ON THE DECOMPOSITION OF CURVATURE TENSOR FIELDS IN CONFORMAL FINSLER SPACES II

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Abstract. The object of the present paper is to decompose curvature tensor fields in conformal Finsler spaces. The authors have also studied properties of the conformal decomposition tensor fields.

1. Introduction

The conformal geometry of generalized metric spaces was studied and developed by M. S. Knebelman [**1**]. R. B. Misra [**3**] has obtained Bianchi identities satisfied by curvature tensor fields in a conformal Finsler space.
The decomposition of recurrent curvature tensors in Finsler spaces was studied by B. B. Sinha and S. P. Singh [**6**]. The present authors [**5**], C. K. Mishra and G. Lodhi [**2**] have decomposed curvature tensors in recurrent conformal Finsler spaces.
In the present paper, we have decomposed curvature tensor fields in conformal Finsler space. The properties of the conformal decomposition tensor fields will also be studied.
We consider an n-dimensional Finsler space $\overline{F}_n$ whose fundamental entities $\overline{F}$ and $\overline{g}_{ij}$ satisfy the following relations with respect to the entities of the Finsler space $\overline{F}_n$ [**1**]:

$$(1) \qquad (a)\ \overline{g}_{ij} = e^{2\sigma} g_{ij}, \quad (b)\ \overline{g}^{ij} = e^{-2\sigma} g^{ij}, \quad (c)\ \overline{F}(x,\dot{x}) = e^{\sigma} F(x,\dot{x})$$

where the factor of proportionality between them is at most a point function.
The covariant derivatives of $X^i(x,\dot{x})$ with respect to x^l in the sense of Berwald are given by [**4**]

$$(2) \qquad X^i_{|l} = \partial_l X^i - (\dot{\partial}_m X^i) G^m_l + X^m \Gamma^{*i}_{ml},$$

$$(3) \qquad X^i_{(l)} = \partial_l X^i - (\dot{\partial}_m X^i) G^m_l + X^m G^i_{ml},$$

where the functions $\Gamma^{*i}_{ml}(x,\dot{x})$ and $G^i_{ml}(x,\dot{x})$ are connection coefficients used by Cartan and Berwald, respectively. These coefficients are homogeneous of degree zero in $\dot{x}^i$.
The respective connection coefficients of the conformal Finsler space $\overline{F}_n$ are given by

$$(4) \qquad \overline{G}^i_{jk} = G^i_{jk} - \sigma_m \dot{\partial}_j \dot{\partial}_k B^{im},$$

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$$\overline{\Gamma}^{*i}_{jk} = \Gamma^{*i}_{jk} + U^i_{jk}, \tag{5}$$

where

$$U^i_{jk}(x,\dot{x}) = 2\sigma_{(i}\delta^i_{k)} - \sigma_m[g^{im}g_{jk} - 2C^i_{r(j}\dot{\partial}_{k)}B^{rm} + g^{ir}C_{jks}\dot{\partial}_r B^{sm}]$$

and $\sigma_m(x) = \partial_m\sigma(x)$, $B^{ij}(x,\dot{x}) = \frac{1}{2}F^2g^{ij} - \dot{x}^i\dot{x}^j$.
Under (1), the curvature tensors $\overline{K}^i_{rjk}$ and $\overline{H}^i_{rjk}$ are transformed to

$$\overline{K}^i_{rjk} = K^i_{rjk} + 2U^i_{r[j|k]} + 2[\sigma_m\dot{\partial}_s(\Gamma^{*i}_{r[j} + U^i_{r[j})\dot{\partial}_{k]}B^{sm} + U^i_{s[k}U^s_{j]r}], \tag{6}$$

$$\begin{aligned}\overline{H}^i_{rjk} = H^i_{rjk} &+ 2\sigma_{m[(j)}\dot{\partial}_{k]}\dot{\partial}_r B^{im} - 2\sigma_m[\dot{\partial}_r(\dot{\partial}_{[j}B^{im})_{(k)]}\\ &-(\dot{\partial}_{[j}B^{im})G^m_{k]rs}] + 2\sigma_m\sigma_s\dot{\partial}_r[(\dot{\partial}_{[j}B^{pm})\dot{\partial}_{k]}\dot{\partial}_p B^{is}],\end{aligned} \tag{7}$$

respectively which give the conformal curvature tensors in $\overline{F}_n$.
The above conformal curvature tensors satisfy the following identities:

$$(a)\,\overline{K}^i_{rjk} = -\overline{K}^i_{rkj}, \quad (b)\,\overline{K}^i_{[rjk]} = 0, \tag{8}$$

$$(a)\,\overline{H}^i_{rjk} = -\overline{H}^i_{rkj}, \quad (b)\,\overline{H}^i_{[rjk]} = 0, \tag{9}$$

$$(a)\,\overline{H}^i_{rjk}\dot{x}^r = \overline{H}^i_{jk}, \quad (b)\,\overline{H}^i_{jk} = -\overline{H}^i_{kj}, \tag{10}$$

$$\overline{H}^i_j = \overline{H}^i_{jk}\dot{x}^j = -\overline{H}^i_{kj}\dot{x}^j, \tag{11}$$

similar to those in F_n.
The Bianchi identities satisfied by $\overline{K}^i_{rjk}$, $\overline{H}^i_{rjk}$, and $\overline{H}^i_{jk}$ are

$$\begin{aligned}&\overline{K}^i_{r[jk|\bar{h}]} + (\dot{\partial}_p\Gamma^{*i}_{r[j})H^p_{kh]} + (\dot{\partial}_pU^i_{r[j})H^p_{kh]} + \{\dot{\partial}_p(\Gamma^{*i}_{r[j} + U^i_{r[j})\}[\{(\dot{\partial}_hB^{pm})_{(k)]}\\ &-(\dot{\partial}_kB^{pm})_{(h)]}\}\sigma_m + \{(\dot{\partial}_kB^{sm})\dot{\partial}_{h]}\dot{\partial}_sB^{pt} - (\dot{\partial}_hB^{sm})\dot{\partial}_{k]}\dot{\partial}_sB^{pt}\}\sigma_m\sigma_t\\ &+\sigma_{|m|(k)}\dot{\partial}_{h]}B^{pm} - \sigma_{|m|(k)}\dot{\partial}_{h]}B^{pm}] = 0,\end{aligned} \tag{12}$$

$$\begin{aligned}\overline{H}^i_{[jk(\bar{h})]} \;=\;& \{H^p_{[kj} - \sigma_{q[(j)}\dot{\partial}_kB^{pq} + \sigma_{q[(k)}\dot{\partial}_jB^{pq}\}G^i_{h]pr} + \sigma_m[\{(\dot{\partial}_{[j}B^{pm})_{(k)}\\ &-(\dot{\partial}_{[k}B^{pm})_{(j)}\}G^i_{h]pr} + \{H^p_{[jk} + \sigma_{q[(j)}\dot{\partial}_kB^{pq} - \sigma_{q[(k)}\dot{\partial}_jB^{pq}\}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_rB^{im}]\\ &-\sigma_m\sigma_t[G^i_{pr[j}\{(\dot{\partial}_kB^{st})\dot{\partial}_{h]}\dot{\partial}_sB^{pm} - (\dot{\partial}_hB^{st})\dot{\partial}_{k]}B^{pm}\}\\ &-(\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{it})\{(\dot{\partial}_kB^{pm})_{(h)]} - (\dot{\partial}_hB^{pm})_{(k)]}\}]\\ &+\sigma_m\sigma_t(\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{iq})\{(\dot{\partial}_kB^{st})\dot{\partial}_{h]}\dot{\partial}_sB^{pm} - (\dot{\partial}_hB^{st})\dot{\partial}_{k]}\dot{\partial}_sB^{pm}\},\end{aligned} \tag{13}$$

and

$$\overline{H}^i_{[jk(\bar{h})]} = 0, \tag{14}$$

where the symbols $|\bar{h}$ and $(\bar{h})$ denote the covariant derivatives with respect to $\acute{x}s$ for the connection coefficients $\overline{\Gamma}^{*i}_{jk}$ and $\overline{G}^i_{jk}$, respectively.

2. Decomposition of the curvature tensor field $\overline{H}^i_{rjk}$

Let us assume the decomposition of the curvature tensor $\overline{H}^i_{rjk}$ in the form

$$\overline{H}^i_{rjk} = \dot{x}^i \overline{\beta}_{rjk}, \tag{15}$$

where $\overline{\beta}_{rjk}$ is a non-zero conformal decomposition tensor field which is homogeneous of degree -1 in $\dot{x}^i$.
Transvection of (15) by $\dot{x}^i$ and using (10), we get

$$\overline{H}^i_{jk} = \dot{x}^i \overline{\beta}_{jk}, \tag{16}$$

where

$$\overline{\beta}_{rjk}\dot{x}^r = \overline{\beta}_{jk}. \tag{17}$$

Commuting the indices j, k in (15) and applying (9), we find

$$\overline{\beta}_{rjk} = -\overline{\beta}_{rkj}. \tag{18}$$

In the view of (10) and (16), we have

$$\overline{\beta}_{jk} = -\overline{\beta}_{kj}. \tag{19}$$

Now considering transvection of (16) by $\dot{x}^j$ and noting (15), we obtain

$$\overline{H}^i_k = \dot{x}^i \overline{\beta}_k \tag{20}$$

where

$$\overline{\beta}_k = \overline{\beta}_{jk}\dot{x}^j. \tag{21}$$

Applying the decomposition (15) in the identity (9), it yields to

$$\overline{\beta}_{[rjk]} = 0. \tag{22}$$

Transvection of the identity (22) by $\dot{x}^r$ and using (18), we find

$$\overline{\beta}_{jk} = 2\overline{\beta}_{[kj]r}\dot{x}^j \dot{x}^r. \tag{23}$$

In the view of (21), the equation (23) yields to

$$\overline{\beta}_k = 2\overline{\beta}_{[kj]r}\dot{x}^j \dot{x}^r. \tag{24}$$

Thus we state the following theorem.

Theorem 2.1. *In a conformal Finsler space $\overline{F}_n$, the conformal decomposition tensor fields $\overline{\beta}_{rjk}, \overline{\beta}_{jk}$, and $\overline{\beta}_k$ satisfy the following identities*

(a) $\overline{\beta}_{rjk} = -\overline{\beta}_{rkj}$,

(b) $\overline{\beta}_{jk} = -\overline{\beta}_{kj}$,

(c) $\overline{\beta}_{[rjk]} = 0$,

(d) $\overline{\beta}_{jk} = 2\overline{\beta}_{[kj]r}\dot{x}^r$,

(e) $\overline{\beta}_k = 2\overline{\beta}_{[kj]r}\dot{x}^j\dot{x}^r$.

Applying the decomposition (15) in the Bianchi identity (13), it takes the form

$$\begin{aligned}\dot{x}^i\overline{\beta}_{r[jk(\bar{h})]} \;=\; & \{H^p_{[kj} - \sigma_{q[(j)}\dot{\partial}_k B^{pq} + \sigma_{q[(k)}\dot{\partial}_j B^{pq}\}G^i_{h]pr} + \sigma_m[\{(\dot{\partial}_{[j}B^{pm})_{(k)} \\ & -(\dot{\partial}_{[k}B^{pm})_{(j)}\}G^i_{h]pr} + \{H^p_{[jk} + \sigma_{q[(j)}\dot{\partial}_k B^{pq} - \sigma_{q[(k)}\dot{\partial}_j B^{pq}\}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_r B^{im}] \\ & -\sigma_m\sigma_t[G^i_{pr[j}\{(\dot{\partial}_k B^{st})\dot{\partial}_{h]}\dot{\partial}_s B^{pm} - (\dot{\partial}_h B^{st})\dot{\partial}_{k]}B^{pm}\} \\ & -(\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{it})\{(\dot{\partial}_k B^{pm})_{(h)]} - (\dot{\partial}_h B^{pm})_{(k)]}\}] \\ & +\sigma_m\sigma_t(\dot{\partial}_p\dot{\partial}_r\dot{\partial}_{[j}B^{iq})\{(\dot{\partial}_k B_{st})\dot{\partial}_{h]}\dot{\partial}_s B^{pm} - (\dot{\partial}_h B^{st})\dot{\partial}_{k]}\dot{\partial}_s B^{pm}\}. \end{aligned} \tag{25}$$

Transvection of (25) by $\dot{x}^r$ and noting (17) and the homogeneity property of $G^i_{rjk}(x,\dot{x})$ and $B^{im}(x,\dot{x})$, we get

$$\overline{\beta}_{[jk(\bar{h})]} = 0, \tag{26}$$

which gives the Bianchi identity of the conformal decomposition tensor $\overline{\beta}_{jk}$.
Further transvection of (26) by $\dot{x}^j$ yields to

$$\overline{\beta}_{k(\bar{h})} - \overline{\beta}_{h(\bar{k})} + \overline{\beta}_{kh(\bar{j})}\dot{x}^j = 0 \tag{27}$$

by the help of (19) and (21).
Hence we state the following theorem.

Theorem 2.2. *In a conformal Finsler space $\overline{F}_n$, the conformal decomposition tensor fields $\overline{\beta}_{rjk}, \overline{\beta}_{jk}$ and $\overline{\beta}_k$ satisfy the identities (25), (26), and (27), respectively.*

In a conformal Finsler space $\overline{F}_n$, if the conformal curvature tensor $\overline{H}^i_{rjk}$ satisfies the relation

$$\overline{H}^i_{rjk\overline{(h)}} = \overline{v}_h\overline{H}^i_{rjk}, \tag{28}$$

then $\overline{F}_n$ defines recurrent conformal Finsler space, which we denote by $\overline{F}^*_n$. The non-zero vector $\overline{v}_h$ is called recurrent conformal vector.
Differentiating (15) with respect to x^h in the sense of Berwald and using (28), we get

$$\overline{\beta}_{rjk\overline{(h)}} = \overline{v}_h\overline{\beta}_{rjk}, \tag{29}$$

since $\dot{x}^i$ is covariant constant. Transvection of (29) by $\dot{x}^r$ and $\dot{x}^j$ successively and using (17) and (21), we obtain

$$\overline{\beta}_{jk\overline{(h)}} = \overline{v}_h\overline{\beta}_{jk}, \tag{30}$$

and

$$\overline{\beta}_{k\overline{(h)}} = \overline{v}_h\overline{\beta}_k. \tag{31}$$

So we state the following theorem.

Theorem 2.3. *In a recurrent conformal Finsler space* $\overline{F}^*_n$*, the conformal decomposition tensor fields* $\overline{\beta}_{rjk}$*,* $\overline{\beta}_{jk}$*, and* $\overline{\beta}_k$ *also behave like recurrent tensor fields.*

3. Decomposition of the curvature tensor field $\overline{K}^i_{rjk}$

In a conformal Finsler space $\overline{F}_n$, we consider the decomposition of the curvature tensor in the form

(32) $$\overline{K}_{rjk} = \overline{l}^i\overline{\Omega}_{rjk},$$

where $\overline{\Omega}_{rjk}$ is the conformal decomposition tensor field and $\overline{l}^i = \dfrac{\dot{x}^i}{\overline{\overline{F}}}$.
By using the equations (8) and (32), we get

(33) $$\overline{\Omega}_{rjk} = -\overline{\Omega}_{rkj}.$$

Transvection of the equation (32) by $\overline{l}^r$, we have

(34) $$\overline{K}^i_{0jk} = \overline{l}^i\overline{\Omega}_{0jk},$$

where

(35) $$\overline{K}^i_{0jk} = \overline{K}_{rjk}\overline{l}^r,$$

and

(36) $$\overline{\Omega}_{0jk} = \overline{\Omega}_{rjk}\overline{l}^r.$$

Interchanging the indices j and k in the equation (34) , we obtain

(37) $$\overline{\Omega}_{0jk} = -\overline{\Omega}_{0kj},$$

since $\overline{K}^i_{0jk}$ is skew-symmetric in the last two covariant indices .
Applying the decomposition (32) in the identity (8), we get

(38) $$\overline{\Omega}_{[rjk]} = 0.$$

Transvection of the equation (37) by $\overline{l}$ and using (33) and (36), we have

(39) $$\overline{\Omega}_{0jk} = 2\overline{\Omega}_{[kj]r}\overline{l}^r.$$

Hence we state the following theorem.

Theorem 3.1. *In a conformal Finsler space* $\overline{F}_n$*, the decomposition tensor fields* $\overline{\Omega}_{rjk}$ *and* $\overline{\Omega}_{0jk}$ *satisfy the following identities:*

(a) $\overline{\Omega}_{rjk} = -\overline{\Omega}_{rkj}$,

(b) $\overline{\Omega}_{0jk} = -\overline{\Omega}_{0kj}$,

(c) $\overline{\Omega}_{[rjk]} = 0$,

(d) $\overline{\Omega}_{0jk} = 2\overline{\Omega}_{[kj]r}\bar{l}^r$.

Applying the decomposition (32) in the Bianchi identity (1.12) satisfied by $\overline{K}^i_{rjk}$ we get

$$\bar{l}^i\overline{\Omega}_{r[jk|\bar{h}]} + (\dot{\partial}_p\Gamma^{*i}_{r[j})H^p_{kh]} + (\dot{\partial}_p U^i_{r[j})H^p_{kh]} + [\dot{\partial}_p(\Gamma^{*i}_{r[j} + U^i_{r[j})][(\dot{\partial}_h B^{pm})_{(k)]}$$
$$-(\dot{\partial}_k B^{pm})_{(h)]})]\sigma_m + [(\dot{\partial}_k B^{sm})\dot{\partial}_{h]}\dot{\partial}_s B^{pt} - (\dot{\partial}_h B^{sm})\dot{\partial}_{k]}\dot{\partial}_s B^{pt}]\sigma_m\sigma_t$$
$$+\sigma_{|m|(k)}\dot{\partial}_{h]}B^{pm} - \sigma_{|m|(h)}\dot{\partial}_{k]}B^{pm}] = 0. \tag{40}$$

Transvection of the equation (40) by $\bar{l}^r$ and using (35), we obtain

$$\bar{l}^i\overline{\Omega}_{0[jk|\bar{h}]} + (\dot{\partial}_p\Gamma^{*i}_{r[j})H^p_{kh]}\bar{l}^r + (\dot{\partial}_p U^i_{r[j})H^p_{kh]}\bar{l}^r + (\dot{\partial}_p(\Gamma^{*i}_{r[j} + U^i_{r[j})[\{(\dot{\partial}_h B^{pm})_{(k)]}$$
$$-(\dot{\partial}_k B^{pm})_{(h)]}\}\sigma_m + \{(\dot{\partial}_k B^{sm})\dot{\partial}_{h]}\dot{\partial}_s B^{pt} - (\dot{\partial}_h B^{sm})\dot{\partial}_{k]}\dot{\partial}_s B^{pt}\}\sigma_m\sigma_t + \sigma_{|m|(k)}\dot{\partial}_{h]}B^{pm}$$
$$-\sigma_{|m|(h)}\dot{\partial}_k B^{pm}]\bar{l}^r = 0.$$

(41)

Accordingly, we state the following theorem.

Theorem 3.2. *In a conformal Finsler space the conformal decomposition tensor fields $\overline{\Omega}_{rjk}$ and $\overline{\Omega}_{0jk}$ satisfy the Bianchi identities* (40) *and* (41)*, respectively.*

In a recurrent conformal Finsler space $\overline{F}^*_n$, Cartans curvature tensor $\overline{K}^i_{rjk}$ also satisfies the recurrent relation

$$\overline{K}^i_{rjk|\bar{h}} = \bar{v}_h\overline{K}^i_{rjk} \tag{42}$$

when the space is non-flat.
Differentiating (32) covariantly with respect to x^k in the sense of Cartan and using (41), we obtain

$$\overline{\Omega}_{rjk|\bar{h}} = \bar{v}_h\overline{\Omega}_{rjk} \tag{43}$$

since $\bar{l}^i$ is covariant constant.
Transvection of the equation (42) by $\bar{l}^i$ and noting (35), we find

$$\overline{\Omega}_{0jk|\bar{h}} = \bar{v}_h\overline{\Omega}_{0jk} \tag{44}$$

Thus we have

Theorem 3.3. *In a recurrent conformal Finsler space $\overline{F}^*_n$, the conformal tensor fields $\overline{\Omega}_{rjk}$ and $\overline{\Omega}_{0jk}$ also behave like recurrence tensor fields.*

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