RELATIVE FIX POINTS OF A CERTAIN CLASS OF COMPLEX FUNCTIONS

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Abstract. We introduce the idea of relative iterations of three functions and by using this, we extend a theorem on fix point of complex functions involving exact factor order.

1. Introduction

A single valued function $f(z)$ of the complex variable z is said to belong to (i) class I if $f(z)$ is entire transcendental, (ii) class II if it is regular in the complex plane punctured at a, b $(a \neq b)$ and has an essential singularity at b, and a singularity at a and if $f(z)$ does not assume the values a and b anywhere in the complex plane except possible at the point a.

We can normalize the functions in class II by taking $a = 0$ and $b = \infty$.

Throughout this paper we shall consider only such normalized functions, whenever we deal with functions in class II.

Iterated $f_n(z)$ of $f(z)$ are defined inductively by

$$f_0(z) = z \quad \text{and} \quad f_{n+1}(z) = f(f_n(z));\ n = 0, 1, 2, \ldots.$$

A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and called a fix point of exact order n if α is a solution of $f_n(z) = z$ but not a solution of $f_k(z) = z$, $k = 1, 2, \ldots, n-1$.

In [**2**], Baker proved the following theorem.

Theorem 1.1. *If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n, except for at most one value of n.*

Bhattacharyya [**4**] extended Theorem 1.1 to the functions belonging to class II as follows:

Theorem 1.2. *If $f(z)$ belongs to class II, then $f(z)$ has infinitely many fix points of exact order n, for every positive integer n.*

Lahiri and Banerjee [**5**] generalized Theorem 1.2 in another direction. For this, they introduced the concept of relative fix point defined as follows:

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Let $f(z)$ and $g(z)$ be functions of complex variable z. Let

$$\begin{aligned}
&f_1(z) = f(z)\\
&f_2(z) = f(g(z)) = f(g_1(z))\\
&f_3(z) = f(g(f(z))) = f(g(f_1(z)))\\
&\vdots\\
&f_n(z) = f(g(f(g...(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even}) ...)))\\
&\qquad = f(g_{n-1}(z)) = f(g(f_{n-2}(z)))
\end{aligned}$$

and so

$$\begin{aligned}
&g_1(z) = g(z)\\
&g_2(z) = g(f(z)) = g(f_1(z))\\
&g_3(z) = g(f_2(z)) = g(f(g_1(z)))\\
&\vdots\\
&g_n(z) = g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}$$

Clearly, all $f_n(z)$ and $g_n(z)$ are functions in class II, if $f(z)$ and $g(z)$ are so.
A point α is called a fix point of $f(z)$ of order n with respect to $g(z)$, if $f_n(\alpha) = \alpha$ and a fix point of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha, k = 1, 2, \ldots, n-1$. Such points α are also called relative fix points.

Theorem 1.3. *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has infinitely many relative fix points of exact order n for every positive integer n provided $\dfrac{T(r, g_n)}{T(r, f_n)}$ is bounded.*

Recently, Banerjee and Mandal [**6**] proved the result of Lahiri and Banerjee [**5**] by introducing the idea of relative fix point of exact factor order n.
A point α is called a relative fix point of $f(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$ and $g_k(\alpha) \neq \alpha$ for all divisors k $(k < n)$ of n.
With this definition Banerjee and Mandal [**6**] proved the following theorem.

Theorem 1.4. *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has infinitely many relative fix points of exact factor order n for every positive integer n, provided $\dfrac{T(r, f_{n-1})}{T(r, f_n)}$ is bounded.*

In the present paper we first define relative iterations for three functions and extend the result of Banerjee and Mandal [**6**] in that direction.
Let $f(z)$, $g(z)$, and $h(z)$ be three functions of the complex variable z and $m \geq 2$ be any fixed positive integer. We set

$$\begin{aligned}
&f_1(z) = f(z)\\
&f_2(z) = f(g(z)) = f(g_1(z))\\
&f_3(z) = f(g(h(z))) = f(g(h_1(z))) = f(g_2(z))\\
&f_4(z) = f(g(h(f(z)))) = f(g(h_2(z))) = f(g_3(z))\\
&\quad\vdots
\end{aligned}$$

$f_n(z) = f(g(h(f...(f(z)$ or $g(z)$ or $h(z)$ according to $n = 3m-2$ or $3m-1$ or $3m$ $)...)))$

$= f(g_{n-1}(z)) = f(g(h_{n-2}(z)))$.

Similarly,

$g_1(z) = g(z)$

$g_2(z) = g(h(z)) = g(h_1(z))$

$g_3(z) = g(h(f(z))) = g(h(f_1(z))) = g(h_2(z))$

$g_4(z) = g(h(f(g(z)))) = g(h(f_2(z))) = g(h_3(z))$

$\vdots$

$g_n(z) = g(h(f(g...(g(z)$ or $h(z)$ or $f(z)$ according to $n = 3m-2$ or $3m-1$ or $3m)...)))$

$= g(h_{n-1}(z)) = g(h(f_{n-2}(z)))$

and so are

$h_1(z) = h(z)$

$h_2(z) = h(f(z)) = h(f_1(z))$

$h_3(z) = h(f(g(z))) = h(f(g_1(z))) = h(f_2(z))$

$h_4(z) = h(f(g(h(z)))) = h(f(g_2(z))) = h(f_3(z))$

$\vdots$

$h_n(z) = h(f(g(h...(h(z)$ or $f(z)$ or $g(z)$ according to $n = 3m-2$ or $3m-1$ or $3m)...)))$

$= h(f_{n-1}(z)) = h(f(g_{n-2}(z)))$.

Clearly all $f_n(z)$, $g_n(z)$, and $h_n(z)$ are functions in class II, if $f(z)$, $g(z)$, and $h(z)$ are so.

The following definition is now introduced.

Definition 1.5. *A point α is called a fix point of $f(z)$ of order n with respect to $g(z)$ and $h(z)$, if $f_n(\alpha) = \alpha$ and a fix point $f(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$, $g_k(\alpha) \neq \alpha$ and $h_k(\alpha) \neq \alpha$ for all divisors k $(k < n)$ of n.*

Example 1.6. *Let $f(z) = z+1$, $g(z) = \dfrac{1}{z-1}$, and $h(z) = \dfrac{2}{z}$. Clearly, $f_3(z) = \dfrac{2}{2-z}$. Here $z = 1 \pm i$ are fix points of $f(z)$ of exact factor order 3.*

Let $f(z)$ be meromorphic in $r_0 \leq |z| < \infty$, $r_0 > 0$. We use the following notations [**1**]:

$n(t, a, f) :=$ number of roots of $f(z) = a$ in $r_0 < |z| \leq t$, counted according to multiplicity,

$$N(r, a, f) := \int_{r_0}^{r} \frac{n(t, a, f)}{t} dt,$$

$n(t, \infty, f) := n(t, f) =$ the number of poles of $f(z)$ in $r_0 < |z| \leq t$, counted with due to multiplicity,

$N(t, \infty, f) := N(t, f)$,

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta$$

and $m(r,a,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta})-a}\right| d\theta.$

With these notations, Jensen's formula can be written as [**1**],

$$m(r,f)+N(r,f)=m(r,\frac{1}{f})+N(r,\frac{1}{f})+O(\log r).$$

Writing $m(r,f)+N(r,f)=T(r,f)$, the above becomes

$$T(r,f)=T(r,\frac{1}{f})+O(\log r)\ .$$

In this case the first fundamental theorem takes the form

$$m(r,a,f)+N(r,a,f)=T(r,f)+O(\log r), \tag{1}$$

where $r_0 \le |z| < \infty$, $r_0 > 0$.

Suppose that $f(z)$ is non-constant. Let $a_1, a_2, \ldots, a_q$, $q \ge 2$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\upsilon| \ge \delta$ for $1 \le \mu \le \upsilon \le q$. Then

$$m(r,f)+\sum_{\upsilon=1}^{q} m(r,a_\upsilon,f) \le 2T(r,f)-N_1(r)+S(r), \tag{2}$$

where

$$N_1(r)=N(r,\frac{1}{f'})+2N(r,f)-N(r,f'),$$

and

$$S(r)=m(r,\frac{f'}{f})+\sum_{\upsilon=1}^{q} m(r,\frac{f'}{f-a_\upsilon})+O(\log r).$$

Adding $N(r,f)+\sum_{\upsilon=1}^{q} N(r,a_\upsilon,f)$ to both sides of (2) and using (1), we obtain

$$(q-1)T(r,f) \le \overline{N}(r,f)+\sum_{\upsilon=1}^{q} \overline{N}(r,a_\upsilon,f)+S_1(r), \tag{3}$$

where $S_1(r)=O(\log T(r,f))$ and $\overline{N}$ corresponds to distinct roots.

Further, because f_n has an essential singularity at ∞, we have $\frac{\log r}{T(r,f_n)} \to 0$ as $r \to \infty$ [**1**].

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. *If n is any positive integer and f, g, and h are functions in class II, then for any $r_0 > 0$ and a positive constant M_1, we have*

$$\frac{T(r,f_{n+p})}{T(r,f_n)} > M_1 \text{ or } \frac{T(r,g_{n+p})}{T(r,f_n)} > M_1 \text{ or } \frac{T(r,h_{n+p})}{T(r,f_n)} > M_1$$

according to $p = 3m$ or $3m-1$ or $3m-2$; $m \in \mathbb{N}$, for all large r, except a set of r intervals of total finite length.

Proof. Case (i). When $p=3m$, $m \in \mathbb{N}$.

In this case we consider the equation $f_{n+p}(z)=a$, where $a \ne 0, \infty$ i.e. $f_p(f_n(z))=a$. This is equivalent to $f_p(w_i)=a$ and $f_n(z)=w_i$, $(i=1,2,\ldots)$.

Because f_p is transcendental, $f_p(w_i) = a$ has infinitely many roots for every complex number a with two exceptions $a = 0, \infty$.
From (1)

$$\begin{aligned} T(r, f_{n+p}) &= m(r, a, f_{n+p}) + N(r, a, f_{n+p}) + O(\log r) \\ &\geq \overline{N}(r, a, f_{n+p}) + O(\log r) \\ &\geq \sum_{i=1}^{M} \overline{N}(r, w_i, f_n) \end{aligned}$$

for a fixed $M(> 3)$.
From (3) taking $a_v = w_i$, $f = f_n$, and $q = M$, we obtain

$$\sum_{i=1}^{M} \overline{N}(r, w_i, f_n) \geq (M-1)T(r, f_n) - \overline{N}(r, f_n) - S_1(r).$$

Since for large r, $S_1(r) \leq T(r, f_n)$, so

$$\sum_{i=1}^{M} \overline{N}(r, w_i, f_n) \geq (M-3)T(r, f_n). \tag{4}$$

Therefore,

$$T(r, f_{n+p}) \geq M_1 T(r, f_n), \text{ where } M_1 = M - 3,$$

outside a set of r intervals of total finite length.
Case (ii). When $p = 3m - 1$, $m \in \mathbb{N}$.
In this case, we consider the equation $g_{n+p}(z) = a$, where $a \neq 0, \infty$ i.e. $g_p(f_n(z)) = a$. This is equivalent to $g_p(w_i') = a$ and $f_n(z) = w_i'$, $(i = 1, 2, \ldots)$.
From (1), by same reasoning as in Case (i), we have

$$\begin{aligned} T(r, g_{n+p}) &= m(r, a, g_{n+p}) + N(r, a, g_{n+p}) + O(\log r) \\ &\geq \overline{N}(r, a, g_{n+p}) + O(\log r) \\ &\geq \sum_{i=1}^{M} \overline{N}(r, w_i', f_n) \end{aligned}$$

for a fixed $M(> 3)$.
Now, we have as in (4)

$$\sum_{i=1}^{M} \overline{N}(r, w_i', f_n) \geq (M-3)T(r, f_n).$$

Therefore, $T(r, g_{n+p}) \geq M_1 T(r, f_n)$, outside a set of r intervals of total finite length.
Case (iii). When $p = 3m - 2$, $m \in \mathbb{N}$.
In this case, we consider the equation $h_{n+p}(z) = a$, where $a \neq 0, \infty$ i.e. $h_p(f_n(z)) = a$. This is equivalent to $h_p(w_i'') = a$ and $f_n(z) = w_i''$, $(i = 1, 2, \ldots)$.

From (1), by same reasoning as in Case (i), we have

$$\begin{aligned} T(r,h_{n+p}) &= m(r,a,h_{n+p}) + N(r,a,h_{n+p}) + O(\log r) \\ &\geq \overline{N}(r,a,h_{n+p}) + O(\log r) \\ &\geq \sum_{i=1}^{M} \overline{N}(r,w_i'',f_n), \end{aligned}$$

for a fixed $M(>3)$.
In this case, as in (4) we also have

$$\sum_{i=1}^{M} \overline{N}(r,w_i'',f_n) \geq (M-3)T(r,f_n).$$

Therefore, $T(r,h_{n+p}) \geq M_1 T(r,f_n)$, outside a set of r intervals of total finite length and this proves the lemma. □
If we simply interchange f, g, and h in cyclic order, then we obtain the following two lemmas.

Lemma 2.2. *If n is any positive integer and f, g, and h are functions in class II, then for any $r_0 > 0$ and a positive constant M_1, we have*

$$\frac{T(r,g_{n+p})}{T(r,g_n)} > M_1 \text{ or } \frac{T(r,h_{n+p})}{T(r,g_n)} > M_1 \text{ or } \frac{T(r,f_{n+p})}{T(r,g_n)} > M_1$$

according to $p = 3m$ or $3m-1$ or $3m-2$;, $m \in \mathbb{N}$, for all large r, except a set of r intervals of total finite length.

Lemma 2.3. *If n is any positive integer and f, g, and h are functions in class II, then for any $r_0 > 0$ and a positive constant M_1*

$$\frac{T(r,h_{n+p})}{T(r,h_n)} > M_1 \text{ or } \frac{T(r,f_{n+p})}{T(r,h_n)} > M_1 \text{ or } \frac{T(r,g_{n+p})}{T(r,h_n)} > M_1$$

according to $p = 3m$ or $3m-1$ or $3m-2$; $m \in \mathbb{N}$, for all large r, except a set of r intervals of total finite length.

3. Main result

Our main result is given in the following theorem.

Theorem 3.1. *If $f(z)$, $g(z)$, and $h(z)$ belong to class II, then $f(z)$ has infinitely many fix points of exact factor order n for every positive integer n (≥ 3) provided $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ are bounded.*

Proof. We consider the function $\phi(z) = \frac{f_n(z)}{z}, r_0 < |z| < \infty$. Then

$$T(r,\phi) = T(r,f_n) + O(\log r). \tag{5}$$

Assume that $f(z)$ has only a finite number of fix points of exact factor order n. Now from (**3**) by taking $q = 2$, $a_1 = 0, a_2 = 1$, we obtain for ϕ,

$$T(r,\phi) \leq \overline{N}(r,\infty,\phi) + \overline{N}(r,0,\phi) + \overline{N}(r,1,\phi) + S_1(r,\phi),$$

where $S_1(r,\phi) = O(\log T(r,\phi))$ outside a set of r intervals of finite length [**3**]. First, we calculate $\overline{N}(r,0,\phi)$. We have $\overline{N}(r,0,\phi) = \int_{r_0}^{r} \frac{\overline{n}(t,0,\phi)}{t} dt$, where $\overline{n}(t,0,\phi)$ is the number of roots of $\phi(z) = 0$ in $r_0 < |z| \leq t$, each multiple root taken once at a time. The distinct roots of $\phi(z) = 0$ in $r_0 < |z| \leq t$ are the roots of $f_n(z) = 0$ in $r_0 < |z| \leq t$. By the definition of functions in class II, $f_n(z)$ has a singularity at $z = 0$, an essential singularity at $z = \infty$, and $f_n(z) \neq 0, \infty$. So $\overline{n}(t,0,\phi) = 0$. Consequently, $\overline{N}(r,0,\phi) = 0$. By similar argument $\overline{N}(r,\infty,\phi) = 0$. So

$$T(r,\phi) \leq \overline{N}(r,1,\phi) + S_1(r,\phi). \tag{6}$$

We now calculate $\overline{N}(r,1,\phi)$. If $\phi(z) = 1$, then $f_n(z) = z$. Due to our iteration process we consider the following three cases.

Case (i). When $n = 3m$, $m \in \mathbb{N}$.

Now by Lemma 2.1, Lemma 2.2, and Lemma 2.3, for all large r, we have

$$\begin{aligned} \overline{N}(r,1,\phi) &= \overline{N}(r,0,f_n - z) \\ &\leq \sum_{j/n,j=1}^{n-2} [\overline{N}(r,0,f_j - z) + \overline{N}(r,0,g_j - z) + \overline{N}(r,0,h_j - z)] + O(\log r). \end{aligned}$$

Remark. The term $O(\log r)$ arises due to the assumption that $f(z)$ has only a finite number of relative fix points of exact factor order n.

$$\begin{aligned} &\leq \sum_{j/n,j=1}^{n-2} [T(r,f_j - z) + O(\log r) + T(r,g_j - z) + O(\log r) + T(r,h_j - z) + O(\log r)] \\ &\quad + O(\log r) \\ &= \sum_{j/n,j=1}^{n-2} [T(r,f_j) + T(r,g_j) + T(r,h_j)] + O(\log r) \\ &= \{T(r,f_{j_1}) + T(r,f_{j_4}) + ... + T(r,f_{j_{3p-2}}) + T(r,f_{j_2}) + T(r,f_{j_5}) + ... + T(r,f_{j_{3q-1}}) \\ &\quad + T(r,f_{j_3}) + T(r,f_{j_6}) + \ldots + T(r,f_{j_{3s}})\} + \{T(r,g_{j_1}) + T(r,g_{j_4}) + \ldots + T(r,g_{j_{3p-2}}) \\ &\quad + T(r,g_{j_2}) + T(r,g_{j_5}) + \ldots + T(r,g_{j_{3q-1}}) + T(r,g_{j_3}) + T(r,g_{j_6}) + \ldots + T(r,g_{j_{3s}})\} \\ &\quad + \{T(r,h_{j_1}) + T(r,h_{j_4}) + \ldots + T(r,h_{j_{3p-2}}) + T(r,h_{j_2}) + T(r,h_{j_5}) + \ldots + T(r,h_{j_{3q-1}}) \\ &\quad + T(r,h_{j_3}) + T(r,h_{j_6}) + \ldots + T(r,h_{j_{3s}})\} + O(\log r), \end{aligned}$$

where $j_1, j_4, \ldots, j_{3p-2}$; $j_2, j_5, \ldots, j_{3q-1}$, and $j_3, j_6, \ldots, j_{3s}$ are divisors of $n = 3m$, and are strictly less than n, and are of the forms $3p-2, 3q-1$, and $3s$, $(p, q, s \in \mathbb{N})$,

$$\begin{aligned}
&= T(r,f_n)\Big[\frac{T(r,f_{j_3})}{T(r,f_n)} + \frac{T(r,f_{j_6})}{T(r,f_n)} + \ldots + \frac{T(r,f_{j_{3s}})}{T(r,f_n)} + \frac{T(r,g_{j_1})}{T(r,f_n)} + \frac{T(r,g_{j_4})}{T(r,f_n)} \\
&\quad + \ldots + \frac{T(r,g_{j_{3p-2}})}{T(r,f_n)} + \frac{T(r,h_{j_2})}{T(r,f_n)} + \frac{T(r,h_{j_5})}{T(r,f_n)} + \ldots + \frac{T(r,h_{j_{3q-1}})}{T(r,f_n)}\Big] \\
&\quad + T(r,g_n)\Big[\frac{T(r,f_{j_2})}{T(r,g_n)} + \frac{T(r,f_{j_5})}{T(r,g_n)} + \ldots + \frac{T(r,f_{j_{3q-1}})}{T(r,g_n)} + \frac{T(r,g_{j_3})}{T(r,g_n)} \\
&\quad + \frac{T(r,g_{j_6})}{T(r,g_n)} + \ldots + \frac{T(r,g_{j_{3s}})}{T(r,g_n)} + \frac{T(r,h_{j_1})}{T(r,g_n)} + \frac{T(r,h_{j_4})}{T(r,g_n)} + \ldots + \frac{T(r,h_{j_{3p-2}})}{T(r,g_n)}\Big] \\
&\quad + T(r,h_n)\Big[\frac{T(r,f_{j_1})}{T(r,h_n)} + \frac{T(r,f_{j_4})}{T(r,h_n)} + \ldots + \frac{T(r,f_{j_{3p-2}})}{T(r,h_n)} + \frac{T(r,g_{j_2})}{T(r,h_n)} + \frac{T(r,g_{j_5})}{T(r,h_n)} \\
&\quad + \ldots + \frac{T(r,g_{j_{3q-1}})}{T(r,h_n)} + \frac{T(r,h_{j_3})}{T(r,h_n)} + \frac{T(r,h_{j_6})}{T(r,h_n)} + \ldots + \frac{T(r,h_{j_{3s}})}{T(r,h_n)}\Big] + O(\log r) \\
&< \frac{n-1}{6n}T(r,f_n) + \frac{n-1}{6n}T(r,g_n) + \frac{n-1}{6n}T(r,h_n) + O(\log r).
\end{aligned}$$

Case(ii). When $n = 3m+1$, $m \in \mathbb{N}$. Now by Lemma 2.1, Lemma 2.2, and Lemma 2.3, for all large r, we have

$$\begin{aligned}
\overline{N}(r,1,\phi) &= \overline{N}(r,0,f_n - z) \\
&\leq \sum_{j/n, j=1}^{n-2} [\overline{N}(r,0,f_j - z) + \overline{N}(r,0,g_j - z) + \overline{N}(r,0,h_j - z)] + O(\log r) \\
&\leq \sum_{j/n, j=1}^{n-2} [T(r,f_j - z) + O(\log r) + T(r,g_j - z) + O(\log r) + T(r,h_j - z) \\
&\quad + O(\log r)] + O(\log r) \\
&= \sum_{j/n, j=1}^{n-2} [T(r,f_j) + T(r,g_j) + T(r,h_j)] + O(\log r) \\
&= \{T(r,f_{j_1}) + T(r,f_{j_4}) + \ldots + T(r,f_{j_{3p-2}}) + T(r,f_{j_2}) + T(r,f_{j_5}) \\
&\quad + \ldots + T(r,f_{j_{3q-1}})\} + \{T(r,g_{j_1}) + T(r,g_{j_4}) + \ldots + T(r,g_{j_{3p-2}}) \\
&\quad + T(r,g_{j_2}) + T(r,g_{j_5}) + \ldots + T(r,g_{j_{3q-1}})\} + \{T(r,h_{j_1}) + T(r,h_{j_4}) \\
&\quad + \ldots + T(r,h_{j_{3p-2}}) + T(r,h_{j_2}) + T(r,h_{j_5}) + \ldots + T(r,h_{j_{3q-1}})\} + O(\log r),
\end{aligned}$$

where $j_1, j_4, \ldots, j_{3p-2}$ and $j_2, j_5, \ldots, j_{3q-1}$ are divisors of $n = 3m+1$ and are strictly less than n, and are of the forms $3p-2$ and $3q-1$ ($p, q \in \mathbb{N}$),

$$\begin{aligned}
&= T(r,f_n)[\frac{T(r,f_{j_1})}{T(r,f_n)} + \frac{T(r,f_{j_4})}{T(r,f_n)} + \ldots + \frac{T(r,f_{j_{3p-2}})}{T(r,f_n)} + \frac{T(r,g_{j_2})}{T(r,f_n)} \\
&\quad + \frac{T(r,g_{j_5})}{T(r,f_n)} + \ldots + \frac{T(r,g_{j_{3q-1}})}{T(r,f_n)}] + T(r,g_n)[\frac{T(r,g_{j_1})}{T(r,g_n)} + \frac{T(r,g_{j_4})}{T(r,g_n)} \\
&\quad + \ldots + \frac{T(r,g_{j_{3p-2}})}{T(r,g_n)} + \frac{T(r,h_{j_2})}{T(r,g_n)} + \frac{T(r,h_{j_5})}{T(r,g_n)} + \ldots + \frac{T(r,h_{j_{3q-1}})}{T(r,g_n)}] \\
&\quad + T(r,h_n)[\frac{T(r,f_{j_2})}{T(r,h_n)} + \frac{T(r,f_{j_5})}{T(r,h_n)} + \ldots + \frac{T(r,f_{j_{3q-1}})}{T(r,h_n)} + \frac{T(r,h_{j_1})}{T(r,h_n)} \\
&\quad + \frac{T(r,h_{j_4})}{T(r,h_n)} + \ldots + \frac{T(r,h_{j_{3p-2}})}{T(r,h_n)}] + O(\log r) \\
&< \frac{n-1}{6n}T(r,f_n) + \frac{n-1}{6n}T(r,g_n) + \frac{n-1}{6n}T(r,h_n) + O(\log r).
\end{aligned}$$

Case(iii). When $n = 3m+2$, $m \in \mathbb{N}$. Now by Lemma 2.1, Lemma 2.2, and Lemma 2.3, for all large r, we have

$$\begin{aligned}
\overline{N}(r,1,\phi) &= \overline{N}(r,0,f_n - z) \\
&\le \sum_{j/n, j=1}^{n-2} [\overline{N}(r,0,f_j - z) + \overline{N}(r,0,g_j - z) + \overline{N}(r,0,h_j - z)] + O(\log r) \\
&\le \sum_{j/n, j=1}^{n-2} [T(r,f_j - z) + O(\log r) + T(r,g_j - z) + O(\log r) + T(r,h_j - z) \\
&\quad + O(\log r)] + O(\log r) \\
&= \sum_{j/n, j=1}^{n-2} [T(r,f_j) + T(r,g_j) + T(r,h_j)] + O(\log r) \\
&= \{T(r,f_{j_1}) + T(r,f_{j_4}) + \ldots + T(r,f_{j_{3p-2}}) + T(r,f_{j_2}) + T(r,f_{j_5}) \\
&\quad + \ldots + T(r,f_{j_{3q-1}})\} + \{T(r,g_{j_1}) + T(r,g_{j_4}) + \ldots + T(r,g_{j_{3p-2}}) + T(r,g_{j_2}) \\
&\quad + T(r,g_{j_5}) + \ldots + T(r,g_{j_{3q-1}})\} + \{T(r,h_{j_1}) + T(r,h_{j_4}) \\
&\quad + \ldots + T(r,h_{j_{3p-2}}) + T(r,h_{j_2}) + T(r,h_{j_5}) + \ldots + T(r,h_{j_{3q-1}})\} + O(\log r),
\end{aligned}$$

where $j_1, j_4, \ldots, j_{3p-2}$ and $j_2, j_5, \ldots, j_{3q-1}$ are divisors of $n = 3m+2$ and are strictly less than n and are of the forms $3p-2$ and $3q-1$ $(p, q \in \mathbb{N}$),

$$\begin{aligned}
&= T(r,f_n)\Big[\frac{T(r,f_{j_2})}{T(r,f_n)} + \frac{T(r,f_{j_5})}{T(r,f_n)} + \ldots + \frac{T(r,f_{j_{3q-1}})}{T(r,f_n)} + \frac{T(r,h_{j_1})}{T(r,f_n)} \\
&\quad + \frac{T(r,h_{j_4})}{T(r,f_n)} + \ldots + \frac{T(r,h_{j_{3p-2}})}{T(r,f_n)}\Big] + T(r,g_n)\Big[\frac{T(r,f_{j_1})}{T(r,g_n)} + \frac{T(r,f_{j_4})}{T(r,g_n)} \\
&\quad + \ldots + \frac{T(r,f_{j_{3p-2}})}{T(r,g_n)} + \frac{T(r,g_{j_2})}{T(r,g_n)} + \frac{T(r,g_{j_5})}{T(r,g_n)} + \ldots + \frac{T(r,g_{j_{3q-1}})}{T(r,g_n)}\Big] \\
&\quad + T(r,h_n)\Big[\frac{T(r,g_{j_1})}{T(r,h_n)} + \frac{T(r,g_{j_4})}{T(r,h_n)} + \ldots + \frac{T(r,g_{j_{3p-2}})}{T(r,h_n)} + \frac{T(r,h_{j_2})}{T(r,h_n)} \\
&\quad + \frac{T(r,h_{j_5})}{T(r,h_n)} + \ldots + \frac{T(r,h_{j_{3q-1}})}{T(r,h_n)}\Big] + O(\log r) \\
&< \frac{n-1}{6n}T(r,f_n) + \frac{n-1}{6n}T(r,g_n) + \frac{n-1}{6n}T(r,h_n) + O(\log r).
\end{aligned}$$

Thus in any case,

$$\overline{N}(r,1,\phi) < \frac{n-1}{6n}T(r,f_n) + \frac{n-1}{6n}T(r,g_n) + \frac{n-1}{6n}T(r,h_n) + O(\log r).$$

So from (6) and since $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ are bounded, we have

$$\begin{aligned}
T(r,\phi) &< \frac{n-1}{6n}T(r,f_n) + \frac{n-1}{6n}T(r,g_n) + \frac{n-1}{6n}T(r,h_n) + O(\log r) + S_1(r,\phi) \\
&= \frac{n-1}{6n}T(r,f_n) + \frac{n-1}{6n}T(r,g_n) + \frac{n-1}{6n}T(r,h_n) + O(\log r) + O(\log T(r,\phi)) \\
&\le T(r,f_n)\Big[\frac{n-1}{6n} + \frac{n-1}{6n}\frac{T(r,g_n)}{T(r,f_n)} + \frac{n-1}{6n}\frac{T(r,h_n)}{T(r,f_n)} + \frac{O(\log(T(r,f_n) + O(\log r)))}{T(r,f_n)} \\
&\quad + \frac{O(\log r)}{T(r,f_n)}\Big] \\
&\le T(r,f_n)\Big[\frac{n-1}{6n} + \frac{n-1}{6n} + \frac{n-1}{6n} + \frac{O(\log(T(r,f_n)(1+\frac{O(\log r)}{T(r,f_n)})))}{T(r,f_n)} + \frac{O(\log r)}{T(r,f_n)}\Big] \\
&< T(r,f_n)\Big[\frac{1}{2} + \frac{O(\log(T(r,f_n)(1+\frac{O(\log r)}{T(r,f_n)})))}{T(r,f_n)} + \frac{O(\log r)}{T(r,f_n)}\Big] = \frac{1}{2}T(r,f_n),
\end{aligned}$$

for all large r.
Therefore, $T(r,\phi) < \frac{1}{2}T(r,f_n)$ for all large r. This contradicts to (5). Hence $f(z)$ has infinitely many relative fix points of exact factor order n.
This proves the theorem. □

Remark. Since fix points of exact order are fix points of exact factor order, if $f(z) = g(z) = h(z)$ then $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ being bounded, Theorem 1.2 covers Theorem 3.1.

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