ON THE EXISTENCE OF RELATIVE FIX POINTS OF A CERTAIN CLASS OF COMPLEX FUNCTIONS

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Abstract. Using the idea of relative iterations of functions we prove a fix point theorem for certain class of complex functions.

1. Introduction

A single valued function $f(z)$ of the complex variable z is said to belong to (i) class I if $f(z)$ is entire transcendental, (ii) class II if it is regular in the complex plane punctured at a, b $(a \neq b)$ and has an essential singularity at b and a singularity at a and if $f(z)$ omits the values a and b except possible at a.
The functions in class II may be normalized by taking $a = 0$ and $b = \infty$.
Throughout this paper we shall consider only such normalized functions, whenever we deal with functions in class II.
Let $f(z)$ be any arbitrary function. Then the iterations are defined inductively by

$$f_0(z) = z \quad \text{and} \quad f_{n+1}(z) = f(f_n(z)), n = 0, 1, 2, \dots .$$

A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and called a fix point of exact order n if α is a solution of $f_n(z) = z$ but not a solution of $f_k(z) = z$, $k = 1, 2, ..., n - 1$.
In this manner, Baker [**2**] proved the following theorem.

Theorem 1.1. *If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n except for atmost one value of n.*

In 1980, Bhattacharyya [**4**] extended Theorem 1.1 to functions in class II as follows:

Theorem 1.2. *If $f(z)$ belongs to class II, then $f(z)$ has an infinity of fix points of exact order n, for every positive integer n.*

In [5] Lahiri and Banerjee introduced a new concept of fix point, called relative fix point (defined below) and using this, proved the result of Bhattacharyya [**4**] .

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Let $f(z)$ and $\phi(z)$ be functions of complex variable z. Let

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(\phi(z)) = f(\phi_1(z))\\
f_3(z) &= f(\phi(f(z))) = f(\phi(f_1(z)))\\
&\vdots\\
f_n(z) &= f(\phi(f(\phi...(f(z) \text{ or } \phi(z) \text{ according as } n \text{ is odd or even } ...)))\\
&= f(\phi_{n-1}(z)) = f(\phi(f_{n-2}(z))),
\end{aligned}$$

and so

$$\begin{aligned}
\phi_1(z) &= \phi(z)\\
\phi_2(z) &= \phi(f(z)) = \phi(f_1(z))\\
\phi_3(z) &= \phi(f_2(z)) = \phi(f(\phi_1(z)))\\
&\vdots\\
\phi_n(z) &= \phi(f_{n-1}(z)) = \phi(f(\phi_{n-2}(z))).
\end{aligned}$$

Clearly all $f_n(z)$ and $\phi_n(z)$ are functions in class II, if $f(z)$ and $\phi(z)$ are so.
A point α is called a fix point of $f(z)$ of order n with respect to $\phi(z)$, if $f_n(\alpha) = \alpha$ and a fixpoint of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha, k = 1, 2, ..., n-1$. Such points α are also called relative fix points.

Theorem 1.3. *If $f(z)$ and $\phi(z)$ belong to class II, then $f(z)$ has an infinity of relative fix points of exact order n for every positive integer n, provided $\dfrac{T(r,\phi_n)}{T(r,f_n)}$ is bounded.*

Recently Banerjee and Jana [**6**] introduced a new concept of fix point, called relative fix point of factor order and using this, extend Theorem 1.1.
A point α is called a relative fix point of $f(z)$ of factor order n if $f_n(\alpha) = \alpha$ but either $f_k(\alpha) \neq \alpha$ or $\phi_k(\alpha) \neq \alpha$ or both, for all divisors k $(k < n)$ of n.

Theorem 1.4. *If $f(z)$ and $\phi(z)$ are transcendental entire functions, then there are relative fix points of factor order n of $f(z)$, except for at most one value of n.*

First we modify the definition of relative fix points of factor order n given by Banerjee and Jana [**6**] and with this modified definition prove the result of Bhattacharyya [**4**].

Definition 1.5. A point α is called a relative fix point of $f(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$ and $\phi_k(\alpha) \neq \alpha$ for all divisors $k(k < n)$ of n.

Example. Let $f(z) = z - 1$ and $\phi(z) = \dfrac{1}{z+1}$. Clearly $f_2(z) = -\dfrac{z}{z+1}$. Here $z = 0, -2$ are relative fix points of exact factor order 2 of $f(z)$.

Remark. *Every relative fix point of exact factor order is also a relative fix point of factor order but converse is not always true.*

Let $f(z)$ be meromorphic in $r_0 \leq |z| < \infty$, $r_0 > 0$. We use the following notations [**1**]:

$n(t, a, f)$ = number of roots of $f(z) = a$ in $r_0 < |z| \leq t$, counted according to multiplicity,

$N(r, a, f) = \displaystyle\int_{r_0}^{r} \frac{n(t, a, f)}{t} dt$,

$n(t, \infty, f) = n(t, f)$ = the number of poles of $f(z)$ in $r_0 < |z| \leq t$, counted due to multiplicity,

$N(t, \infty, f) = N(t, f)$,

$m(r, f) = \displaystyle\frac{1}{\pi} \int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta$,

and, $m(r, a, f) = \displaystyle\frac{1}{\pi} \int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta}) - a}\right| d\theta$.

With these notations, Jensen's formula can be written as [**1**],

$$m(r, f) + N(r, f) = m(r, \frac{1}{f}) + N(r, \frac{1}{f}) + O(\log r).$$

Writing $m(r, f) + N(r, f) = T(r, f)$, the above becomes

$$T(r, f) = T\Big(r, \frac{1}{f}\Big) + O(\log r).$$

In this case the first fundamental theorem takes the form

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(\log r), \tag{1}$$

where $r_0 \leq |z| < \infty$, $r_0 > 0$.
Suppose that $f(z)$ is non-constant. Let $a_1, a_2, ..., a_q$, $q \geq 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\upsilon| \geq \delta$ for $1 \leq \mu \leq \upsilon \leq q$. Then

$$m(r, f) + \sum_{\upsilon=1}^{q} m(r, a_\upsilon, f) \leq 2T(r, f) - N_1(r) + S(r), \tag{2}$$

where

$$N_1(r) = N(r, \frac{1}{f'}) + 2N(r, f) - N(r, f')$$

and

$$S(r) = m(r, \frac{f'}{f}) + \sum_{\upsilon=1}^{q} m(r, \frac{f'}{f - a_\upsilon}) + O(\log r).$$

Adding $N(r,f)+\sum_{\upsilon=1}^{q} N(r,a_\upsilon,f)$ to both sides of (2) and using (1), we obtain

$$(q-1)T(r,f) \leq \overline{N}(r,f)+\sum_{\upsilon=1}^{q}\overline{N}(r,a_\upsilon,f)+S_1(r), \tag{3}$$

where $S_1(r)=O(\log T(r,f))$.
Therefore,

$$\sum_{\upsilon=1}^{q}\overline{N}(r,a_\upsilon,f) \geq (q-1)T(r,f)-\overline{N}(r,f)-S_1(r), \tag{4}$$

where $\overline{N}$ corresponds to distinct roots.
Further, because f_n has an essential singularity at ∞, we have [**1**], $\frac{\log r}{T(r,f_n)} \to 0$ as $r \to \infty$.

2. LEMMAS

The following lemmas will be needed in the sequel.

Lemma 2.1. [**5**] *If n is any positive integer and f and ϕ are functions in class II, then for any $r_0 > 0$ and a positive constant M, we have*

$$\frac{T(r,f_{n+p})}{T(r,f_n)} > M \quad \text{or} \quad \frac{T(r,\phi_{n+p})}{T(r,f_n)} > M$$

according as p is even or odd, for all large r except a set of r intervals of total finite length.

If we interchange simply f and ϕ then we obtain the following lemma.

Lemma 2.2. *If n is any positive integer and f and ϕ are functions in class II, then for any $r_0 > 0$ and M, a positive constant*

$$\frac{T(r,\phi_{n+p})}{T(r,\phi_n)} > M \quad \text{or} \quad \frac{T(r,f_{n+p})}{T(r,\phi_n)} > M$$

according as p is even or odd, for all large r except a set of r intervals of total finite length.

3. MAIN RESULT

The main result of this paper is the following theorem.

Theorem 3.1. *If $f(z)$ and $\phi(z)$ belong to class II, then $f(z)$ has an infinity of relative fix points of exact factor order n for every positive integer n, provided $\frac{T(r,f_{n-1})}{T(r,f_n)}$ is bounded.*

Proof. We may assume that $n > 1$, because if $n = 1$, the theorem has no relevance. We consider the function $g(z) = \dfrac{f_n(z)}{z}, r_0 < |z| < \infty$. Then

$$T(r, g) = T(r, f_n) + O(\log r). \tag{5}$$

Assume that $f(z)$ has only a finite number of relative fix points of exact factor order n. Now from (3) by taking $q = 2$, $a_1 = 0, a_2 = 1$, we obtain,

$$T(r, g) \le \overline{N}(r, \infty, g) + \overline{N}(r, 0, g) + \overline{N}(r, 1, g) + S_1(r, g),$$

where $S_1(r, g) = O(\log T(r, g))$ outside a set of r intervals of finite length [**3**].
First we calculate $\overline{N}(r, 0, g)$. We have

$$\overline{N}(r, 0, g) = \int_{r_0}^{r} \frac{\overline{n}(t, 0, g)}{t} dt,$$

where $\overline{n}(t, 0, g)$ is the number of roots of $g(z) = 0$ in $r_0 < |z| \le t$, each multiple root taken once at a time. The distinct roots of $g(z) = 0$ in $r_0 < |z| \le t$ are the roots of $f_n(z) = 0$ in $r_0 < |z| \le t$. By the definition of functions in class II, $f_n(z)$ has a singularity at $z = 0$ and an essential singularity at $z = \infty$ and $f_n(z) \ne 0, \infty$. So $\overline{n}(t, 0, g) = 0$. Consequently, $\overline{N}(r, 0, g) = 0$. By similar argument $\overline{N}(r, \infty, g) = 0$. So

$$T(r, g) \le \overline{N}(r, 1, g) + S_1(r, g). \tag{6}$$

We now calculate $\overline{N}(r, 1, g)$. If $g(z) = 1$, then $f_n(z) = z$.
Due to our definition two cases arise.

Case (i). When n is even.
Now by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} \overline{N}(r, 1, g) &= \overline{N}(r, 0, f_n - z) \\ &\le \sum_{j/n, j=1}^{n-2} [\overline{N}(r, 0, f_j - z) + \overline{N}(r, 0, \phi_j - z)] + O(\log r) \end{aligned}$$

(The term $O(logr)$ arises due to the assumption that $f(z)$ has only a finite number of relative fix points of exact factor order n.)

$$\begin{aligned}
&\leq \sum_{j/n,j=1}^{n-2}[T(r,f_j-z)+O(\log r)+T(r,\phi_j-z)+O(\log r)]+O(\log r)\\
&= \sum_{j/n,j=1}^{n-2}[T(r,f_j-z)+T(r,\phi_j-z)]+O(\log r)\\
&= \sum_{j/n,j=1}^{n-2}[T(r,f_j)+T(r,\phi_j)]+O(\log r)\\
&= \{T(r,f_{j_1})+T(r,f_{j_3})+...+T(r,f_{j_{2p-1}})+T(r,\phi_{j_2})+T(r,\phi_{j_4})+...+T(r,\phi_{j_{2q}})\}\\
&\quad +\{T(r,f_{j_2})+T(r,f_{j_4})+...+T(r,f_{j_{2q}})+T(r,\phi_{j_1})\\
&\quad +T(r,\phi_{j_3})+...+T(r,\phi_{j_{2p-1}})\}+O(\log r),
\end{aligned}$$

(where $j_1, j_3, ..., j_{2p-1}$ are odd divisors of n and $j_2, j_4, ..., j_{2q}$ are even divisors of n and strictly less than n.)

$$\begin{aligned}
&= T(r,f_n)\Big[\frac{T(r,f_{j_2})}{T(r,f_n)}+\frac{T(r,f_{j_4})}{T(r,f_n)}+....+\frac{T(r,f_{j_{2q}})}{T(r,f_n)}+\frac{T(r,\phi_{j_1})}{T(r,f_n)}+\frac{T(r,\phi_{j_3})}{T(r,f_n)}\\
&\quad +....+\frac{T(r,\phi_{j_{2p-1}})}{T(r,f_n)}\Big]+T(r,f_{n-1})\Big[\frac{T(r,f_{j_1})}{T(r,f_{n-1})}+\frac{T(r,f_{j_3})}{T(r,f_{n-1})}+....+\frac{T(r,f_{j_{2p-1}})}{T(r,f_{n-1})}\\
&\quad +\frac{T(r,\phi_{j_2})}{T(r,f_{n-1})}+\frac{T(r,\phi_{j_4})}{T(r,f_{n-1})}+....+\frac{T(r,\phi_{j_{2q}})}{T(r,f_{n-1})}\Big]+O(\log r)\\
&< \frac{n-1}{4n}T(r,f_n)+\frac{n+1}{4n}T(r,f_{n-1})+O(\log r),
\end{aligned}$$

for all large r.

Case(ii). When n is odd, by Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}
\overline{N}(r,1,g) &= \overline{N}(r,0,f_n-z)\\
&\leq \sum_{j/n,j=1}^{n-2}[\overline{N}(r,0,f_j-z)+\overline{N}(r,0,\phi_j-z)]+O(\log r)\\
&\leq \sum_{j/n,j=1}^{n-2}[T(r,f_j-z)+O(\log r)+T(r,\phi_j-z)+O(\log r)]+O(\log r)\\
&= \sum_{j/n,j=1}^{n-2}[T(r,f_j-z)+T(r,\phi_j-z)]+O(\log r)\\
&= \sum_{j/n,j=1}^{n-2}[T(r,f_j)+T(r,\phi_j)]+O(\log r)\\
&= T(r,f_n)\sum_{j/n,j=1}^{n-2}\frac{T(r,f_j)}{T(r,f_n)}+T(r,f_{n-1})\sum_{j/n,j=1}^{n-2}\frac{T(r,\phi_j)}{T(r,f_{n-1})}+O(\log r)\\
&< \frac{n-1}{4n}T(r,f_n)+\frac{n+1}{4n}T(r,f_{n-1})+O(\log r),
\end{aligned}$$

for all large r. Thus in any case,

$$\overline{N}(r,1,g)<\frac{n-1}{4n}T(r,f_n)+\frac{n+1}{4n}T(r,f_{n-1})+O(\log r).$$

So from (6) and since $\dfrac{T(r,f_{n-1})}{T(r,f_n)}$ is bounded, we have

$$\begin{aligned}
T(r,g) &< \frac{n-1}{4n}T(r,f_n)+\frac{n+1}{4n}T(r,f_{n-1})+O(\log r)+S_1(r,g)\\
&= \frac{n-1}{4n}T(r,f_n)+\frac{n+1}{4n}T(r,f_{n-1})+O(\log r)+O(\log T(r,g))\\
&= T(r,f_n)\Big[\frac{n-1}{4n}+\frac{n+1}{4n}\frac{T(r,f_{n-1})}{T(r,f_n)}+\frac{O(\log r)}{T(r,f_n)}+\frac{O(\log T(r,g))}{T(r,f_n)}\Big]\\
&\leq T(r,f_n)\Big[\frac{n-1}{4n}+\frac{n+1}{4n}+\frac{O(\log r)}{T(r,f_n)}+\frac{O(\log(T(r,f_n)+O(\log r)))}{T(r,f_n)}\Big],\\
&= T(r,f_n)\Big[\frac{1}{2}+\frac{O(\log r)}{T(r,f_n)}+\frac{O(\log(T(r,f_n)(1+\frac{O(\log r)}{T(r,f_n)})))}{T(r,f_n)}\Big]\\
&= \frac{1}{2}T(r,f_n),
\end{aligned}$$

for all large r.

Therefore, $T(r,g)<\dfrac{1}{2}T(r,f_n)$. This contradicts to (5). Hence $f(z)$ has infinitely many relative fix points of exact factor order n. This proves the theorem. □

References

[**1**] Bieberbach L., *Theorie der Gewöhnlichen Differentialgleichungen*, Berlin, 1953.
[**2**] Baker I.N., *The existence of fix points of entire functions*, Math. Zeit., 73 (1960), 280–284.
[**3**] Hayman W.K., *Meromorphic functions*, The Oxford University Press, 1964.
[**4**] Bhattacharyya P., *An extention of a theorem of Baker*, Publicationes Mathematicae Debrecen, 27 (1980), 273–277.
[**5**] Lahiri B.K., Banerjee D., *On the existence of relative fix points*, Istanbul Univ. Fen Fak. Mat. Dergisi, 55-56 (1996-1997), 283–292.
[**6**] Banerjee D., Jana S., *Relative fix points of factor order of transcendental entire functions*, Indian Journal of Mathematics, 52(1) (2010), 209–216.

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