Available online: March 17, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 782–793 (2020)
DOI: 10.31801/cfsuasmas.606591
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS SERIES A1

GENERALIZED FUZZY SUBHYPERSPACES BASED ON FUZZY POINTS

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Abstract. We define $(\in, \in \vee q^{\delta})$-fuzzy subhyperspaces and $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspaces, as a generalization of fuzzy subhyperspaces, $(\in, \in \vee q)$-fuzzy subhyperspaces and $(\in, \in \vee q_k)$-fuzzy subhyperspaces. In this way, we show that $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspaces are the largest family of generalized fuzzy subhyperspaces based on concepts of belongingness and quasi-coincidence. Moreover, we study some properties and investigate the difference of generalized fuzzy subhyperspaces, supported by examples.

1. Introduction

The theory of fuzzy set was initiated by Zadeh [23] in 1965. It was extended to algebra by Rosenfeld [17] with defining fuzzy subgroups. Then other fuzzy algebraic structures have been investigated, such as fuzzy semigroups, fuzzy ideals, fuzzy vector spaces and so on. For more information about fuzzy algebraic structures refer to [15] and [16].

Algebraic hyperstructures was introduced by Marty [14] in 1934, when he defined hypergroups. Similarly fuzzy algebraic hyperstructures were investigated in many branches ([8]). Ameri [1] introduced fuzzy subhyperspaces of hypervector spaces in the sense of Scafatti-Tallini [18]. Fuzzy subhyperspaces were studied more in [2], [3] and [13].

After defining the concept of $(\in, \in \vee q)$-fuzzy subgroups by Bhakat and Das [4] as an important generalization of Rosenfeld's fuzzy subgroups, this notion and its another type, $(\in, \in \vee q_k)$-fuzzy subgroups, were studied on many algebraic structures (see [9]). In context of hyperstructures theory, as an extension of fuzzy subhyperstructures, Davvaz and Corsini defined $(\in, \in \vee q)$-fuzzy subhyperquasigroups in [5]. Furthermore, semihypergroups were characterized by $(\in, \in \vee q_k)$-fuzzy hyperideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals in [19] and [20], respectively. Moreover,

Received by the editors: August 19, 2019; Accepted: February 22, 2020.

2010 *Mathematics Subject Classification.* 08A72, 20N20.

Key words and phrases. Hypervector space, fuzzy subhyperspace, $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace, $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace.

$(\in, \in \vee q)$-fuzzy n-ary subhypergroups in [12] and [22], $(\in, \in \vee q)$-fuzzy and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary subpolygroups in [11] and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroups in [21], had been studied. Also, this concept and related topics were investigated on H_v-rings in [6], hypermodules in [24] and (m,n)-ary hypermodules in [7].

A new generalization of $(\in, \in \vee q)$-fuzzy subgroups was defined by Jun et al. ([10]) which called $(\in, \in \vee q^\delta)$-fuzzy subgroups. Now, in this paper, we introduce new generalizations of a fuzzy subhyperspace. In this regards, $(\in, \in \vee q^\delta)$-fuzzy subhyperspaces and $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspaces as generalizations of fuzzy subhyperspaces, $(\in, \in \vee q)$-fuzzy subhyperspaces and also $(\in, \in \vee q_k)$-fuzzy subhyperspaces are defined. It is shown that these notions construct a bigger family for generalized fuzzy subhyperspaces and also indicated that subhyperspaces are characterized by them. Moreover, connections and differences of them are studied, supported by illustrative examples.

2. Preliminaries

In this section we present some definitions and properties of hypervector spaces and fuzzy subhyperspaces that we shall use in later.

Definition 1. [18] *Let K be a field, $(V,+)$ be an Abelian group and $P_*(V)$ be the set of all non-empty subsets of V. We define a hypervector space over K to be the quadruplet $(V,+,\circ,K)$, where "$\circ$" is an external hyperoperation*

$$\circ : K \times V \longrightarrow P_*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

- (H_1) $a \circ (x+y) \subseteq a \circ x + a \circ y$, *right distributive law,*
- (H_2) $(a+b) \circ x \subseteq a \circ x + b \circ x$, *left distributive law,*
- (H_3) $a \circ (b \circ x) = (ab) \circ x$,
- (H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
- (H_5) $x \in 1 \circ x$,

where in (H_1), $a \circ x + a \circ y = \{p + q : p \in a \circ x, q \in a \circ y\}$. Similarly it is in (H_2). Also in (H_3), $a \circ (b \circ x) = \bigcup\limits_{t \in b \circ x} a \circ t$.

V is called strongly right distributive, if we have equality in (H_1). In a similar way we define the strongly left distributive hypervector spaces. V is called strongly distributive, if it is strongly right and left distributive.

A non-empty subset W of V is called a subhyperspace of V if W is itself a hypervector space with the external hyperoperation on V, i.e. for all $a \in K$ and $x, y \in W$, $x - y \in W$ and $a \circ x \subseteq W$.

In the sequel of this paper, V denotes a hypervector space over the field K, unless otherwise is specified.

Example 2. [2] *In classical vector space* $(\mathbb{R}^3, +, ., \mathbb{R})$ *we define:*

$$\begin{cases} \circ: & \mathbb{R} \times \mathbb{R}^3 \longrightarrow P_*(\mathbb{R}^3) \\ & a \circ (x_0, y_0, z_0) = L, \end{cases}$$

where L *is a line with the parametric equations:*

$$L: \begin{cases} x = ax_0, \\ y = ay_0, \\ z = t. \end{cases}$$

Then $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ *is a strongly left distributive hypervector space.*

Definition 3. [1] *A fuzzy subset* μ *of* V *is called a fuzzy subhyperspace of* V*, if for all* $a \in K$ *and* $x, y \in V$*, the following conditions are satisfied:*

1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
2) $\bigwedge_{t \in a \circ x} \mu(t) \geq \mu(x)$.

Example 4. *(modified example 2.16, of* [3]*) Consider the hypervector space* $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ *in Example 2. Define a fuzzy subset* μ *of* V *by the following:*

$$\mu(x, y, z) = \begin{cases} t_3 & (x, y, z) \in \{0\} \times \{0\} \times \mathbb{R}, \\ t_2 & (x, y, z) \in \mathbb{R} \times \{0\} \times \mathbb{R} \backslash \{0\} \times \{0\} \times \mathbb{R}, \\ t_1 & otherwise, \end{cases}$$

where $0 \leq t_1 < t_2 < t_3 \leq 1$*. Then* μ *is a fuzzy subhyperspace of* V*.*

3. (α, β)-Fuzzy Subhyperspaces

A fuzzy subset μ of a hypervector space V defined by

$$\mu(y) = \begin{cases} t \ (\neq 0), & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with the *support* x and the *value* t and is denoted by x_t. For a fuzzy point x_t and the fuzzy subset μ we write
(1) $x_t \in \mu \Leftrightarrow \mu(x) \geq t$.
(2) $x_t q \mu \Leftrightarrow \mu(x) + t > 1$.
(3) $x_t q_k \mu \Leftrightarrow \mu(x) + t + k > 1$, for $k \in [0, 1)$.
(4) $x_t q^\delta \mu \Leftrightarrow \mu(x) + t > \delta$, for $\delta \in (0, 1]$.
(5) $x_t q_k^\delta \mu \Leftrightarrow \mu(x) + t + k > \delta$, for $(k, \delta) \in [0, 1) \times (0, 1]$.
In case (1) we say that x_t is belong to μ and in (2) x_t is quasi-coincident with the fuzzy subset μ. For a fuzzy point x_t, we write $x_t \in \vee q\mu$ ($x_t \in \wedge q\mu$) if $x_t \in \mu$ or $x_t q\mu$ ($x_t \in \mu$ and $x_t q\mu$). Similarly, we have $x_t \in \vee q_k \mu$ and $x_t \in \wedge q_k \mu$. Also, for $\alpha \in \{\in, q, q_k, \in \vee q, \in \wedge q, \ldots\}$, the notation $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold.

Definition 5. *A fuzzy subset* μ *of* V *is called an* $(\in, \in \vee q_k^\delta)$*-fuzzy subhyperspace of* V*, if for all* $t, r \in [0, 1)$*,* $x, y \in V$ *and* $a \in K$*:*

$k\delta 1)$ $x_t \in \mu$ *and* $y_r \in \mu$ *imply that* $(x - y)_{t \wedge r} \in \vee q_k^\delta \mu$*;*

$k\delta 2)$ *$x_t \in \mu$ implies that $z_t \in \vee q_k^{\delta}\mu$, for all $z \in a \circ x$.*

μ is called an $(\in, \in \vee q)$-fuzzy subhyperspace of V, if $\delta = 1$ and $k = 0$. It is called an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V, if $\delta = 1$. Also, μ is called an $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace of V, if $k = 0$.

Theorem 6. *A fuzzy subset μ of V is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$ the following conditions hold:*

$k\acute{\delta}1)$ $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$;

$k\acute{\delta}2)$ $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2}$.

Proof. Let μ be an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V. Assume that $(k\acute{\delta}1)$ is not valid, i.e. there exist $x, y \in V$ such that $\mu(x-y) < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$. Then $\mu(x-y) < t \leq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for some $t \in (0,1]$. Thus $t \leq \mu(x)$ and $t \leq \mu(y)$, and so $x_t, y_t \in \mu$. Hence $(x-y)_t \in \vee q_k^{\delta}\mu$. But $\mu(x-y) < t$ and also $\mu(x-y) + t < t + t \leq \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. It follows that $\mu(x-y) + t + k \leq \delta$. Therefore, $(x-y)_t \overline{\in \vee q_k^{\delta}}\mu$, which is a contradiction. Consequently, $(k\acute{\delta}1)$ is valid.

Now if there exist some $x, z \in V$ and $a \in K$, such that $z \in a \circ x$ and $\mu(z) < \mu(x) \wedge \frac{\delta-k}{2}$, then $\mu(z) < t \leq \mu(x) \wedge \frac{\delta-k}{2}$, for some $t \in (0,1]$. Thus $t \leq \mu(x)$ and so $x_t \in \mu$. Hence $z_t \in \vee q_k^{\delta}\mu$. But $\mu(z) < t$ and also $\mu(z)+t < t+t \leq \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. Thus $\mu(z)+t+k \leq \delta$. Therefore $z_t \overline{\in \vee q_k^{\delta}}\mu$, which is a contradiction. Consequently, $(k\acute{\delta}2)$ is valid.

Conversely, let $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Thus by $(k\acute{\delta}1)$, $\mu(x-y) \geq t \wedge r \wedge \frac{\delta-k}{2}$. If $t \wedge r \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t \wedge r$ and so $(x-y)_{t \wedge r} \in \mu$. If $t \wedge r > \frac{\delta-k}{2}$, then $\mu(x-y) \geq \frac{\delta-k}{2}$ and $\mu(x-y) + (t \wedge r) > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$. Thus $\mu(x-y) + (t \wedge r) + k > \delta$. Hence $(x-y)_{t \wedge r} q_k^{\delta}\mu$. Therefore $(x-y)_{t \wedge r} \in \vee q_k^{\delta}\mu$. Similarly, $(k\acute{\delta}2)$ implies $(k\delta 2)$. □

Corollary 7. *Let $\mu \in FS(V)$, i.e. μ is a fuzzy subset of V. Then μ is*

1) *an $(\in, \in \vee q)$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge 0.5$;*
2) *an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{1-k}{2}$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{1-k}{2}$;*
3) *an $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace of V if and only if for all $x, y \in V$ and $a \in K$, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta}{2}$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta}{2}$.*

Example 8. *Consider the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 2. Define a fuzzy subset μ of V by the following:*

$$\mu(x,y,z) = \begin{cases} \frac{1}{2} & (x,y,z) = (0,0,0), \\ \frac{1}{3} & (x,y,z) \in \mathbb{R} \times \{0\} \times \{0\} \setminus (0,0,0), \\ \frac{1}{5} & o.w. \end{cases}$$

Then μ is an $(\in, \in \vee q_{0.5}^{0.7})$-fuzzy subhyperspace of V, but it is not an $(\in, \in \vee q_{0.3}^{0.8})$-fuzzy subhyperspace of V, since the condition $(k\acute{\delta}2)$ is not valid (if $x = (0,0,0)$ and $z = (0,0,2)$, then for all $a \in \mathbb{R}$, $z \in a \circ x$, so $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{\delta-k}{2}$).

In the next example one can see that an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace is not an $(\in, \in \vee q)$-fuzzy subhyperspace or $(\in, \in \vee q_k)$-fuzzy subhyperspace or $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace of V, in general.

Example 9. *Consider the $(\in, \in \vee q_{0.5}^{0.7})$-fuzzy subhyperspace μ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8. Then by Corollary 7, it follows that:*

1) *μ is not an $(\in, \in \vee q)$-fuzzy subhyperspace of V, because if $x = (0,0,0)$ and $z = (0,0,3)$, then for all $a \in \mathbb{R}$ and $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge 0.5$;*
2) *μ is not an $(\in, \in \vee q_{0.5})$-fuzzy subhyperspace of V, because if $x = (0,0,0)$ and $z = (0,0,2)$, then for all $a \in \mathbb{R}$ and $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{1-k}{2}$;*
3) *μ is not an $(\in, \in \vee q^{0.7})$-fuzzy subhyperspace of V, because if $x = (1,0,0)$, $a = 2$ and $z = (2,0,5)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{\delta}{2}$.*

Theorem 10. *Let $\mu \in FS(V)$, $\delta \in (0,1]$ and $k \in [0,1)$. Then μ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V if and only if $\mu_t (\neq \emptyset)$ is a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$.*

Proof. Suppose μ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V, $t \in (0, \frac{\delta-k}{2}]$ and $x, y \in \mu_t$. Then by Theorem 6,

$$\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2} \geq t \wedge t \wedge \frac{\delta-k}{2} = t.$$

Thus $x - y \in \mu_t$. Moreover, for all $a \in K$, $z \in a \circ x$ and $x \in \mu_t$, we have $\mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2} \geq t$, which means that $a \circ x \subseteq \mu_t$. Hence μ_t is a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$.

Conversely, let μ_t be a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$ and let $(k\acute{\delta}1)$ is not valid. Then there exist $x, y \in V$ and $t \in (0,1)$ such that

$$\mu(x-y) < t < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}.$$

Thus $x, y \in \mu_t$ for some $0 < t \leq \frac{\delta-k}{2}$, but $x - y \notin \mu_t$, which is a contradiction. Hence $(k\acute{\delta}1)$ is valid. Similarly, we can show $(k\acute{\delta}2)$ is valid. Therefore, by Theorem 6, μ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V. $\square$

Corollary 11. *Let $\mu \in FS(V)$. Then μ is*

1) *an $(\in, \in \vee q)$-fuzzy subhyperspace of V if and only if $\mu_t (\neq \emptyset)$ is a subhyperspace of V, for all $t \in (0, 0.5]$;*

2) *an* $(\in, \in \vee q_k)$*-fuzzy subhyperspace of* V *if and only if* $\mu_t (\neq \emptyset)$ *is a subhyperspace of* V*, for all* $t \in (0, \frac{1-k}{2}]$*;*
3) *an* $(\in, \in \vee q^{\delta})$*-fuzzy subhyperspace of* V *if and only if* $\mu_t (\neq \emptyset)$ *is a subhyperspace of* V*, for all* $t \in (0, \frac{\delta}{2}]$.

Theorem 12. *Let* $\mu \in FS(V)$*. Then* $\mu_t (\neq \emptyset)$ *is a subhyperspace of* V*, for all* $t \in (\frac{\delta-k}{2}, 1]$*, if and only if*

(*i*) $\mu(x-y) \vee \frac{\delta-k}{2} \geq \mu(x) \wedge \mu(y)$*, for all* $x, y \in V$*;*
(*ii*) $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee \frac{\delta-k}{2} \geq \mu(x)$*, for all* $x \in V$ *and* $a \in K$*.*

Proof. Let $\mu_t (\neq \emptyset)$ be a subhyperspace of V, for all $t \in (\frac{\delta-k}{2}, 1]$. If there exist $x, y \in V$ such that

$$\mu(x-y) \vee \frac{\delta-k}{2} < \mu(x) \wedge \mu(y),$$

then $t_0 = \mu(x) \wedge \mu(y) \in (\frac{\delta-k}{2}, 1]$ and $x, y \in \mu_{t_0}$. Thus $x-y \in \mu_{t_0}$ and so $\mu(x-y) \geq t_0$, which is a contradiction with $\mu(x-y) \vee \frac{\delta-k}{2} < t_0$. Hence (i) holds. Similarly, condition (ii) will be obtained.

Conversely, assume that $t \in (\frac{\delta-k}{2}, 1]$ and $x, y \in \mu_t$. Then

$$\mu(x-y) \vee \frac{\delta-k}{2} < \mu(x) \wedge \mu(y) \geq t > \frac{\delta-k}{2}.$$

Thus $\mu(x-y) \geq t$ and so $x-y \in \mu_t$. Now let $a \in K$, $x \in \mu_t$ and $z \in a \circ x$. Then

$$\mu(z) \vee \frac{\delta-k}{2} \geq \bigwedge_{z \in a \circ x} \mu(z) \vee \frac{\delta-k}{2} \geq \mu(x) \geq t,$$

which implies that $\mu(z) \geq t$, for all $z \in a \circ x$. Hence $a \circ x \subseteq \mu_t$, for all $t \in (\frac{\delta-k}{2}, 1]$. Therefore, μ_t is a subhyperspace of V. □

Corollary 13. *Let* $\mu \in FS(V)$*. Then*

1) $\mu_t (\neq \emptyset)$ *is a subhyperspace of* V *for all* $t \in (0.5, 1]$ *if and only if for all* $x, y \in V$ *and* $a \in K$*,* $\mu(x-y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ *and* $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee 0.5 \geq \mu(x)$*;*
2) $\mu_t (\neq \emptyset)$ *is a subhyperspace of* V *for all* $t \in (\frac{1-k}{2}, 1]$ *if and only if for all* $x, y \in V$ *and* $a \in K$*,* $\mu(x-y) \vee \frac{1-k}{2} \geq \mu(x) \wedge \mu(y)$ *and* $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee \frac{1-k}{2} \geq \mu(x)$*;*
3) $\mu_t (\neq \emptyset)$ *is a subhyperspace of* V *for all* $t \in (\frac{\delta}{2}, 1]$ *if and only if for all* $x, y \in V$ *and* $a \in K$*,* $\mu(x-y) \vee \frac{\delta}{2} \geq \mu(x) \wedge \mu(y)$ *and* $\bigwedge\limits_{z \in a \circ x} \mu(z) \vee \frac{\delta}{2} \geq \mu(x)$.

Theorem 14. *A non-empty subset* S *of* V *is a subhyperspace of* V *if and only if* χ_S *is an* $(\in, \in \vee q_k^{\delta})$*-fuzzy subhyperspace of* V*.*

Proof. Let S be a subhyperspace of V and $t \in (0, \frac{\delta-k}{2}]$. If $x, y \in \chi_{S_t}$, then $\chi_S(x), \chi_S(y) \geq t$. Thus $\chi_S(x) = \chi_S(y) = 1$ and so $x - y \in S$. Hence $\chi_S(x-y) = 1 \geq t$, i.e. $x - y \in \chi_{S_t}$. Similarly, $a \circ x \subseteq \chi_{S_t}$, for all $a \in K$ and $x \in \chi_{S_t}$. Consequently, χ_{S_t} is a subhyperspace of V, for all $t \in (0, \frac{\delta-k}{2}]$. Therefore, by Theorem 10, χ_S is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V.

Conversely, let χ_S be an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V, $a \in K$ and $x \in S$. Then by Theorem 6, for all $z \in a \circ x$, $\bigwedge\limits_{z \in a \circ x} \chi_S(z) \geq \chi_S(x) \wedge \frac{\delta-k}{2} = 1 \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2}$. Since $\delta \in (0, 1]$ and $k \in [0, 1)$, so $\chi_S(z) = 1$, for all $z \in a \circ x$. Thus $a \circ x \subseteq S$. Similarly, $x + y \in S$, for all $x, y \in S$. Therefore, S is a subhyperspace of V. □

It is well-known that the characterization function of any subhyperspace is a fuzzy subhyperspace. Hence the following corollary is obtained from Theorem 14:

Corollary 15. *A non-empty subset S of V is a subhyperspace of V if and only if χ_S is an $(\in, \in \vee q)$-fuzzy subhyperspace of V if and only if χ_S is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V if and only if χ_S is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V.*

Proposition 16. *Let $\delta \in (0, 1]$, $k \in [0, 1)$. Then*

1) *Every $(\in, \in \vee q_k)$-fuzzy subhyperspace of V is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V, if $\delta + k < 1$;*
2) *Every $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V, if $\delta + k > 1$;*
3) *For $\delta + k = 1$, μ is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V if and only if it is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V;*
4) *Every $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V is an $(\in, \in \vee q)$-fuzzy subhyperspace of V, if $\delta = k + 1$;*
5) *Every $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V is a fuzzy subhyperspace of V, if $\delta = k$.*

Proof. 1) Let μ be an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V. Then by Corollary 7(2), for all $x, y \in V$ and $a \in K$, it follows that:

$$\mu(x - y) \geq \mu(x) \wedge \mu(y) \wedge \frac{1-k}{2} \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta}{2},$$

and

$$\bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{1-k}{2} \geq \mu(x) \wedge \frac{\delta}{2}.$$

Thus by Corollary 7(3), μ is an $(\in, \in \vee q^\delta)$-fuzzy subhyperspace of V.

2) The proof is completed by Corollary 7, similarly.

3) One can conclude by Corollary 7(2) and Corollary 7(3).

4) It is straightforward by Theorem 6 and Corollary 7(1).

5) The proof is obtained by Theorem 6 and Definition 3. □

The following example shows that the converse of assertions (1) and (2) of Proposition 16 are not generally true.

Example 17. *Consider the fuzzy subset μ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8. Then by Corollary 7, it follows that:*

1) *μ is an $(\in, \in \vee q^{0.2})$-fuzzy subhyperspace of V, but it is not an $(\in, \in \vee q_{0.4})$-fuzzy subhyperspace of V, because if $x = (2, 0, 0)$, $a = 4$ and $z = (8, 0, 3)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{1-k}{2}$ ($\delta = 0.2,\ k = 0.4$).*
2) *μ is an $(\in, \in \vee q_{0.7})$-fuzzy subhyperspace of V, but it is not an $(\in, \in \vee q^{0.5})$-fuzzy subhyperspace of V, because if $x = (3, 0, 0)$, $a = 1$ and $z = (3, 0, 2)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \ngeq \mu(x) \wedge \frac{\delta}{2}$ ($\delta = 0.5,\ k = 0.7$).*

Theorem 18. *If μ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subhyperspace of V, such that $\mu(x) \leq \frac{\delta-k}{2}$, for all $x \in V$, then μ is a fuzzy subhyperspace of V.*

Proof. By Theorem 6, for all $x, y \in V$ and $a \in K$, $\mu(x - y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2} = \mu(x) \wedge \mu(y)$ and $\bigwedge\limits_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta-k}{2} = \mu(x)$. So μ is a fuzzy subhyperspace of V. □

Note that in the $(\in, \in \vee q_{0.5}^{0.7})$-fuzzy subhyperspace μ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8, $\mu(x) \nleq \frac{\delta-k}{2}$, for some $x \in V$ and μ is not a fuzzy subhyperspace of V.

Corollary 19. *Let $\mu \in FS(V)$. Then*

1) *If μ is an $(\in, \in \vee q)$-fuzzy subhyperspace of V, such that $\mu(x) < 0.5$, for all $x \in V$, then μ is a fuzzy subhyperspace of V;*
2) *If μ is an $(\in, \in \vee q_k)$-fuzzy subhyperspace of V, such that $\mu(x) < \frac{1-k}{2}$, for all $x \in V$, then μ is a fuzzy subhyperspace of V;*
3) *If μ is an $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace of V, such that $\mu(x) < \frac{\delta}{2}$, for all $x \in V$, then μ is a fuzzy subhyperspace of V.*

Proposition 20. *Let $0 < \delta_2 \leq \delta_1 \leq 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q_k^{\delta_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q_k^{\delta_2})$-fuzzy subhyperspace of V.*

Proof. By Theorem 6, $\mu(x - y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta_1-k}{2}$ and $\mu(z) \geq \mu(x) \wedge \frac{\delta_1-k}{2}$, for all $x, y \in V$, $a \in K$ and $z \in a \circ x$. Since $\delta_1 \geq \delta_2$, thus $\mu(x-y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta_2-k}{2}$ and $\mu(z) \geq \mu(x) \wedge \frac{\delta_2-k}{2}$. Hence the proof is completed by Theorem 6. □

In the following example it can be seen that the converse of Proposition 20, is not generally valid.

Example 21. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q_{0.5}^{0.8})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q_{0.5}^{0.95})$-fuzzy subhyperspace of V, while $\delta_2 = 0.8 \leq \delta_1 = 0.95$.*

Corollary 22. *Let $0 < \delta_2 \leq \delta_1 \leq 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q^{\delta_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q^{\delta_2})$-fuzzy subhyperspace of V.*

Proof. It is straightforward by Corollary 7 and Proposition 20. □

Next example shows that the converse of Corollary 22, is not valid, in general.

Example 23. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q^{0.3})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q^{0.5})$-fuzzy subhyperspace of V, while $\delta_2 = 0.3 \leq \delta_1 = 0.5$.*

Proposition 24. *Let $0 \leq k_1 \leq k_2 < 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q^{\delta}_{k_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q^{\delta}_{k_2})$-fuzzy subhyperspace of V.*

Proof. It is completed by a similar manner of the proof of Proposition 20. □

The converse of Proposition 24, is not valid in general. See the following example:

Example 25. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q^{0.7}_{0.4})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q^{0.7}_{0.2})$-fuzzy subhyperspace of V, while $k_1 = 0.2 \leq k_2 = 0.4$.*

The following corollary is immediately concluded by Corollary 7 and Proposition 24.

Corollary 26. *Let $0 \leq k_1 \leq k_2 < 1$ and $\mu \in FS(V)$. If μ is an $(\in, \in \vee q_{k_1})$-fuzzy subhyperspace of V, then it is an $(\in, \in \vee q_{k_2})$-fuzzy subhyperspace of V.*

The converse of Corollary 26, is not valid in general. See the next example:

Example 27. *Consider the fuzzy subset μ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then μ is an $(\in, \in \vee q_{0.7})$-fuzzy subhyperspace of V and μ is not an $(\in, \in \vee q_{0.5})$-fuzzy subhyperspace of V, while $k_1 = 0.5 \leq k_2 = 0.7$.*

Theorem 28. *Let S be a subhyperspace of V. Then for every $t \in (0, \frac{\delta - k}{2}]$, there exists an $(\in, \in \vee q^{\delta}_{k})$-fuzzy subhyperspace μ of V such that $\mu_t = S$.*

Proof. Let $t \in (0, \frac{\delta - k}{2}]$ and define a fuzzy subset μ of V as

$$\mu(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\mu_t = S$. Now if there exist $x, y \in V$ such that $\mu(x-y) < \mu(x) \wedge \mu(y) \wedge \frac{\delta - k}{2}$, then $\mu(x - y) = 0$ and $\mu(x) = \mu(y) = t$, which is a contradiction. Thus $\mu(x - y) \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta - k}{2}$, for all $x, y \in V$. Similarly, $\bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{\delta - k}{2}$, for all $a \in K$. Hence μ is an $(\in, \in \vee q^{\delta}_{k})$-fuzzy subhyperspace of V, by Theorem 6. □

Corollary 29. *Let S be a subhyperspace of V. Then*

1) *For every $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$-fuzzy subhyperspace μ of V, such that $\mu_t = S$;*
2) *For every $t \in (0, \frac{1-k}{2}]$, there exists an $(\in, \in \vee q_k)$-fuzzy subhyperspace μ of V, such that $\mu_t = S$;*
3) *For every $t \in (0, \frac{\delta}{2}]$, there exists an $(\in, \in \vee q^{\delta})$-fuzzy subhyperspace μ of V, such that $\mu_t = S$.*

Theorem 30. *If μ_i is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V, for all $i \in I$, then $\mu = \cap_{i\in I}\mu_i$ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V.*

Proof. Let $x_t, y_r \in \mu$, for $x, y \in V$ and $t, r \in (0,1)$ and $(x-y)_{t\wedge r} \overline{\in \vee q_k^\delta}\mu$. Then $\mu(x-y) < t \wedge r$ and $\mu(x-y) + t \wedge r \leq \delta - k$, which imply that $\mu(x-y) < \frac{\delta-k}{2}$. Now, put $I_1 = \{i \in I \mid (x-y)_{t\wedge r} \in \mu_i\}$ and $I_2 = \{i \in I \mid (x-y)_{t\wedge r} q_k^\delta \mu_i\} \cap \{j \in I \mid (x-y)_{t\wedge r} \overline{\in} \mu_j\}$. Then $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. If $I_2 = \emptyset$, then $(x-y)_{t\wedge r} \in \mu_i$, for all $i \in I$, which implies that $\mu(x-y) \geq t \wedge r$, that is a contradiction. Hence $I_2 \neq \emptyset$, and so for every $i \in I_2$, $\mu_i(x-y) < t \wedge r$ and $\mu_i(x-y) + t \wedge r > \delta - k$, that is $t \wedge r > \frac{\delta-k}{2}$. Thus from $x_t, y_r \in \mu$, we can obtain $\mu_i(x) \wedge \mu_i(y) > \mu(x) \wedge \mu(y) > t \wedge r > \frac{\delta-k}{2}$. Now, set $\alpha = \mu_i(x-y) < \frac{\delta-k}{2}$ and take $\alpha < \beta < \frac{\delta-k}{2}$. Then $x_\alpha, y_\beta \in \mu_i$ but $\mu_i(x-y) = \alpha < \beta$ and $\mu_i(x-y) + \beta < \delta - k$. This contradicts that μ_i is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V. Thus $\mu_i(x-y) \geq \frac{\delta-k}{2}$, which is a contradiction. Hence $(x-y)_{t\wedge r} \in \vee q_k^\delta \mu$. Similarly, the condition $(k\delta 2)$ will be proven. Therefore, $\mu = \cap_{i\in I}\mu_i$ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V. □

For any fuzzy set μ of V and $t \in (0,1]$, we denote

$$(\mu)_t = \{x \in V \mid x_t\ q_k^\delta\ \mu\} \quad \text{and} \quad [\mu]_t = \{x \in V \mid x_t\ \in \vee q_k^\delta\ \mu\}.$$

Obviously, $[\mu]_t = \mu_t \cup (\mu)_t$.

Theorem 31. *Let $\mu \in FS(V)$. Then μ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V if and only if $[\mu]_t$ is a subhyperspace of V, for all $t \in (0,1]$.*

Proof. let μ be an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V and $x, y \in [\mu]_t$, for $t \in (0,1]$. Then $\mu(x) \geq t$ or $\mu(x) + t > \delta - k$, and $\mu(y) \geq t$ or $\mu(y) + t > \delta - k$. Using Theorem 6,

1) for $\mu(x), \mu(y) \geq t$, $\mu(x-y) \geq t \wedge \frac{\delta-k}{2}$. If $t > \frac{\delta-k}{2}$, then $\mu(x-y) + t > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, and so $(x-y)_t q_k^\delta \mu$. If $t \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t$ and so $(x-y)_t \in \mu$. Thus $x - y \in [\mu]_t$. Similarly, we can show $a \circ x \subseteq [\mu]_t$, for all $a \in K$, in this case.

2) for $\mu(x) \geq t$ and $\mu(y) + t > \delta - k$, $\mu(x-y) \geq t \wedge \delta - k - t \wedge \frac{\delta-k}{2}$. If $t > \frac{\delta-k}{2}$, then $\mu(x-y) > \delta - k - t$, and so $(x-y)_t q_k^\delta \mu$. If $t \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t$ and so $(x-y)_t \in \mu$. Thus $x - y \in [\mu]_t$. Similarly, we can show $a \circ x \subseteq [\mu]_t$, for all $a \in K$, in this case.

3) for $\mu(x) + t > \delta - k$ and $\mu(y) \geq t$, we can prove similar to the case (2).

4) for $\mu(x) + t > \delta - k$ and $\mu(y) + t > \delta - k$, if $t > \frac{\delta-k}{2}$, then $\mu(x-y) > \delta - k - t$, and if $t \leq \frac{\delta-k}{2}$, then $\mu(x-y) \geq t$. Thus $(x-y)_t \in \vee q_k^\delta \mu$ and so $x - y \in [\mu]_t$. Similarly, $a \circ x \subseteq [\mu]_t$, for all $a \in K$.

Therefore, $[\mu]_t$ is a subhyperspace of V.

Conversely, let μ be a fuzzy subset of V and there exist $x, y \in V$ such that $\mu(x-y) < \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for $\delta \in (0,1]$ and $k \in [0,1)$. Then $\mu(x-y) < t \geq \mu(x) \wedge \mu(y) \wedge \frac{\delta-k}{2}$, for some $t \in (0,1)$. Thus $x, y \in [\mu]_t$ and so $x - y \in [\mu]_t$. But

$\mu(x-y) < t$ and $\mu(x-y)+t \geq \delta - k$, which is a contradiction. Hence (1) of Theorem 6, and similarly the assertion (2) of Theorem 6, are valid. Therefore, μ is an $(\in, \in \vee q_k^\delta)$-fuzzy subhyperspace of V. □

4. Conclusion

We define $\lambda(x)+t > \delta$ and $\lambda(x)+t+k > \delta$ as new connections between a fuzzy point and a fuzzy subset on a hypervector space to generalize the concept of fuzzy subhyperspaces. These new connections help us to find new generalizations for fuzzy subhyperspaces and specially the largest family of them based on the concepts of belongingness and quasi-coincidence. This study can be extended to other algebraic structures and hyperstructures, in future. The following figure shows how we extend the family of generalized fuzzy subhyperspaces:

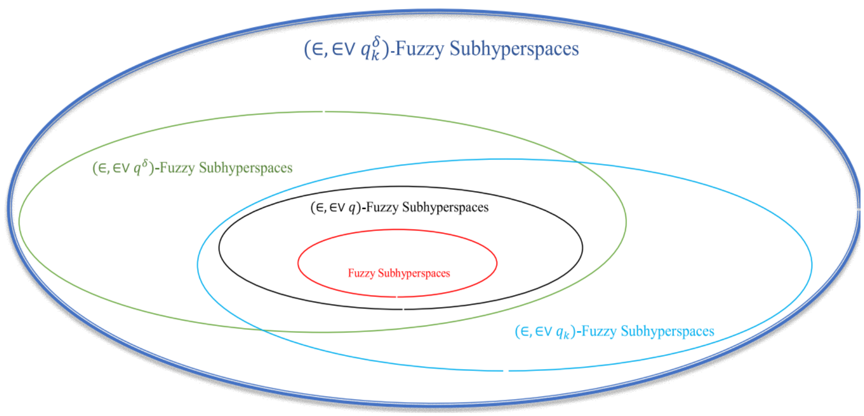

Figure 1. Generalizations of fuzzy subhyperspces

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