A VERSION OF LAGRANGE'S THEOREM FOR SOME CLASSES OF FUNCTIONS OF MANY VARIABLES

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Abstract. The famous mean motion problem which goes back to Lagrange is as follows: to prove that any exponential polynomial with exponents on the imaginary axis has an average speed for the amplitude, whenever the variable moves along a horizontal line. It was completely proved by B. Jessen and H. Tornehave in 1945. Actually, this result is a consequence of almost periodicity in Weyl's sense of amplitude increments over segments of length 1. Here we consider the problem for some classes of almost periodic functions of several variables.

1. Introduction

Consider an exponential polynomial

$$f(z)=\sum_{j=1}^{N} c_j e^{i\lambda_j z}, \quad c_j\in\mathbb{C}, \quad \lambda_j\in\mathbb{R}. \tag{1}$$

J.L. Lagrange [**13**] assumes that for each fixed $y\in\mathbb{R}$ there exist the limits

$$c^+(y)=\lim_{\beta-\alpha\to\infty}\frac{\Delta_{\alpha<x<\beta}\arg^+ f(x+iy)}{\beta-\alpha}, \tag{2}$$

and

$$c^-(y)=\lim_{\beta-\alpha\to\infty}\frac{\Delta_{\alpha<x<\beta}\arg^- f(x+iy)}{\beta-\alpha}, \tag{3}$$

so-called mean motions along real axis. Here $\arg^+ f(x+iy)$ and $\arg^- f(x+iy)$ are branches of $\arg f(z)$, which are continuous in x on every interval without zeros of f and have the jumps $-p\pi$ and $+p\pi$, respectively, at zeros of multiplicity p, and $\Delta_{\alpha<x<\beta}\arg^{\pm} f(x+iy)$ are increments of the functions $\arg^{\pm} f(x+iy)$ on $(\alpha,\,\beta)$.

J.L. Lagrange proves his conjecture when the absolute value of one of the coefficients in (1) is greater than the sum of absolute values of other coefficients. Moreover, if this is the case, then

$$\arg^+ f(x+iy)=c^+x+O(1), \quad \arg^- f(x+iy)=c^-x+O(1) \quad (x\to\infty), \tag{4}$$

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besides, the mean motions $c^+(y)$ and $c^-(y)$ are equal. Also, J.L. Lagrange shows that the equalities (4) hold for the case $N=2$ with arbitrary terms in (1) too, but c^+ and c^- may be different (for example, in the case $f(z)=\sin z$ at the point $y=0$).

Note that the equalities (4) are false for sums (1) in the case $N>2$ (F. Bernstein [**3**]). On the other hand, H. Bohr in [**2**] proves (4) with $c^+=c^-$ for almost periodic functions f on $\mathbb{R}$ under the condition

$$|f|\ge\kappa>0. \tag{5}$$

Moreover, in this case the terms $O(1)$ in (4) are almost periodic functions as well. Next, B. Jessen [**9**] proves that limits (2) and (3) exist for all $y\in(a,\,b)$ outside of a countable set for holomorphic almost periodic functions in a strip $\{z=x+iy:\ a<y<b\}$. Also, he establishes a connection of mean motions with zero distribution of f.

Lagrange's Conjecture for exponential polynomials is proved by H. Weil [**20**] in the case of linearly independent $\lambda_1,\dots,\lambda_N$ over $\mathbb{Z}$, and by B. Jessen and H. Tornehave [**10**] in the general case (for an easy presentation see [**5**]). Actually, they prove that the functions

$$\Delta_{-1/2<t<1/2}\arg^+ f(x+t+iy),\qquad \Delta_{-1/2<t<1/2}\arg^- f(x+t+iy) \tag{6}$$

are bounded in $x\in\mathbb{R}$ and have mean values

$$c^{\pm}(y)=\lim_{\beta-\alpha\to\infty}\frac{1}{\beta-\alpha}\int_\alpha^\beta \Delta_{-1/2<t<1/2}\arg^{\pm} f(x+t+iy)\,dx.$$

Now (2) and (3) follow immediately from the equality

$$\arg^{\pm} f(\beta+iy)-\arg^{\pm} f(\alpha+iy)=\int_{\beta-1/2}^{\beta+1/2}\arg^{\pm} f(t+iy)\,dt-$$

$$\int_{\alpha-1/2}^{\alpha+1/2}\arg^{\pm} f(t+iy)\,dt+O(1)=\int_\alpha^\beta \Delta_{-1/2<t<1/2}\arg^{\pm} f(t+x+iy)\,dx+O(1).$$

Bohr's result has a multidimensional version. Namely, for any almost periodic function $f(x)$, $x\in\mathbb{R}^p$ with condition (5) we have

$$f(x)=\exp\{i\langle c,x\rangle+g(x)\},$$

where g is almost periodic, c is a vector from $\mathbb{R}^p$. In [**15**], [**16**], [**4**], and [**6**] one can find various relations between mean motions and zero distribution for holomorphic almost periodic functions in tube domains. Nevertheless, we do not know the studies about Lagrange's conjecture for exponential polynomials in several variables, although there are a lot of papers devoted to properties of such polynomials (see, for example, [**7**], [**11**], [**12**], [**14**]).

In the present paper we introduce analogues of functions (6) for arbitrary exponential polynomials in R^p and prove that these functions are bounded and almost periodic in the sense of Weyl. Hence they have averages over $\mathbb{R}^p$ as usual almost periodic functions. As a consequence, we get a multidimensional version of Lagrange's theorem on mean motion.

Notations. For $z=(z_1,\dots,z_p)\in\mathbb{C}^p$, put $'z=(z_2,\dots,z_p)$ and $z=x+iy$, $x,y\in\mathbb{R}^p$, $'z='x+i'y$, $'x,'y\in\mathbb{R}^{p-1}$. By $\langle x,\,y\rangle$ or $\langle z,\,w\rangle$ denote the scalar product for vectors from

$\mathbb{R}^p$ or the Hermitian scalar product for $z, w \in \mathbb{C}^p$, $|a| = \sqrt{\langle a, a\rangle}$. Next, a finite sum of the form

$$P(z) = \sum_{j=1}^{S} c_j e^{i\langle z,\, \lambda^j\rangle}, \quad c_j \in \mathbb{C}, \quad \lambda^j \in \mathbb{R}^p, \quad \lambda^j \neq \lambda^{j'} \quad \text{for} \quad j \neq j', \tag{7}$$

is called a (generalized) trigonometric polynomial in $\mathbb{C}^p$.

Definition 1.1 (for $p = 1$ see [**1**], for $p > 1$ see [**18**]). *A locally integrable function $f(x)$ in $\mathbb{R}^p$ is called almost periodic in the sense of Weyl, if for any $\varepsilon > 0$ there is a trigonometric polynomial (7) such that*

$$\|f - P\|_W = \limsup_{\min_j(\beta_j - \alpha_j) \to \infty} \prod_{1 \le j \le p} (\beta_j - \alpha_j)^{-1} \int_{\alpha_j < x_j < \beta_j,\, j=1,\dots,p} |f(x) - P(x)| dx < \varepsilon.$$

Clearly, here we can replace a trigonometric polynomial by a uniformly almost periodic function. Since any trigonometric polynomial has a mean value, we see that every almost periodic function in the sense of Weyl also has a mean value

$$\lim_{\min_j(\beta_j - \alpha_j) \to \infty} \prod_{1 \le j \le p} (\beta_j - \alpha_j)^{-1} \int_{\alpha_j < x_j < \beta_j,\, j=1,\dots,p} f(x)\, dx. \tag{8}$$

Let $P(z) \not\equiv 0$ be a trigonometric polynomial of the form (7). Put

$$\begin{aligned} E &= \{'z \in \mathbb{C}^{p-1} : P(z_1, 'z) \equiv 0 \quad \forall z_1 \in \mathbb{C}\} \\ &= \{'z \in \mathbb{C}^{p-1} : 0 = P(0, 'z) = P'_{z_1}(0, 'z) = P''_{(z_1)^2}(0, 'z) = \dots\}. \end{aligned}$$

Since E is a closed analytic set, for any $'y \in \mathbb{R}^{p-1}$ the set $E \cap (R^{p-1} + i'y)$ has a zero $(p-1)$-dimensional Lebesgue measure.

Theorem 1.2. *For any fixed $y = (y_1, 'y) \in \mathbb{R}^p$ the functions*

$$\Delta_{-1/2<t<1/2} \arg^+ P(x_1 + t + iy_1, 'x + i'y), \quad \Delta_{-1/2<t<1/2} \arg^- P(x_1 + t + iy_1, 'x + i'y) \tag{9}$$

are uniformly bounded in $x = (x_1, 'x)$ on the set $\mathbb{R}^p \setminus \{(x_1, 'x) : 'x + i'y \in E\}$ and almost periodic in the sense of Weyl in the variable $x \in \mathbb{R}^p$.

Remark. *Note that the functions (9) are defined for any $'z = 'x + i'y \notin E$, hence they are defined for any fixed $y \in \mathbb{R}^p$ almost everywhere in $x \in \mathbb{R}^p$.*

Taking into account the existence of mean values and arguing as above, we obtain the following consequence of Theorem 1.2.

Theorem 1.3. *For any trigonometric polynomial (7) and each $y \in \mathbb{R}^p$ there exist the limits*

$$\lim_{\min_j(\beta_j - \alpha_j) \to \infty} \prod_{1 \le j \le p} (\beta_j - \alpha_j)^{-1} \int_{\Pi^{(p-1)}('\alpha, '\beta)} \Delta_{\alpha_1 < x_1 < \beta_1} \arg^+ P(x + iy)\, d\,'x,$$

and

$$\lim_{\min_j(\beta_j-\alpha_j)\to\infty} \prod_{1\le j\le p} (\beta_j-\alpha_j)^{-1} \int_{\Pi^{(p-1)}('\alpha,'\beta)} \Delta_{\alpha_1<x_1<\beta_1} \arg^- P(x+iy)\, d\,'x.$$

Here $\Pi^{(p-1)}('\alpha,'\beta)=\{'x\in\mathbb{R}^{p-1}:\ \alpha_j<x_j<\beta_j,\ j=2,\dots,p\}$.

The proof of Theorem 1.2 is based on the following lemma.

Lemma 1.4 (for $p=1$ see [**19**])**.** *Suppose* $g(u)$, $u=(u_1,\dots,u_N)$ *is a* 2π*-periodic function in each variable* $u_1,\dots,u_N$, *and* $\mu^1,\dots,\mu^N\in\mathbb{R}^p$ *are linearly independent over* $\mathbb{Z}$. *If* $g(u)$ *is integrable in the sense of Riemann on* $[0,\,2\pi]^N$, *then* $g(\langle\mu^1,x\rangle,\dots,\langle\mu^N,x\rangle)$ *is almost periodic in the sense of Weyl.*

Proof. The proof of Lemma 1.4 is very close to one in the one–dimensional case. For reader's convenience, we give it here.
We may assume that g is a real-valued function. If g is a trigonometric polynomial of the form

$$\sum_{k\in\mathbb{Z}^N} b_k e^{i\langle k,\,u\rangle}, \tag{10}$$

then its average $\mathbf{M}g$ over the cube $[0,\,2\pi]^N$ equals the coefficient b_0. Since $k_1\mu^1+\dots+k_N\mu^N=0$ only for the case $k=(k_1,\dots,k_N)=0$, we see that the mean value (8) of $g(\langle\mu^1,x\rangle,\dots,\langle\mu^N,x\rangle)$ equals to b_0 as well.
Furthermore, an arbitrary continuous 2π-periodic in each variable function g can be uniformly approximated by polynomials (11). Hence in this case $g(\langle\mu^1,x\rangle,\dots,\langle\mu^N,x\rangle)$ is uniformly almost periodic.
Finally, for any Riemann integrable function g and any $\varepsilon>0$ there are continuous 2π-periodic in each variable functions $g_\varepsilon(u)\le g(u)$ and $g^\varepsilon(u)\ge g(u)$ such that

$$\mathbf{M}g^\varepsilon\le\mathbf{M}g+\varepsilon,\quad \mathbf{M}g_\varepsilon\ge\mathbf{M}g-\varepsilon.$$

We get

$$\limsup_{\min_j(\beta_j-\alpha_j)\to\infty}\prod_j(\beta_j-\alpha_j)^{-1}\int_{\Pi^{(p)}(\alpha,\beta)} g(\langle\mu^1,x\rangle,\dots,\langle\mu^N,x\rangle)\,dm_p(x)\le\mathbf{M}g^\varepsilon,$$

$$\liminf_{\min_j(\beta_j-\alpha_j)\to\infty}\prod_j(\beta_j-\alpha_j)^{-1}\int_{\Pi^{(p)}(\alpha,\beta)} g(\langle\mu^1,x\rangle,\dots,\langle\mu^N,x\rangle)\,dm_p(x)\ge\mathbf{M}g_\varepsilon.$$

Therefore, $\|g-g_\varepsilon\|_W<2\varepsilon$. The lemma is proved.

□

We also need the following simple assertion.

Lemma 1.5. *For any real numbers* $\gamma_1,\dots,\gamma_n$, *there is a constant* $C<\infty$ *such that the number of zeros in the segment* $[-1,\,1]$ *of an arbitrary trigonometric polynomial* $g(s)\not\equiv0$ *of* *the form*

$$q(s)=\sum_{k=1}^{n} a_k e^{i\gamma_k s}, \quad a_k\in\mathbb{C}, \tag{11}$$

does not exceed C.

Proof. Collecting similar terms, we may assume that $\gamma_k\neq\gamma_l$ for $k\neq l$. Also, we may suppose $\max_k|a_k|=1$. The family of all trigonometric polynomials (11) under these conditions is a compact set with respect to the uniform convergence on compacta in $\mathbb{C}$. Since the functions $e^{i\gamma'_k s}$ are linearly independent over $\mathbb{C}$, the family does not contain the function $g(s)\equiv 0$. Using Hurwitz' theorem, we obtain an easy proof of the lemma by contradiction. □

Proof of Theorem 1.2.

Let $P(z)$ be a trigonometric polynomial (7) and $\mu^1,\dots,\mu^N$ be a basis of $\mathrm{Lin}_{\mathbb{Z}}\{\lambda^1,\dots,\lambda^S\}$. Therefore,

$$\lambda^j=\sum_{r=1}^{N} k_{r,j}\mu^r, \quad k_{r,j}\in\mathbb{Z}, \quad 1\le j\le S,\, 1\le r\le N.$$

Set

$$F(z,w)=\sum_{j=1}^{S} c_j\exp\{i\langle z,\lambda^j\rangle+i\sum_{r=1}^{N}k_{r,j}w_r\},$$

for $w=(w_1,\dots,w_N)\in\mathbb{C}^N$. The function $F(z,\,u)$, $u\in\mathbb{R}^N$ is 2π-periodic in each variable $u_1,\dots,u_N$ and

$$F(T+iy,\langle\mu^1,x\rangle,\dots,\langle\mu^N,x\rangle)=P(x+iy+T)\quad\forall T\in\mathbb{R}^p. \tag{12}$$

Fix $y=y^{(0)}\in\mathbb{R}^p$. Since $P(z)\not\equiv 0$, we get $F(z_1,i'y^{(0)},w)\not\equiv 0$ in the variables z_1 and w. Therefore the set

$$\begin{aligned} M &= \{w\in\mathbb{C}^N : F(z_1,i'y^{(0)},w)=0 \quad\forall z_1\in\mathbb{C}\}\\ &= \{w\in\mathbb{C}^N : 0=F(0,i'y^{(0)},w)=F'_{z_1}(0,i'y^{(0)},w)=F''_{z_1}(0,i'y^{(0)},w)=\dots\}\end{aligned}$$

is closed and analytic in $\mathbb{C}^N$. Hence N-dimensional Lebesgue measure of the set $M\cap R^N$ is zero. Set

$$I^+(u)=\Delta_{-1/2<x_1<1/2}\arg^+ F(x_1+iy_1^{(0)},i'y^{(0)},u),$$

$$I^-(u)=\Delta_{-1/2<x_1<1/2}\arg^- F(x_1+iy_1^{(0)},i'y^{(0)},u),$$

for $u\in\mathbb{R}^N\setminus M$. Let us check that the functions $I^+(u)$ and $I^-(u)$ are uniformly bounded and continuous almost everywhere in $u\in[0,\,2\pi]^N$.

If $F(x_1+iy_1^{(0)},i'y^{(0)},u^{(0)})\neq 0$ for all $x_1\in[-1/2,\,1/2]$, then the function

$$F'_z(x_1+iy_1^{(0)},i'y^{(0)},u)/F(x_1+iy_1^{(0)},i'y^{(0)},u)$$

is continuous and uniformly bounded in $x \in [-1/2, 1/2]$ and u belonging to a neighborhood of $u^{(0)}$. Hence the functions $I^+(u)$ and $I^-(u)$ coincide, are uniformly bounded and continuous in this neighborhood.

Suppose that $u^{(0)} \notin M \cap \mathbb{R}^N$ and $F(x_1^{(1)} + iy_1^{(0)}, i'y^{(0)}, u^{(0)}) = 0$ at a point $x_1^{(1)} \in [-1/2, 1/2]$. Since $F(z_1, i'y^{(0)}, u^{(0)}) \not\equiv 0$ in the variable z_1, we can use the Weierstrass Preparation Theorem (see, for example, [**8**]). Hence there are $\varepsilon > 0$, $\delta > 0$, and pseudopolynomial

$$P_1(z_1, i'y^{(0)}, w) = (z_1 - x_1^{(1)} - iy_1^{(0)})^r + a_1(w)(z_1 - x_1^{(1)} - iy_1^{(0)})^{r-1} + \cdots + a_r(w) \tag{13}$$

with holomorphic coefficients $a_j(w)$ in the ball $\{w : |w - u^{(0)}| < \varepsilon\}$ such that

$$a_j(u^{(0)}) = 0, \quad j = 1, \dots, r, \tag{14}$$

and

$$F(z_1, i'y^{(0)}, w) = P_1(z_1, i'y^{(0)}, w) F_1(z_1, i'y^{(0)}, w), \quad F_1(z_1, i'y^{(0)}, w) \neq 0,$$

in the set $\{(z_1, w) : |w - u^{(0)}| < \varepsilon, \, |z_1 - x_1^{(1)} - iy_1^{(0)}| < \delta\}$.

For a small ε each solution $\tilde{z}_1$ of the equation $P_1(z_1, i'y^{(0)}, w) = 0$ belongs to the disc $|z_1 - x_1^{(1)} - iy_1^{(0)}| < \delta$. Hence the function $F_1 = F/P_1$ is holomorphic in the set $\{(z_1, w) : z_1 \in \mathbb{C}, \, |w - u^{(0)}| < \varepsilon\}$.

Let $x_1^{(2)}$ be another point of the segment $[-1/2, 1/2]$ such that $F(x_1^{(2)} + iy_1^{(0)}, i'y^{(0)}, u^{(0)}) = 0$. Using the Weierstrass Preparation Theorem for $F_1(z_1, i'y^{(0)}, w)$ in a neighborhood of the point $(x_1^{(2)} + iy_1^{(0)}, u^{(0)})$, we get

$$F_1(z_1, i'y^{(0)}, w) = P_2(z_1, i'y^{(0)}, w) F_2(z_1, i'y^{(0)}, w).$$

Here $P_2(z_1, i'y^{(0)}, w)$ has the form (13) with $x_1^{(2)}$ instead of $x_1^{(1)}$, the function $F_2(z_1, i'y^{(0)}, w)$ is holomorphic in the set $\{(z_1, w) : z_1 \in \mathbb{C}, \, |w - u^{(0)}| < \varepsilon\}$ and has no zeros in the set $\{(z_1, w) : |w - u^{(0)}| < \varepsilon, \, |z_1 - x_1^{(2)} - iy_1^{(0)}| < \delta\}$. Proceeding in the same way, we get the representation

$$F(z_1, i'y^{(0)}, w) = P_1(z_1, i'y^{(0)}, w) \cdot \dots \cdot P_s(z_1, i'y^{(0)}, w) G(z_1, i'y^{(0)}, w), \tag{15}$$

where the pseudopolynomials P_j have form (13) with various points $\tilde{x}_1 \in [-1/2, 1/2]$ instead of $x_1^{(1)}$. Their coefficients satisfy (14) and the holomorphic function $G(z_1, i'y^{(0)}, w)$ does not vanish in a neighborhood of the set $\{(z_1, w) : x_1 \in [-1/2, 1/2], \, y_1 = y_1^{(0)}, \, w = u^{(0)}\}$. Each pseudopolynomial P_j is a product of irreducible pseudopolynomials of form (13) with conditions (14) (see, for example, [**8**]), therefore we may assume that all pseudopolynomials P_j in (15) are irreducible. Also, we can rewrite (15) in the form

$$F(z_1, i'y^{(0)}, w) = (z_1 - b_1(w))^{t_1} \cdots (z_1 - b_k(w))^{t_k} G(z_1, i'y^{(0)}, w), \tag{16}$$

where $b_n(w)$, $n = 1, \dots, k$ are analytic in some neighborhood U of the point $u^{(0)}$.

Since the functions $\{F(s + iy_1^{(0)}, i'y^{(0)}, u)\}_{u \in \mathbb{R}^N}$ satisfy the condition of Lemma 1.5, we see that the number of zeros $t_1 + t_2 + \cdots + t_k$ is bounded from above uniformly in $u \in \mathbb{R}^N \setminus M$. An increment of the amplitude of any linear multiplier along any segment is at most π, hence the functions $I^+(u)$ and $I^-(u)$ are bounded uniformly in $u \in \mathbb{R}^N$.

Note that the discriminant $d_P(w)$ of a pseudopolynomial P of the form (13) is a holomorphic function in U. If P is irreducible, then $d_P(w) \not\equiv 0$ (see, for example, [**8**]). Set

$$M_1 = \{w \in U : F(-1/2, i'y^{(0)}, w)F(1/2, i'y^{(0)}, w)d_{P_1}(w) \cdots d_{P_r}(w) = 0\}.$$

Note that N-dimensional Lebesgue measure of the set $M_1 \cap R^N$ is zero. Take a point $u^{(1)} \in (U \setminus M_1) \cap \mathbb{R}^N$. There is a neighborhood $U_1 \subset \mathbb{C}^N$ of $u^{(1)}$ such that for each point $w \in U_1$ every pseudopolynomial $P_m(z, i'y^{(0)}, w)$ has only simple zeros in $z_1 \in \mathbb{C}$. Hence for $w \in U_1$ representation (16) holds with mutual different functions $b_n(w)$. We shall prove that each function $\Delta_{-1/2<x<1/2} \arg^{\pm}(x + iy - b_n(u))$ is continuous at points of the set $U_1 \cap \mathbb{R}^N$. Note that $F(\pm 1/2 + iy_1^{(0)}, i'y^{(0)}, u^{(1)}) \neq 0$. Hence for sufficiently small U_1 the functions $b_n(w)$ take U_1 to the set $\{z_1 : |x_1| < 1/2, |y_1 - y_1^{(0)}| < \delta\}$. Put

$$b'_n(w) = (b_n(w) + \overline{b_n(\bar{w})})/2, \quad b''_n(w) = (b_n(w) - \overline{b_n(\bar{w})})/2i.$$

Clearly, for all $u \in \mathbb{R}^N$ we have $b'_n(u) = \mathrm{Re} b_n(u)$, $b''_n(u) = \mathrm{Im} b_n(u)$.
If $b''_n(w) \not\equiv y_1^{(0)}$ for $w \in U_1$, then N-dimensional Lebesgue measure of the set $\{w : b''_n(w) = y_1^{(0)}\} \cap \mathbb{R}^N$ is zero. Hence for almost all points $u \in U_1 \cap \mathbb{R}^N$ the function $x_1 + iy_1^{(0)} - b_n(u)$ does not vanish for $x_1 \in [-1/2, 1/2]$, and the functions $\Delta_{-1/2<x_1<1/2} \arg^+(x_1 + iy_1^{(0)} - b_n(u))$ and $\Delta_{-1/2<x_1<1/2} \arg^-(x_1 + iy_1^{(0)} - b_n(u))$ are continuous and coincide almost everywhere on $U_1 \cap \mathbb{R}^N$.
Now consider the case $b''_n(w) \equiv y_1^{(0)}$ for $w \in U_1$. The function $b'_n(u)$ is continuous in u, therefore the function $x_1 + iy_1^{(0)} - b_n(u) = x_1 - b'_n(u)$ of the variable x_1 has exactly one simple zero for all u in a neighborhood U_2 of a point $u^{(2)} \in U_1 \cap \mathbb{R}^N$. Hence, $\Delta_{-1/2<x_1<1/2} \arg^+(x_1 + iy_1 - b_n(u)) \equiv -\pi$ and $\Delta_{-1/2<x_1<1/2} \arg^-(x_1 + iy_1 - b_n(u)) \equiv \pi$ for all $u \in U_2$.
Since $u^{(0)}$ is an arbitrary point of $[0, 2\pi]^N \setminus M$, we see that the functions $I^+(u)$ and $I^-(u)$ are bounded and continuous almost everywhere in the cube $[0, 2\pi]^N$. Therefore, these functions are integrable in the sense of Riemann over the cube. By Lemma 1.4 we obtain the assertion of Theorem 1.2.

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