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GRAND LORENTZ SEQUENCE SPACE AND ITS MULTIPLICATION OPERATOR

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Abstract. In this paper, we introduce the grand Lorentz sequence spaces $\ell^{\theta}_{p,q)}$ and study on some topological properties. Also, we characterize some properties of the multiplication operator, such as compactness, Fredholmness etc., defined on $\ell^{\theta}_{p,q)}$.

1. Introduction

Let (X, S, μ) be a $\sigma-$finite measure space and let g be a complex-valued measurable function defined on X. The *non-increasing rearrangement* g^* of g is defined by

$$g^*(s) = \inf \{t > 0 : F_\mu(t) \leq s\}, \;\; s \geq 0,$$

where $F_\mu(t) = \mu \{x \in X : |g(x)| > t\}$, $t \geq 0$, is the *distribution function* of g. If μ is counting measure on $S = 2^{\mathbb{N}}$, then we can write the *distribution function* and *the non-increasing rearrangement of a complex-valued sequence* (x_n), respectively, as follows;

$$F_\mu(t) = \mu \{n \in \mathbb{N} : |x_n| > t\}, t \geq 0$$

and

$$x_{\phi(n)} = \inf \{t > 0 : F_\mu(t) \leq n - 1\}$$

if $n-1 \leq t < n$ with $F_\mu(t) < \infty$. By the definition of non-increasing rearrangement, we can interpret that $\left(x_{\phi(n)}\right)$ can be obtained by permuting $(|x_n|)_{n\in R}$, where $R = \{n \in \mathbb{N} : x_n \neq 0\}$, in the decreasing order. Here, $x_{\phi(n)} = 0$ for $n > \mu(R)$ if $\mu(R) < \infty$ [2].

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Lorentz introduced the classical Lorentz space $\Lambda_{q,w}$, $0<q<\infty$, which the space of all measurable functions f defined on $(0,1)$ with

$$\|f\|_{\Lambda_{q,w}} = \left(\int_0^1 (f^*(x))^q w(x)dx \right)^{\frac{1}{q}},$$

where f^* is the non-increasing rearrangement of f and w is a weight function [12], [13]. The space $\Lambda_{q,w}$ and its special case $L^{p,q}$, $0<q,p\leq\infty$, have been widely studied by many authors. For more details see [3], [5], [7].

The Lorentz sequence spaces $\ell_{p,q}$ is the space of all complex-valued sequences $x=(x_n)$ such that

$$\|x\|_{p,q} = \begin{cases} \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1} (x_{\phi(n)})^q \right)^{\frac{1}{q}}, & 1\leq p\leq\infty,\ 1\leq q<\infty \\ \sup_n n^{\frac{1}{p}} x_{\phi(n)}, & 1\leq p<\infty,\ q=\infty \end{cases}$$

is finite, where $\left(x_{\phi(n)}\right)$ is non-increasing rearrangement of x. The spaces $\ell_{p,q}$ have been used to introduce and investigate some classes of operators, like $(p,q)-$nuclear, $(p,q;r)-$absolutely summing operator [14]. Kato [11] characterized the dual space of $\ell_{p,q}\{E\}$, where E is a Banach space. See also [2], [10], [15].

The idea of grand spaces was raised by Iwaniec and Sbordone [8]. They introduced the grand Lebesgue spaces $L^{p)}$ for $1<p<\infty$. Samko and Umarkhadzhiev [17] studied some properties of grand Lebesgue spaces on sets of infinite measure. Jain and Kumari [9] introduced the grand Lorentz spaces $\Lambda_{q),w}$, $0<q<\infty$ and studied on its basic properties. Also, they characterized boundedness of maximal operator on the space $\Lambda_{q),w}$. Later, Rafeiro and others [16] introduced the grand Lebesgue sequence space $\ell^{p),\theta} = \ell^{p),\theta}(X)$ by the norm

$$\|x\|_{\ell^{p),\theta}(X)} = \sup_{\varepsilon>0} \left(\varepsilon^\theta \sum_{k\in X} |x_k|^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|x\|_{\ell^{p(1+\varepsilon)}(X)}$$

where X is one of the sets $\mathbb{Z}^n$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ for $1\leq p<\infty$, $\theta>0$. They studied various operators of harmonic analysis, e. g. maximal, convolution, Hardy etc.

In this paper, we are inspired by this work and introduce the grand Lorentz sequence spaces $\ell^\theta_{p,q)}$ as follows; let $\theta>0$. The grand Lorentz sequence space $\ell^\theta_{p,q)}$ is the set of all sequences $a=(a_n)$ such that $\|a\|_{p,q),\theta}<\infty$, where $\|a\|_{p,q),\theta}$ is defined by

$$\begin{cases} \sup_{\varepsilon>0} \left(\varepsilon^\theta \sum_{n=1}^{\infty} \left(n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)} \right)^{q(1+\varepsilon)} n^{-1} \right)^{\frac{1}{q(1+\varepsilon)}}, & 1\leq p\leq\infty, 1\leq q<\infty \\ \sup_{n\geq1} n^{\frac{1}{p}} a_{\phi(n)}, & 1\leq p<\infty, q=\infty \end{cases}$$

where $\left(a_{\phi(n)}\right)$ is the non-increasing rearrangement of the sequence $a=(a_n)$. In case $p=q$, the grand Lorentz sequence space $\ell^\theta_{p,q)}$ coincides with the grand Lebesgue space $\ell^{p),\theta}(\mathbb{N})$. In this work, we study on some topological properties and inclusion theorems of the space $\ell^{\theta}_{p,q)}$. Also, we characterize some properties of multiplication operator on the $\ell^{\theta}_{p,q)}$.

We will need the following lemma:

Lemma 1. *(Hardy, Littlewood and Polya) Let* (r_n^*) *and* $({}^*r_n)$ *be the non-increasing and non-decreasing rearrangements of a finite sequence* (r_n) *of positive numbers. Then, we have for any two sequences* (a_n) *and* (b_n) *of positive numbers such that*

$$\sum_n a_n^{**} b_n \leq \sum_n a_n b_n \leq \sum_n a_n^* b_n^*$$

[6].

2. Main Results

2.1. Grand Lorentz Sequence Space.

Theorem 2. *The grand Lorentz sequence space* $\ell^{\theta}_{p,q)}$ *is a normed space for* $1 \leq q \leq p \leq \infty$ *and a quasi-normed space for* $1 \leq p < q \leq \infty$.

Proof. By definition of the norm of $\ell^{\theta}_{p,q)}$, we can write

$$\|a\|_{p,q),\theta} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \|a\|_{p,q(1+\varepsilon)} . \tag{1}$$

Let $1 \leq q < p \leq \infty$. For any $a, b \in \ell^{\theta}_{p,q)}$, since $n^{\frac{q}{p}-1}$ is decreasing sequence of positive numbers and so by Lemma 1, we have

$$\begin{aligned}
\|a+b\|_{p,q),\theta} &= \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(a_{\vartheta(n)} + b_{\vartheta(n)}\right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&= \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} \left(n^{\left(\frac{q}{p}-1\right)\frac{1}{q(1+\varepsilon)}} \left(a_{\vartheta(n)} + b_{\vartheta(n)}\right) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\leq \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(a_{\vartheta(n)}\right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\quad + \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(b_{\vartheta(n)}\right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\leq \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(a_{\phi(n)}\right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\quad + \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(b_{\psi(n)}\right)^{q(1+\varepsilon)} \right)^{\frac{1}{q(1+\varepsilon)}}
\end{aligned}$$

$$= \quad \|a\|_{p,q),\theta} + \|b\|_{p,q),\theta}$$

where $\left(a_{\vartheta(n)} + b_{\vartheta(n)}\right)$, $\left(a_{\phi(n)}\right)$ and $\left(b_{\psi(n)}\right)$ are the non-increasing rearrangements of $(a_n + b_n)$, (a_n) and (b_n), respectively.

Let $1 \leq p < q < \infty$. Then, we have $p < q(1+\varepsilon)$ for $\varepsilon > 0$ and hence $\|a\|_{p,q(1+\varepsilon)}$ is a quasi-norm. Thus, we get

$$\begin{aligned} \|a+b\|_{p,q),\theta} &= \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \|a+b\|_{p,q(1+\varepsilon)} \\ &\leq \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \left(2^{\frac{1}{p}} \left(\|a\|_{p,q(1+\varepsilon)} + \|b\|_{p,q(1+\varepsilon)}\right)\right) \\ &\leq 2^{\frac{1}{p}} \left(\|a\|_{p,q),\theta} + \|b\|_{p,q),\theta}\right). \end{aligned}$$

For $1 \leq p < \infty$ and $q = \infty$, we have $\|a\|_{p,\infty),\theta} = \|a\|_{p,\infty}$. The proof is completed. □

Remark 3. *Let $\alpha > 0$ and let us take the sequence*

$$(a_n) = \left(n^{\frac{-1}{p}} (\ln(n+1))^{-\alpha}\right)$$

as in [16]*. It is easy to see that the sequence (a_n) is decreasing and thus the non-increasing rearrangement of (a_n) is itself. Therefore, we have*

$$\sum_{n=1}^{\infty} \left(n^{\frac{1}{p}} n^{\frac{-1}{p}} (\ln(n+1))^{-\alpha}\right)^q n^{-1} = \sum_{n=1}^{\infty} n^{-1} (\ln(n+1))^{-\alpha q}.$$

If $\alpha > \frac{1}{q}$, then $(a_n) \in \ell_{p,q}$. Using similar technique as in [16]*, we get $(a_n) \in \ell^{\theta}_{p,q)}$ if and only if $\alpha \geq \frac{1-\theta}{q}$. Thus, we get $(a_n) \in \ell^{\theta}_{p,q)}$ and $(a_n) \notin \ell_{p,q}$ whenever $\frac{1-\theta}{q} \leq \alpha \leq \frac{1}{q}$.*

Definition 4. *The vanishing grand Lorentz sequence space $\mathring{\ell}^{\theta}_{p,q)}$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, consists of all sequences $(a_n) \in \ell^{\theta}_{p,q)}$ such that*

$$\lim_{\varepsilon \to 0} \varepsilon^{\theta} \sum_{n=1}^{\infty} \left(n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)}\right)^{q(1+\varepsilon)} n^{-1} = 0.$$

Lemma 5. *The space $\mathring{\ell}^{\theta}_{p,q)}$ is a closed subspace of the space $\ell^{\theta}_{p,q)}$.*

Proof. The proof can be obtained by using similar technique as in [16]. □

Remark 6. *It is enough to take the supremum in (1) on the finite interval for ε, which means*

$$\|a\|_{p,q),\theta} = \sup_{0<\varepsilon<\frac{1}{W(1/e)}} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \|a\|_{p,q(1+\varepsilon)}$$

where $W(t)$ is the Lambert function. Note that $\frac{1}{W(1/e)} \approx 3.59$ (see [4]*,* [16]*).*

Lemma 7. *Let $a = (a_n) \in \ell^{\theta}_{p,q)}$, $1 \le p, q < \infty$ and $\theta > 0$. Then, we have the following inequalities for all $n \in \mathbb{N}$:*

$$a_{\phi(n)} \le h\left(\frac{1}{W(e^{-1})}\right)^{\frac{-\theta}{q}} \left(\frac{p}{q}R(\varepsilon_0)\right)^{\frac{-1}{q}} n^{\frac{-1}{p}} \|a\|_{p,q),\theta}$$

if $1 \le p \le q < \infty$ and

$$a_{\phi(n)} \le h\left(\frac{1}{W(e^{-1})}\right)^{-\frac{\theta}{q}} n^{\frac{1}{q}-\frac{1}{p}} \|a\|_{p,q),\theta}$$

if $1 \le q < p \le \infty$, where $h(x) = x^{\frac{1}{1+x}}$, $R(x) = (1+x)^{-\frac{1}{1+x}}$ and $\varepsilon_0 \approx 1,7182$.

Proof. Let $a = (a_n) \in \ell^{\theta}_{p,q)}$ and let $1 \le p \le q < \infty$. Since $p \le q(1+\varepsilon)$, we have by Lemma 2 in [11] that

$$\begin{aligned}
\|a\|_{p,q),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)} \\
&\ge \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(n^{\frac{1}{p}} \left(\frac{p}{q(1+\varepsilon)}\right)^{\frac{1}{q(1+\varepsilon)}} a_{\phi(n)} \right) \\
&\ge \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (1+\varepsilon)^{-\frac{1}{q(1+\varepsilon)}} n^{\frac{1}{p}} a_{\phi(n)} \\
&= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (R(\varepsilon))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}. \\
&\ge \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (R(\varepsilon_0))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}. \\
&= h\left(\frac{1}{W(e^{-1})}\right)^{\frac{\theta}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} (R(\varepsilon_0))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}.
\end{aligned}$$

Here $R(x) = (1+x)^{-\frac{1}{1+x}}$ attains the minimum at the point $\varepsilon_0 \approx 1,7182$.
Let $1 \le q < p < \infty$. Then, since $n^{\frac{q}{p}-1}$ is decreasing, we have

$$\begin{aligned}
\|a\|_{p,q),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)} \\
&\ge \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\sum_{n=1}^{k} \left(n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)} \right)^{q(1+\varepsilon)} n^{-1} \right)^{\frac{1}{q(1+\varepsilon)}} \\
&\ge a_{\phi(k)} \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(\sum_{n=1}^{k} n^{\frac{q}{p}-1} \right)^{\frac{1}{q(1+\varepsilon)}}
\end{aligned}$$

$$\geq \quad a_{\phi(k)} \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left(k^{\frac{q}{p}-1}\right)^{\frac{1}{q(1+\varepsilon)}}$$

$$\geq \quad h\left(\frac{1}{W(e^{-1})}\right)^{\frac{\theta}{q}} n^{\frac{1}{p}-\frac{1}{q}} a_{\phi(k)}.$$

□

Theorem 8. *The space $\ell^{\theta}_{p,q)}$ is complete for $1 \leq p, q \leq \infty$.*

Proof. Let $a^{(s)} = \left(a_n^{(s)}\right) \in \ell^{\theta}_{p,q)}$ such that

$$\lim_{s,t\to\infty} \left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta} = 0.$$

For $q = \infty$, the proof is clear. Let $q < \infty$. Then, there exists a natural number s_0 such that

$$\left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta} < \eta$$

whenever $s, t \geq s_0$. By Lemma 3, we have

$$\begin{aligned}
\left|a_k^{(s)} - a_k^{(t)}\right| &\leq h\left(\frac{1}{W(e^{-1})}\right)^{-\frac{\theta}{q}} \begin{cases} k^{\frac{1}{q}-\frac{1}{p}} \left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta}, & q < p \\ \left(\frac{p}{q} R(\varepsilon_0)\right)^{-\frac{1}{q}} k^{-\frac{1}{p}} \left\|a^{(s)} - a^{(t)}\right\|_{p,q),\theta}, & p \leq q \end{cases} \\
&< h\left(\frac{1}{W(e^{-1})}\right)^{-\frac{\theta}{q}} \begin{cases} k^{\frac{1}{q}-\frac{1}{p}} \eta, & q < p \\ \left(\frac{p}{q} R(\varepsilon_0)\right)^{-\frac{1}{q}} k^{-\frac{1}{p}} \eta, & p \leq q \end{cases}
\end{aligned}$$

where $h(x) = x^{\frac{1}{1+x}}$, $R(x) = (1+x)^{-\frac{1}{1+x}}$. This shows that $\left(a_k^{(s)}\right)$ is a Cauchy sequence in $\mathbb{C}$. Thus, we have $(a_k) \in \mathbb{C}$ such that $\lim_{s\to\infty} \left|a_k^{(s)} - a_k\right| = 0$. By using the equality (1) with classical method, we get $\ell^{\theta}_{p,q)}$ is a complete space. □

Lemma 9. *Let $1 \leq p < \infty,\ 1 \leq q < q_1 \leq \infty$. Then, we have the following*

$$\ell^{\theta}_{p,q)} \subset \ell^{\theta}_{p,q_1)}.$$

Proof. Let $a = (a_n) \in \ell^{\theta}_{p,q)}$ and $p < q$. Then, we have by Proposition 2 in [11] that

$$\begin{aligned}
\|a\|_{p,q_1),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q_1}} \|a\|_{p,q_1(1+\varepsilon)} \\
&\leq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q_1}} \left(\frac{q(1+\varepsilon)}{p}\right)^{\frac{1}{q(1+\varepsilon)}-\frac{1}{q_1}} \|a\|_{p,q(1+\varepsilon)} \\
&\leq \left(\frac{q}{p}\left(1 + \frac{1}{W(e^{-1})}\right)\right)^{\frac{1}{q}-\frac{1}{q_1}} \|a\|_{p,q),\theta} \\
&< \infty.
\end{aligned}$$

where $h(x) = x^{\frac{1}{1+x}}$. The inclusion can be obtained by similar way for $p \geq q$ with Lemma 3. □

Theorem 10. *Let either* $1 \leq p < p_1 \leq \infty,\ 1 \leq q < \infty$ *or* $1 \leq p < p_1 < \infty,\ q = \infty$. *Then, the inclusion*

$$\ell^{\theta}_{p,q)} \subset \ell^{\theta}_{p_1,q)}$$

holds.

Proof. Let $a \in \ell^{\theta}_{p,q)}$. Then, we have

$$\begin{aligned} \|a\|_{p_1,q),\theta} &= \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p_1,q(1+\varepsilon)} \\ &\leq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)} \\ &= \|a\|_{p,q),\theta} \\ &< \infty \end{aligned}$$

which shows $a \in \ell^{\theta}_{p_1,q)}$. □

Corollary 11. *Let* $1 \leq p_1 < p \leq q < q_1 \leq \infty$. *Then, the inclusions*

$$\ell^{p_1),\theta} \subset \ell^{\theta}_{p,q)} \subset \ell^{q_1),\theta}$$

hold.

Theorem 12. *The grand Lorentz sequence space* $\ell^{\theta}_{p,q)}$ *is strictly convex for* $1 < p < \infty$ *and* $1 < q < \infty$.

Proof. Let $a, b \in \ell^{\theta}_{p,q)}$ such that $\|a\|_{p,q),\theta} = \|b\|_{p,q),\theta} = 1$ and $\left\|\frac{a+b}{2}\right\|_{p,q),\theta} = 1$. Then, we have by using similar technique as in [1] that

$$\begin{aligned} 1 &= \left\|\frac{a+b}{2}\right\|_{p,q),\theta} = \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \left\|\frac{a+b}{2}\right\|_{p,q(1+\varepsilon)} \\ &\leq \sup_{0<\varepsilon<\frac{1}{W(e^{-1})}} \varepsilon^{\frac{\theta}{q(1+\varepsilon)}} \left(\frac{\|a\|_{p,q(1+\varepsilon)} + \|b\|_{p,q(1+\varepsilon)}}{2}\right) \\ &\leq \left(\frac{\|a\|_{p,q),\theta} + \|b\|_{p,q),\theta}}{2}\right) \\ &= 1 \end{aligned}$$

which shows $a = b$. □

2.2. **Multiplication Operator.** In this section, we characterize some properties of the multiplication operators on $\ell^{\theta}_{p,q)}$. Let $v = (v_n)$ be a complex-valued sequence and let us define the linear transformation M_v on the sequence space X into the linear space of all complex-valued sequences by

$$M_v(x) = vx = (v_n x_n).$$

If the linear transformation M_v is bounded with range in X, then it is called *multiplication operator* on X.

Theorem 13. *Let $v = (v_n)$ be a complex-valued sequence. Then, M_v is a multiplication operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$ if and only if v is a bounded sequence.*

Proof. Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$ and let $q < \infty$. Then, there exists a positive number $K > 0$ such that

$$\|M_v(a)\|_{p,q),\theta} \leq K \|a\|_{p,q),\theta}$$

for all $a \in \ell^{\theta}_{p,q)}$. Let us define

$$e_n^{(k)} = \begin{cases} s^{-\frac{\theta}{p}}, & k = n \\ 0, & k \neq n \end{cases}$$

where $s = \left(\frac{1}{W(e^{-1})}\right)^{\frac{W(e^{-1})}{1+W(e^{-1})}}$ for all $n \in \mathbb{N}$. Then, the non-increasing rearrangement of $\left(e_n^{(k)}\right)$ is

$$e_{\phi(n)}^{(k)} = \begin{cases} s^{-\frac{\theta}{p}} & , n = 1 \\ 0 & , n \neq 1 \end{cases}.$$

Then, we have $\left(e_n^{(k)}\right) \in \ell^{\theta}_{p,q)}$ with $\left\|e^{(k)}\right\|_{p,q),\theta} = 1$. By the boundedness of M_v, it can be written $\left\|M_v e^{(k)}\right\|_{p,q),\theta} \leq K \left\|e^{(k)}\right\|_{p,q),\theta} = K$. Thus, we get

$$\begin{aligned} \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{n=1}^{\infty}\left(n^{\frac{1}{p(1+\varepsilon)}} v_{\psi(n)} e_{\psi(n)}^{(k)}\right)^{q(1+\varepsilon)} n^{-1}\right)^{\frac{1}{q(1+\varepsilon)}} &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(v_{\psi(1)} e_{\psi(1)}^{(k)}\right)^{q(1+\varepsilon)}\right)^{\frac{1}{q(1+\varepsilon)}} \\ &= s^{-\frac{\theta}{p}} \sup_{\varepsilon>0}\left(\varepsilon^{\frac{\theta}{q(1+\varepsilon)}} v_{\psi(1)}\right) \\ &\leq K \end{aligned}$$

which gives that $v_{\psi(1)} \leq K.s^{-\frac{\theta}{q}+\frac{\theta}{p}}$. This shows that v is bounded. If $q = \infty$, the proof is similar as was used in the classical Lorentz sequence spaces.

Conversely, let v be a bounded sequence. Then, there exists $T > 0$ such that $|v_k| \leq T$ for all $k \in \mathbb{N}$. Thus, we get

$$\|M_v a\|_{p,q),\theta} = \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=1}^{\infty}\left(k^{\frac{1}{p(1+\varepsilon)}} v_{\psi(k)} a_{\psi(k)}\right)^{q(1+\varepsilon)} k^{-1}\right)^{\frac{1}{q(1+\varepsilon)}}$$

$$\leq \quad T \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=1}^{\infty} \left(k^{\frac{1}{p(1+\varepsilon)}} a_{\psi(k)} \right)^{q(1+\varepsilon)} k^{-1} \right)^{\frac{1}{q(1+\varepsilon)}}$$
$$= \quad T \left\| a \right\|_{p,q),\theta}$$

for $q < \infty$. If $q = \infty$, then

$$\sup_{k \in \mathbb{N}} k^{\frac{1}{p}} v_{\psi(k)} a_{\psi(k)} \leq T \left\| a \right\|_{p,q),\theta} .$$

□

Theorem 14. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$. Then, M_v is invertible if and only if there exists $\mu > 0$ such that $|v_n| \geq \mu . s^{-\frac{\theta}{q} + \frac{2\theta}{p}}$, where $s = \left(\frac{1}{W(e^{-1})} \right)^{\frac{W(e^{-1})}{1+W(e^{-1})}}$ for all $n \in \mathbb{N}$.*

Proof. Let M_v be invertible operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$. Then, there exists $\rho > 0$ such that

$$\left\| M_v a \right\|_{p,q),\theta} \geq \mu \left\| a \right\|_{p,q),\theta}$$

for all $a \in \ell^{\theta}_{p,q)}$. Thus, for $\left(e^{(k)}_n \right) \in \ell^{\theta}_{p,q)}$, we get

$$\left\| M_v e^{(k)} \right\|_{p,q),\theta} = s^{\frac{\theta}{q} - \frac{\theta}{p}} \left| v_k \right| \geq \mu s^{\frac{\theta}{p}}$$

which gives $|v_k| \geq s^{-\frac{\theta}{q} + \frac{2\theta}{p}} \mu$. Conversely, let define $z_k = (v_k)^{-1}$. By using Theorem 5, the proof can be obtained. □

Theorem 15. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$, $1 \leq p, q \leq \infty$. Then, a necessary and sufficient condition for M_v to have closed range is that for some $\varrho > 0$*

$$|v_n| \geq \varrho$$

for each $n \in R = \{ n \in \mathbb{N} : v_n \neq 0 \}$.

Proof. Assume that $|v_n| \geq \varrho$ for $\varrho > 0$ and for all $n \in R$. Let $q < \infty$ and let $g^{(k)}, g \in \ell^{\theta}_{p,q)}$ such that $M_v g^{(k)} \to g$ as $k \to \infty$. Then, we write

$$\lim_{m,n \to \infty} \left\| M_v g^{(m)} - M_v g^{(n)} \right\|_{p,q),\theta} = 0.$$

Put $x^{(mn)} = g^{(m)} - g^{(n)}$. Thus, we have

$$\left\{ l \in \mathbb{N} : \left| x^{(mn)}_l \right| > \frac{r}{\varrho} \right\} \subseteq \left\{ l \in \mathbb{N} : \left| v_l x^{(mn)}_l \right| > r \right\}$$

for each $r > 0$ and so $\varrho x^{(mn)}_{\phi(l)} \leq v_{\psi(l)} x^{(mn)}_{\psi(l)}$, where $x^{(mn)}_{\phi(l)}$ and $v_{\psi(l)} x^{(mn)}_{\psi(l)}$ are the non-increasing rearrangement of the sequences $\left(x^{(mn)}_l \right)$ and $\left(v_l x^{(mn)}_l \right)$, respectively.

Thus, we have

$$\begin{aligned}\left\|vx^{(mn)}\right\|_{p,q),\theta} &= \left\|M_v g^{(m)} - M_v g^{(n)}\right\|_{p,q),\theta} \\ &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{l\in R}\left(l^{\frac{1}{p(1+\varepsilon)}} v_{\psi(l)} x^{(mn)}_{\psi(l)}\right)^{q(1+\varepsilon)} l^{-1}\right)^{\frac{1}{q(1+\varepsilon)}} \\ &\geq \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{l\in R}\varrho^{q(1+\varepsilon)}\left(l^{\frac{1}{p(1+\varepsilon)}} x^{(mn)}_{\phi(l)}\right)^{q(1+\varepsilon)} l^{-1}\right)^{\frac{1}{q(1+\varepsilon)}} \\ &= \varrho\left\|x^{(mn)}\right\|_{p,q),\theta}.\end{aligned}$$

Since $\left\|vx^{(mn)}\right\|_{p,q),\theta} \to 0$ as $m,n\to\infty$, we have $x^{(mn)}\to 0$ as $m,n\to\infty$. This means that $g^{(m)}$ is a Cauchy sequence in $\ell^{\theta}_{p,q)}|_R$, where $\ell^{\theta}_{p,q)}|_R = \left\{a=(a_k)\in \ell^{\theta}_{p,q)} : a_k = 0 \text{ if } k\in\mathbb{N}\backslash R\right\}$ is a closed subspace of $\ell^{\theta}_{p,q)}$. Thus, we get $f\in \ell^{\theta}_{p,q)}|_R$ such that $g^{(m)}\to f$ as $m\to\infty$. Since M_v is bounded linear operator, we can write $M_v g^{(m)} \to M_v f$. This gives $M_v f = g$. Because of $Ker\,(M_v) = \ell^{\theta}_{p,q)}|_{\mathbb{N}\backslash R}$, M_v has closed range.

Conversely, assume that M_v has closed range and there exists $(l_n)\in R$ such that $|v_{l_n}| < \frac{1}{n}$. Let

$$e^{(l_n)}_m = \begin{cases} s^{-\frac{\theta}{p}}, & m = l_n \\ 0, & m\neq l_n\end{cases}$$

where $s = \left(\frac{1}{W(e^{-1})}\right)^{\frac{W(e^{-1})}{1+W(e^{-1})}}$ and let $q<\infty$. Then, $\left\|e^{(l_n)}\right\|_{p,q),\theta} = 1$. Thus, we get

$$\begin{aligned}\left\|M_v e^{(l_n)}\right\|_{p,q),\theta} &= \left\|ve^{(l_n)}\right\|_{p,q),\theta} \\ &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{m=1}^{\infty}\left(m^{\frac{1}{p(1+\varepsilon)}} v_{\psi(m)} e^{(l_n)}_{\psi(m)}\right)^{q(1+\varepsilon)} m^{-1}\right)^{\frac{1}{q(1+\varepsilon)}} \\ &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(v_{\psi(1)} e^{(l_n)}_{\psi(1)}\right)^{q(1+\varepsilon)}\right)^{\frac{1}{q(1+\varepsilon)}} \\ &= s^{\frac{\theta}{p}-\frac{\theta}{q}} v_{l_n} \\ &< \frac{1}{n} s^{\frac{\theta}{p}-\frac{\theta}{q}}\left\|e^{(l_n)}\right\|_{p,q),\theta}\end{aligned}$$

which means M_v is not bounded different from zero. Thus, $|v_n|\geq\varrho$ for some $\varrho>0$ and all $n\in R$. For the case $q=\infty$ the proof can be obtained by similar way. □

Theorem 16. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$. Then, M_v is compact if and only if $|v_n|\to 0$ as $n\to\infty$.*

Proof. The proof can be obtained by the similar way used in the classical Lorentz sequence space. □

Corollary 17. *Let M_v be a multiplication operator on $\ell^{\theta}_{p,q)}$. Then, M_v is Fredholm if and only if the set $\mathbb{N}\backslash R$ has finite elements and there exists $\rho > 0$ such that*

$$|v_n| \geq \varrho$$

for all $n \in \mathbb{N}$, where $R = \{n \in \mathbb{N} : v_n \neq 0\}$.

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