^+(\rho)$-disks).

Lemma 3.11. *Let a sequence $A = \{a_n, n = 1, 2, \dots\}$ be a weakly regular sequence in $\mathbb{C}_+$ of a proximate order $\rho(r)$. If the condition* (C_+) *holds then*

$$\Phi_z(\alpha) \leq K\alpha \tag{15}$$

and if the condition (C'_+) *is true then*

$$\Phi_z(\alpha) \leq K\alpha^2\,, \tag{16}$$

for some $K > 0$.

Proof. Let us assume that the condition (C'_+) holds and take a point z which does not belong to any of $C_R^+([\rho(r), \infty))$-disks of an exceptional set. Let us take the disk $C(z, \alpha|z|)$ with center at the point z of the radius $\alpha|z|$. Since the center of this disk does not belong to any of $C_R^+([\rho(r), \infty))$-disks of the exceptional set then radii of the exceptional disks with centers in this disk are less than $\alpha|z|$. Since the exceptional disks do not intersect then sum of their areas is less than the area of $C(z, 2\alpha|z|)$, i.e.

$$\sum_{a_n \in C(z, 2\alpha|z|} d^2 \sin(\arg a_n)|a_n|^{2-\rho(|a_n|)} \leq 4\alpha^2|z|^2\,. \tag{17}$$

If the point $a_n = r_n e^{i\theta_n} \in C(z, \alpha|z|)$ then

$$(1-\alpha)|z| \leq |a_n| \leq (1+\alpha)|z|.$$

From this inequality and (17), we obtain (16). □

The proof of Theorem 3.7 follows from Lemma 3.11 and (12).
The proof of Theorem 3.8 follows from Theorem 3.7.

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