SOME FORMULAS FOR THE ACTION OF STEENROD POWERS ON COHOMOLOGY RING OF $K\left(\mathbb{Z}_p^n, 2\right)$

BEKİR TANAY AND TARKAN ÖNER

Abstract. In this study we give some formulas for the action of Steenrod powers on certain monomials and some polynomials having these monomials as a factor in the polynomial algebra $\mathbf{P}(n) = \mathbb{Z}_p[x_1, \ldots, x_n]$, $\deg(x_i) = 2$, $i = 1, \ldots, n$ and p is an odd prime. We also give some new family of hit polynomials.

1. Introduction and Preliminaries

Steenrod square $Sq^k : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+k}(X;\mathbb{Z}_2)$ and Steenrod power $P^k : H^n(X;\mathbb{Z}_p) \longrightarrow H^{n+2k(p-1)}(X;\mathbb{Z}_p)$ operations are cohomology operations. They were introduced by Norman Steenrod [**1, 2**]. These operations are used to solve some problems in algebraic topology [**3, 4**]. Steenrod algebra is generated by these operations and the structure of this algebra was studied by various mathematicians [**5**]-[**10**]. This algebra acts on the cohomology ring $H^*(X;\mathbb{Z}_p)$. These actions are determined by the following propositions.

Proposition 1.1. [**11**] *For* $\alpha, \alpha_1, \alpha_2 \in H^*(X;\mathbb{Z}_2)$,
i) Sq^0 *is the identity morphism,*
ii) $Sq^k(\alpha) = \alpha^2$ *if* $k = \deg(\alpha)$,
iii) $Sq^k(\alpha) = 0$ *if* $k > \deg(\alpha)$,
iv) The Cartan formula

$$Sq^k(\alpha_1 \cup \alpha_2) = \sum_{i+j=k} Sq^i(\alpha_1)\, Sq^j(\alpha_2)$$

holds.

Proposition 1.2. [**11**] *For* $\alpha, \alpha_1, \alpha_2 \in H^*(X;\mathbb{Z}_p)$,
i) P^0 *is the identity morphism,*
ii) $P^k(\alpha) = \alpha^p$ *if* $2k = \deg(\alpha)$,
iii) $P^k(\alpha) = 0$ *if* $2k > \deg(\alpha)$,
iv) The Cartan formula

$$P^k(\alpha_1 \cup \alpha_2) = \sum_{i+j=k} P^i(\alpha_1)\, P^j(\alpha_2)$$

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holds.

For the topological space $X = \prod_{i=1}^{n} \mathbb{R}P^{\infty}$, the cohomology ring $H^*(X;\mathbb{Z}_2)$ is the polynomial algebra $\wp(n) = \mathbb{F}_2[x_1,\ldots,x_n] = \oplus_{d\geq 0}\wp^d(n), deg(x_i) = 1, i = 1,\ldots,n$ [**12**] and the action of Steenrod squares on $\wp(n)$ as follows by the Proposition 1.1.

Proposition 1.3. *For the homogeneous element f in $\wp(n)$ we have*
i) Sq^0 is the identity morphism,
ii) $Sq^k(f) = f^2$ if $k = \deg(f)$,
iii) $Sq^k(f) = 0$ if $k > \deg(f)$,
iv) The Cartan formula

$$Sq^k(fg) = \sum_{i+j=k} Sq^i(f)\,Sq^j(g),$$

where f, g are homogeneous elements in $\wp(n)$ holds.

Similarly, the cohomology ring $H^*(X;\mathbb{Z}_p)$ is the polynomial algebra $\mathbf{P}(n) = \mathbb{F}_p[x_1,\ldots,x_n]$ $= \oplus_{d\geq 0}\mathbf{P}^d(n)$, $deg(x_i) = 2, i = 1,\ldots,n$ where $X = K(\mathbb{Z}_p^n; 2)$ [**13**]. The action of Steenrod powers on $\mathbf{P}(n)$ is given as follows by the Proposition 1.2.

Proposition 1.4. *For the homogeneous element f in $\mathbf{P}(n)$ we have*
i) P^0 is the identity morphism,
ii) $P^k(f) = f^p$ if $2k = \deg(f)$,
iii) $P^k(f) = 0$ if $2k > \deg(f)$,
iv) The Cartan formula

$$P^k(fg) = \sum_{i+j=k} P^i(f)\,P^j(g),$$

where f, g are homogeneous elements in $\mathbf{P}(n)$ holds.

In [**14**], Janfada gave useful formulas for the action of Steenrod squares on the monomials of the polynomial algebra $\wp(n)$ and by using these formulas, he also gave an application on hit polynomials.
Aim of this study is to give similar formulas given in [**14**] for Steenrod powers P^k.
To obtain the action of P^kon powers of a generator of $\mathbf{P}(n)$, we need the followings.

Definition 1.5. [**15**] Summation of all Steenrod powers

$$P = \sum_{k\geq 0} P^k$$

is called total Steenrod power.

Lemma 1.6. [**15**] *If $f \in \mathbf{P}^2(n)$, then we have $P(f) = f + f^p$.*

Total Steenrod power defines an action on $\mathbf{P}(n)$ by the property (*iii*) of Proposition 1.4, since only a finite number of P^k can be nonzero on a given polynomial. By using Cartan formula, it can be shown that $P(fg) = P(f)P(g)$. So $P : \mathbf{P}(n) \longrightarrow \mathbf{P}(n)$ becomes a ring homomorphism. By using this property we have the following lemma.

Lemma 1.7. [**15**] *If* $f \in \mathbf{P}^2(n)$, *then we have* $P^k(f^r) = \binom{r}{k} f^{(p-1)k+r}$.

In particular, if we take $f = x_i \in \mathbf{P}^2(n)$ in Lemma 1.7, we have the following corollary.

Corollary 1.8. *If* $x \in \mathbf{P}^2(n)$, *then we have*

$$P^k(x_i^r) = \binom{r}{k} x_i^{(p-1)k+r}.$$

Hence we have a formula for the action of P^k on powers of generators. But since Steenrod power operations are not ring homomorphisms, we cannot extend this corollary to any monomial.
The aim of this study is to give a formula for the action of P^k on the monomials $x_1^{m_1 p^t} ... x_n^{m_n p^t}$ where $m_i \geq 0$ and $t \geq 1$ for some special values of k. Moreover if a polynomial

$$g = \left(x_1^{m_1 p^t} \dots x_n^{m_n p^t}\right) f \tag{1}$$

is given, by using Cartan formula we have

$$P^k(g) = \sum_{i+j=k} P^i\left(x_1^{m_1 p^t} \dots x_n^{m_n p^t}\right) P^j(f).$$

After having formulas on $P^i\left(x_1^{m_1 p^t} \dots x_n^{m_n p^t}\right)$ for some special values of k, we only need to know the value of $P^j(f)$ to calculate monomial $P^k(g)$.
This result will be used to obtain new hit polynomials by using certain hit polynomials.
If we take g as a monomial $x_1^{e_1} \dots x_n^{e_n}$, then for $m_i \geq 0$ and certain t_i we have

$$g = x_1^{e_1} \dots x_n^{e_n} = \left(x_1^{m_1 p^t} \dots x_n^{m_n p^t}\right)\left(x_1^{a_1} \dots x_n^{a_n}\right),$$

where $x_1^{a_1} \dots x_n^{a_n}$ corresponds f in the equation (1). We will use this result in applications.
Throughout the paper, we will use the following notations for simplicity:

$$x^a = x_1^{a_1} \dots x_n^{a_n},$$
$$x^{m(p^t)} = x_1^{m_1 p^t} \dots x_n^{m_n p^t}.$$

2. Results

We start with the following results which can be found in [**16**] for the action of Steenrod squares on $\mathbb{Z}_2[x_1, \dots, x_n]$.

Theorem 2.1. *For* $f \in \mathbf{P}(n)$ *and* $k, s \geq 0$,

$$P^k(f^p) = \begin{cases} [P^s(f)]^p & , k = sp, \\ 0 & , \text{otherwise.} \end{cases}$$

Proof. Since P is a ring homomorphism we have $P(f^p) = [P(f)]^p$. From the left hand side, we have

$$\begin{aligned} P(f^p) &= \sum_{k \geq 0} P^k(f^p) \\ &= P^0(f^p) + P^1(f^p) + P^2(f^p) + \cdots, \end{aligned}$$

and from the right hand side, we have

$$[P(f)]^p = \left[\sum_{k\geq 0} P^k(f)\right]^p = \sum_{k\geq 0} \left[P^k(f)\right]^p$$
$$= \left[P^0(f)\right]^p + \left[P^1(f)\right]^p + \left[P^2(f)\right]^p + \cdots .$$

Since the terms having the same exponents must be equal, the claim is true. □

Theorem 2.2. *For $f \in \mathbf{P}(n)$ and $k, t \geq 0$,*

$$P^k\left(f^{p^t}\right) = \begin{cases} [P^s(f)]^{p^t} & , k = sp^t, \\ 0 & , \textit{otherwise.} \end{cases}$$

Proof. We prove by induction on t. For $t = 0$, the result is obvious. For $t = 1$, it is true by Theorem 2.1. Assume that the result is true for smaller values than t. Since we can write

$$P^k\left(f^{p^t}\right) = P^k\left(f^{p^{t-1}p}\right) = P^k\left(\left[f^{p^{t-1}}\right]^p\right)$$

for t, by Theorem 2.1 we have

$$P^k\left(f^{p^t}\right) = \begin{cases} \left[P^{s_1}\left(f^{p^{t-1}}\right)\right]^p & , k = s_1 p \\ 0 & , \text{otherwise} \end{cases}$$

Then we have

$$P^{s_1}\left(f^{p^{t-1}}\right) = \begin{cases} [P^s(f)]^{p^{t-1}} & , s_1 = sp^{t-1} \\ 0 & , \text{otherwise} \end{cases}$$

by the assumption of induction. These prove the theorem. □

Theorem 2.3. *For $f, g \in \mathbf{P}(n)$ and $k, s \geq 0$,*

$$P^k\left(gf^{p^t}\right) = \sum_{i+sp^t=k} P^i(g)\,[P^s(f)]^{p^t}.$$

Proof. This is a consequence of the Cartan formula and Theorem 2.2. □

Theorem 2.4. *Let $n \in \mathbb{Z}^+$. The following relation*

$$P^k\left(x_1^{e_1}\ldots x_n^{e_n}\right) = \sum_{i_1+\cdots+i_n=k} P^{i_1}\left(x_1^{e_1}\right)\ldots P^{i_n}\left(x_n^{e_n}\right)$$

holds.

Proof. We prove by induction on n. For $n = 2$, it is true by Cartan formula. Assume that the following is true for n.

$$P^k\left(x_1^{e_1}\ldots x_n^{e_n}\right) = \sum_{i_1+\cdots+i_n=k} P^{i_1}\left(x_1^{e_1}\right)\ldots P^{i_n}\left(x_n^{e_n}\right).$$

For $n+1$, we can write

$$\begin{aligned} P^k\left(x_1^{e_1}\dots x_n^{e_n}x_{n+1}^{e_{n+1}}\right) &= \sum_{i+i_{n+1}=k} P^i\left(x_1^{e_1}\dots x_n^{e_n}\right)P^{i_{n+1}}\left(x_{n+1}^{e_{n+1}}\right) \\ &= \sum_{i+i_{n+1}=k}\left[\sum_{i_1+\dots+i_n=i} P^{i_1}\left(x_1^{e_1}\right)\dots P^{i_n}\left(x_n^{e_n}\right)\right]P^{i_{n+1}}\left(x_{n+1}^{e_{n+1}}\right) \\ &= \sum_{i_1+\dots+i_{n+1}=k} P^{i_1}\left(x_1^{e_1}\right)\dots P^{i_{n+1}}\left(x_{n+1}^{e_{n+1}}\right). \end{aligned}$$

Hence proof is completed. □

Lemma 2.5. *Let $f\in\mathbf{P}^2(n)$, $t\geq 1$ and $0\leq r\leq p-1$. Then*

$$P^k\left(f^{rp^t}\right)=\begin{cases}\binom{r}{s}f^{(p-1)k+rp^t} & , k=sp^t, 1\leq s\leq r,\\ 0 & , \textit{otherwise.}\end{cases}$$

Proof. By Theorem 2.2 and Lemma 1.7, we have

$$\begin{aligned} P^k\left(f^{rp^t}\right)=P^k\left([f^r]^{p^t}\right) &= \begin{cases}[P^s(f^r)]^{p^t} & , k=sp^t, 1\leq s\leq r\\ 0 & , \text{otherwise}\end{cases} \\ &= \begin{cases}\left[\binom{r}{s}f^{(p-1)s+r}\right]^{pt} & , k=sp^t, 1\leq s\leq r\\ 0 & , \text{otherwise}\end{cases} \\ &= \begin{cases}\binom{r}{s}f^{(p-1)sp^t+rp^t} & , k=sp^t, 1\leq s\leq r\\ 0 & , \text{otherwise}\end{cases} \\ &= \begin{cases}\binom{r}{s}f^{(p-1)k+rp^t} & , k=sp^t, 1\leq s\leq r\\ 0 & , \text{otherwise.}\end{cases} \end{aligned}$$

□

By Lemma 2.5, we have the following corollary.

Corollary 2.6. *Let $x_i\in\mathbf{P}^2(n)$, $t\geq 1$ and $0\leq r\leq p-1$. Then*

$$P^k\left(x_i^{rp^t}\right)=\begin{cases}\binom{r}{s}x_i^{(p-1)k+rp^t} & , k=sp^t, 1\leq s\leq r,\\ 0 & , \textit{otherwise.}\end{cases}$$

Next corollary is an extension of Corollary 2.6 to n variables.

Corollary 2.7. *Let $t\geq 1$, $0\leq r_i\leq p-1$ and $x_1^{r_1p^t}\dots x_n^{r_np^t}\in P(n)$. Then*

$$P^k\left(x^{r(p^t)}\right)=\begin{cases}\sum\limits_{s_1+\dots+s_n=s}\binom{r_1}{s_1}x_1^{(p-1)s_1p^t+r_1p^t}\dots\binom{r_n}{s_n}x_n^{(p-1)s_np^t+r_np^t} & , k=sp^t,\ 1\leq s_j\leq r_j,\\ 0 & , \textit{otherwise.}\end{cases}$$

Proof. By Theorem 2.4 and Corollary 2.6, we have the followings:

$$P^k\left(x^{r(p^t)}\right) = \sum_{i_1+\cdots+i_n=k} P^{i_1}\left(x_1^{r_1p^t}\right)\dots P^{i_n}\left(x_n^{r_np^t}\right)$$
$$= \begin{cases} \sum_{i_1+\cdots+i_n=k} \binom{r_1}{s_1} x_1^{(p-1)s_1p^t+r_1p^t}\dots\binom{r_n}{s_n}x_n^{(p-1)s_np^t+r_np^t} & , i_j=s_jp^t, 1\le s_j\le r_j, \\ 0 & , \text{otherwise.}\end{cases}$$
$$= \begin{cases} \sum_{s_1+\cdots+s_n=s} \binom{r_1}{s_1} x_1^{(p-1)s_1p^t+r_1p^t}\dots\binom{r_n}{s_n}x_n^{(p-1)s_np^t+r_np^t} & , k=sp^t, 1\le s_j\le r_j, \\ 0 & , \text{otherwise.}\end{cases}$$

□

Remark. *If we take $k = p^t(s = 1)$ in Corollary 2.7, we have*

$$P^{p^t}\left(x^{r(p^t)}\right) = \sum_{s_1+\cdots+s_n=1}\binom{r_1}{s_1}x_1^{(p-1)s_1p^t+r_1p^t}\dots\binom{r_n}{s_n}x_n^{(p-1)s_np^t+r_np^t}$$
$$= \binom{r_1}{1}x_1^{(p-1)1p^t+r_1p^t}\binom{r_1}{0}x_2^{(p-1)0p^t+r_2p^t}\dots\binom{r_n}{0}x_n^{(p-1)0p^t+r_np^t}+$$
$$\vdots$$
$$+\binom{r_1}{0}x_1^{(p-1)0p^t+r_1p^t}\dots\binom{r_{n-1}}{0}x_{n-1}^{(p-1)0p^t+r_{n-1}p^t}\binom{r_n}{1}x_n^{(p-1)1p^t+r_np^t}$$
$$= r_1x_1^{(p-1)p^t+r_1p^t}.x_2^{r_2p^t}\dots x_n^{r_np^t}+$$
$$\vdots$$
$$+x_1^{r_1p^t}\dots x_{n-1}^{r_{n-1}p^t}r_nx_n^{(p-1)p^t+r_np^t}$$
$$= x_1^{r_1p^t}\dots x_n^{r_np^t}\left(r_1x_1^{(p-1)p^t}+\cdots+r_nx_n^{(p-1)p^t}\right). \tag{2}$$

Theorem 2.8. *Let $t \ge 1$, $m_i = q_ip$, $q_i \ge 1$, $1 < i < n$. Then*

$$P^k\left(x^{m(p^t)}\right) = \begin{cases} \left[P^s\left(x_1^{q_1}\dots x_n^{q_n}\right)\right]^{p^{t+1}} & , k = sp^{t+1}, 1 \le s \le m_1+\cdots+m_n, \\ 0 & , \textit{otherwise.}\end{cases}$$

Proof. Since $m_i = q_ip$ for all i,

$$x_1^{m_1p^t}\dots x_n^{m_np^t} = x_1^{q_1pp^t}\dots x_n^{q_npp^t} = x_1^{q_1p^{t+1}}\dots x_n^{q_np^{t+1}} = \left(x_1^{q_1}\dots x_n^{q_n}\right)^{p^{t+1}}.$$
$$P^k\left(x^{m(p^t)}\right) = P^k\left(\left(x_1^{q_1}\dots x_n^{q_n}\right)^{p^{t+1}}\right)$$

then by Theorem 2.2 we have

$$P^k\left(x^{m(p^t)}\right) = \begin{cases} \left[P^s\left(x_1^{q_1}\dots x_n^{q_n}\right)\right]^{p^{t+1}} & , k = sp^{t+1}, 1 \le s \le m_1+\cdots+m_n \\ 0 & , \text{otherwise.}\end{cases}$$

The condition $1 \le s \le m_1 + \cdots + m_n$ comes from Proposition 1.4 (*iii*). □

Theorem 2.9. *Let $t \geq 1$, $q_i \geq 1$, $m_i = q_i p + r_i$, $1 \leq r_i \leq p-1$ for $i = 1, \dots, h$, and $m_i = q_i p$ for $i = h+1, \dots, n$. Then*

$$P^k\left(x^{m(p^t)}\right) = \begin{cases} \sum\limits_{s_1 p^{t+1} + s_2 p^t = k} \left[P^{s_1}\left(x_1^{q_1} \dots x_n^{q_n}\right)\right]^{p^{t+1}} \left[P^{s_2}\left(x_1^{r_1} \dots x_h^{r_h}\right)\right]^{p^t} & , 1 \leq s_1 \leq q_1 + \dots + q_n \\ & , 0 \leq s_2 \leq r_1 + \dots + r_h \\ 0 & , \textit{otherwise}. \end{cases}$$

Proof. Since $m_i = q_i p + r_i$, $1 \leq r_i \leq p-1$ for $i = 1, \dots, h$ and $m_i = q_i p$ for $i = h+1, \dots, n$, we have

$$\begin{aligned} x_1^{m_1 p^t} \dots x_n^{m_n p^t} &= x_1^{(q_1 p + r_1)p^t} \dots x_h^{(q_h p + r_h)p^t} x_{h+1}^{q_{h+1} p p^t} x_n^{q_n p p^t} \\ &= x_1^{q_1 p^{t+1}} \dots x_n^{q_n p^{t+1}} x_1^{r_1 p^t} \dots x_h^{r_h p^t} = \left(x_1^{q_1} \dots x_n^{q_n}\right)^{p^{t+1}} \left(x_1^{r_1} \dots x_h^{r_h}\right)^{p^t} \end{aligned}$$

then by Cartan formula we have

$$P^k\left(x^{m(p^t)}\right) = \sum_{i+j=k} P^i\left(\left(x_1^{q_1} \dots x_n^{q_n}\right)^{p^{t+1}}\right) P^j\left(\left(x_1^{r_1} \dots x_h^{r_h}\right)^{p^t}\right).$$

By Theorem 2.2

$$P^k\left(x^{m(p^t)}\right) = \begin{cases} \sum\limits_{s_1 p^{t+1} + s_2 p^t = k} \left[P^{s_1}\left(x_1^{q_1} \dots x_n^{q_n}\right)\right]^{p^{t+1}} \left[P^{s_2}\left(x_1^{r_1} \dots x_h^{r_h}\right)\right]^{p^t} & , 1 \leq s_1 \leq q_1 + \dots + q_n \\ & , 0 \leq s_2 \leq r_1 + \dots + r_h \\ 0 & , \text{otherwise}. \end{cases}$$

The conditions $1 \leq s_1 \leq q_1 + \dots + q_n$ and $0 \leq s_2 \leq r_1 + \dots + r_h$ yield from Proposition 1.4 *(iii)*. □

From Theorem 2.8, we have the following corollary for $k \leq p^{t+1}$.

Corollary 2.10. *Let $t \geq 1$,$m_i = q_i p$,$q_i \geq 1$. Then*
for $k = 0$

$$P^0\left(x^{m(p^t)}\right) = x_1^{m_1 p^t} \dots x_n^{m_n p^t},$$

for $0 < k < p^{t+1}$

$$P^k\left(x^{m(p^t)}\right) = 0,$$

for $k = p^{t+1}$

$$P^{p^{t+1}}\left(x^{m(p^t)}\right) = \left[P^1\left(x_1^{q_1} \dots x_n^{q_n}\right)\right]^{p^{t+1}}.$$

From Theorem 2.9 and equation (2), we have the following corollary for $k \leq p^t$.

Corollary 2.11. *Let $t \geq 1$, $q_i \geq 1$, $m_i = q_i p + r_i$, $1 \leq r_i \leq p-1$ for $i = 1, \dots, h$, and $m_i = q_i p$ for $i = h+1, \dots, n$. Then*
for $k = 0$

$$P^0\left(x^{m(p^t)}\right) = x_1^{m_1 p^t} \dots x_n^{m_n p^t},$$

for $0 < k < p^t$

$$P^k\left(x^{m(p^t)}\right) = 0,$$

for $k = p^t$

$$P^{p^t}\left(x^{m(p^t)}\right) = x_1^{m_1 p^t} \dots x_n^{m_n p^t} \left(r_1 x_1^{(p-1)p^t} + \dots + r_h x_h^{(p-1)p^t}\right).$$

Following theorem is one of the main results mentioned in the introduction.

Theorem 2.12. *Let $f \in \mathbf{P}(n)$. Then for $x^{m(p^t)} f \in \mathbf{P}(n)$, we have the following formulas:*
i)

$$P^0\left(x^{m(p^t)} f\right) = x^{m(p^t)} f,$$

ii) $t \geq 1$,$m_i = q_i p$, $q_i \geq 1$

$$P^k\left(x^{m(p^t)} f\right) = \begin{cases} x^{m(p^t)} P^k(f) & , 0 < k < p^{t+1} \\ x^{m(p^t)} P^{p^{t+1}}(f) + \left[P^1\left(x_1^{q_1} \ldots x_n^{q_n}\right)\right]^{p^{t+1}} f & , k = p^{t+1} \end{cases}$$

iii) $t \geq 1$, $q_i \geq 1$ for $i = 1, \ldots, h$, $m_i = q_i p + r_i$, $1 \leq r_i \leq p-1$ and for $i = h+1, \ldots, n$, $m_i = q_i p$,

$$P^k\left(x^{m(p^t)} f\right) = \begin{cases} x^{m(p^t)} P^k(f) & , 0<k<p^t \\ x^{m(p^t)} P^{p^t}(f) + x^{m(p^t)}\left(r_1 x_1^{(p-1)p^t} + \ldots + r_h x_h^{(p-1)p^t}\right) f & , k=p^t \end{cases}$$

Proof. The equality *i)* is obvious. Let us analyze *ii)* in two cases.
Case 1: Let $0 < k < p^{t+1}$. By Cartan formula, we have the following equation

$$\begin{aligned} P^k\left(x^{m(p^t)} f\right) &= \sum_{i+j=k} P^i\left(x^{m(p^t)}\right) P^j(f) \\ &= P^0\left(x^{m(p^t)}\right) P^k(f) + \sum_{\substack{i+j=k \\ 0<i<k}} P^i\left(x^{m(p^t)}\right) P^j(f) \end{aligned}$$

and then by Corollary 2.10, we have $P^i\left(x^{m(p^t)}\right) = 0$ for $0 < i < p^{t+1}$. Hence we can write

$$P^k\left(x^{m(p^t)} f\right) = x^{m(p^t)} P^k(f).$$

Case 2: Let $k = p^{t+1}$. By Cartan formula, we have the following equation

$$\begin{aligned} P^{p^{t+1}}\left(x^{m(p^t)} f\right) &= \sum_{i+j=p^{t+1}} P^i\left(x^{m(p^t)}\right) P^j(f) \\ &= P^0\left(x^{m(p^t)}\right) P^{p^{t+1}}(f) + \sum_{\substack{i+j=p^{t+1} \\ 0<i<p^{t+1}}} P^i\left(x^{m(p^t)}\right) P^j(f) + \\ &\quad + P^{p^{t+1}}\left(x^{m(p^t)}\right) P^0(f) \end{aligned}$$

and then by Corollary 2.10, we have $P^i\left(x^{m(p^t)}\right) = 0$ for $0 < i < p^{t+1}$. For $i = p^{t+1}$, we have $P^{p^{t+1}}\left(x^{m(p^t)}\right) = \left[P^1\left(x_1^{q_1} \ldots x_n^{q_n}\right)\right]^{p^{t+1}}$. Hence we get

$$P^{p^{t+1}}\left(x^{m(p^t)} f\right) = x^{m(p^t)} P^{p^{t+1}}(f) + \left[P^1\left(x_1^{q_1} \ldots x_n^{q_n}\right)\right]^{p^{t+1}} f.$$

We also analyze *iii)* in two cases.

Case 1: Let $0 < k < p^t$. By Cartan formula, we have the following equation

$$\begin{aligned} P^k\left(x^{m\left(p^t\right)} f\right) &= \sum_{i+j=k} P^i\left(x^{m\left(p^t\right)}\right) P^j\left(f\right) \\ &= P^0\left(x^{m\left(p^t\right)}\right) P^k\left(f\right) + \sum_{\substack{i+j=k \\ 0<i<k}} P^i\left(x^{m\left(p^t\right)}\right) P^j\left(f\right) \end{aligned}$$

and then by Corollary 2.11, we have $P^i\left(x^{m\left(p^t\right)}\right) = 0$ for $0 < i < p^t$. Hence we get

$$P^k\left(x^{m\left(p^t\right)} f\right) = x^{m\left(p^t\right)} P^k\left(f\right).$$

Case 2 : Let $k = p^t$. By Cartan formula, we have the following equation

$$\begin{aligned} P^t\left(x^{m\left(p^t\right)} f\right) &= \sum_{i+j=p^t} P^i\left(x^{m\left(p^t\right)}\right) P^j\left(f\right) \\ &= P^0\left(x^{m\left(p^t\right)}\right) P^{p^t}\left(f\right) + \sum_{\substack{i+j=p^t+ \\ 0<i<p^t}} P^i\left(x^{m\left(p^t\right)}\right) P^j\left(f\right) \\ &\quad + P^{p^t}\left(x^{m\left(p^t\right)}\right) P^0\left(f\right) \end{aligned}$$

and then by Corollary 2.11, for $0 < i < p^t$ we have $P^i\left(x^{m\left(p^t\right)}\right) = 0$. For $i = p^t$, we have $P^{p^t}\left(x^{m\left(p^t\right)}\right) = x_1^{m_1 p^t} \dots x_n^{m_n p^t}\left(r_1 {x_1}^{(p-1)p^t} + \dots + r_h {x_h}^{(p-1)p^t}\right)$. Hence we can write

$$P^{p^t}\left(x^{m\left(p^t\right)} f\right) = x^{m\left(p^t\right)} P^{p^t}\left(f\right) + x^{m\left(p^t\right)}\left(r_1 x_1^{(p-1)p^t} + \ldots + r_h x_h^{(p-1)p^t}\right) f.$$

□

In particular, if we take $f = x^a \in \mathbf{P}\left(n\right)$ in Theorem 2.12, we have the following corollary.

Corollary 2.13. *For the monomial* $x^e = x^{m\left(p^t\right)} x^a \in \mathbf{P}\left(n\right)$, *we have*

i)

$$P^0\left(x^{m\left(p^t\right)} x^a\right) = x^{m\left(p^t\right)} x^a,$$

ii) $t \geq 1$,$m_i = q_i p$, $q_i \geq 1$,

$$P^k\left(x^{m\left(p^t\right)} x^a\right) = \begin{cases} x^{m\left(p^t\right)} P^k\left(x^a\right) & , 0<k<p^{t+1} \\ x^{m\left(p^t\right)} P^{p^{t+1}}\left(x^a\right) + \left[P^1\left(x_1^{q_1} \dots x_n^{q_n}\right)\right]^{p^{t+1}} x^a & , k=p^{t+1} \end{cases}$$

iii) $t \geq 1$, $q_i \geq 1$ *for* $i = 1, \ldots, h$, $m_i = q_i p + r_i$, $1 \leq r_i \leq p-1$ *and for* $i = h+1, \ldots, n$, $m_i = q_i p$,

$$P^k\left(x^{m\left(p^t\right)} x^a\right) = \begin{cases} x^{m\left(p^t\right)} P^k\left(x^a\right) & , 0<k<p^t \\ x^{m\left(p^t\right)} P^{p^t}\left(x^a\right) + x^{m\left(p^t\right)}\left(r_1 x_1^{(p-1)p^t} + \ldots + r_h x_h^{(p-1)p^t}\right) x^a & , k=p^t \end{cases}$$

Example. *Let* $p = 3$ *and* $x^{55} y^{31} z^{81} \in \mathbf{P}\left(3\right)$. *Write this monomial as*

$$\begin{aligned} x^{55} y^{31} z^{81} &= \left(x^{1+2.3^3} y^{4+1.3^3} z^{3.3^3}\right) \\ &= \left(x^{2.3^3} y^{1.3^3} z^{3.3^3}\right)\left(x^1 y^4\right). \end{aligned}$$

Here $t = 3$. Then by Corollary 2.13.iii, for $0 < k < 27 = 3^3$

$$P^k\left(x^{55}y^{31}z^{81}\right) = P^k\left(\left(x^{2.3^3}y^{1.3^3}z^{3.3^3}\right)\left(x^1y^4\right)\right)$$
$$= \left(x^{6.3^2}y^{3.3^2}z^{9.3^2}\right)P^k\left(x^1y^4\right).$$

For $k = 27 = 3^3$, we have

$$\begin{aligned} P^{27}\left(x^{55}y^{31}z^{81}\right) &= P^{27}\left(\left(x^{2.3^3}y^{1.3^3}z^{3.3^3}\right)\left(x^1y^4\right)\right) \\ &= \left(x^{2.3^3}y^{1.3^3}z^{3.3^3}\right)P^{27}\left(x^1y^4\right) + x^{2.3^3}y^{1.3^3}z^{3.3^3}(2x^{2.27} + \\ &+ 1y^{2.27} + 3z^{2.27})\left(x^1y^4\right) \\ &= 2x^{109}y^{31}z^{81} + x^{55}y^{85}z^{81}. \end{aligned}$$

If we write $x^{55}y^{31}z^{81}$ as $x^{55}y^{31}z^{81} = \left(x^{2.3.3^2}y^{1.3.3^2}z^{3.3.3^2}\right)\left(x^1y^4\right)$ then $t = 2$ and by Corollary 2.13.ii, for $0 < k < 27 = 3^{2+1}$

$$P^k\left(x^{55}y^{31}z^{81}\right) = P^k\left(\left(x^{2.3^3}y^{1.3^3}z^{3.3^3}\right)\left(x^1y^4\right)\right)$$
$$= \left(x^{6.3^2}y^{3.3^2}z^{9.3^2}\right)P^k\left(x^1y^4\right).$$

For $k = 27 = 3^{2+1}$

$$\begin{aligned} P^{27}\left(x^{55}y^{31}z^{81}\right) &= \left(x^{6.3^2}y^{3.3^2}z^{9.3^2}\right)P^{27}\left(x^1y^4\right) + \left[P^1\left(x^2y^1z^3\right)\right]^{27}\left(x^1y^4\right) \\ &= (2x^4y^{31}z^3 + x^2y^3z^3 + 3x^2y^1z^3)^{27}\left(x^1y^4\right) \\ &= 2x^{109}y^{31}z^{81} + x^{55}y^{85}z^{81}. \end{aligned}$$

3. Application to Hit Problem

Definition 3.1. [Hit Polynomial] A homogeneous element $f \in \mathbf{P}^d(n)$ is said to be hit if it can be written as

$$f = \sum_{k>0} P^k\left(f_k\right), \tag{3}$$

where $\deg\left(f_k\right) < d$ and this equation (3) is called the hit equation.

The following propositions are consequences of Theorem 2.12.

Proposition 3.2. *Let $t \geq 1$ and $f \in \mathbf{P}^d(n)$. Then f is hit via*

$$f = \sum_{0<k<p^{t+1}} P^k\left(f_k\right),$$

if and only if $g = x^{m\left(p^t\right)}f$ is hit via

$$g = \sum_{0<k<p^{t+1}} P^k\left(x^{m\left(p^t\right)}f_k\right),$$

where $m_i = q_ip$, $q_i \geq 1$.

Proposition 3.3. *Let $t \geq 1$ and $f \in \mathbf{P}^d(n)$. Then f is hit via*

$$f = \sum_{0<k<p^t} P^k(f_k),$$

if and only if $g = x^{m(p^t)} f$ is hit via

$$g = \sum_{0<k<p^t} P^k\left(x^{m(p^t)} f_k\right),$$

where $q_i \geq 1$, $m_i = q_i p + r_i$, $1 \leq r_i \leq p-1$ for $i = 1, \ldots, h$, and $m_i = q_i p$ for $i = h+1, \ldots, n$.

Hence by Proposition 3.2 and 3.3, we can get new hit polynomials from the old ones satisfying the conditions given in propositions.

Example. *Let $p = 3$. Consider the hit polynomial*

$$\begin{aligned} f(x,y) &= x^{21}y^9 + 2x^2y^{28} + 2x^4y^{26} \\ &= P^{10}\left(x^7y^3\right) + P^1\left(x^2y^{26}\right). \end{aligned}$$

The polynomial

$$g(x,y) = x^{1.3.3^2} y^{2.3.3^2}\left(x^{21}y^9 + 2x^2y^{28} + 2x^4y^{26}\right)$$

is hit by Proposition 3.2 since $1 < 27 = 3^{2+1} = 3^{t+1}$ and $10 < 27$ where $t = 2$. The hit equation of g is as follows

$$\begin{aligned} g(x,y) &= x^{27}y^{54}\left(x^{21}y^9 + 2x^2y^{28} + 2x^4y^{26}\right) \\ &= x^{27}y^{54}\left(P^{10}\left(x^7y^3\right) + P^1\left(x^2y^{26}\right)\right) \\ &= P^{10}\left(\left(x^{27}y^{54}\right)\left(x^7y^3\right)\right) + P^1\left(\left(x^{27}y^{54}\right)\left(x^2y^{26}\right)\right) \\ &= P^{10}\left(x^{34}y^{57}\right) + P^1\left(x^{29}y^{80}\right). \end{aligned}$$

Example. *Let $p = 3$. Consider the hit polynomial*

$$\begin{aligned} g(x,y) &= x^{54}y^{66} + x^{51}y^{69} \\ &= x^{5.3^2}y^{7.3^2}\left(x^9y^3 + x^6y^6\right) \\ &= P^4\left(x^{48}y^{64}\right) + P^1\left(x^{51}y^{67}\right). \end{aligned}$$

The polynomial

$$f(x,y) = x^9y^3 + x^6y^6$$

is hit by Proposition 3.3 since $1 < 9 = 3^2 = 3^t$ and $4 < 9$ where $t = 2$. The hit equation of f is as follows

$$f(x,y) = P^4\left(x^3y^1\right) + P^1\left(x^6y^4\right).$$

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Bekir TANAY, Department of Mathematics, Muğla Sıtkı Koçman University, Muğla 48000, Turkey, *e-mail*: `btanay@mu.edu.tr`

Tarkan ÖNER, Department of Mathematics, Muğla Sıtkı Koçman University, Muğla 48000, Turkey, *e-mail*: `tarkanoner@mu.edu.tr`