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# ON SOME NEW $FK$ SPACES OBTAINED FROM SUMMABILITY MATRIX

MAHMUT KARAKUŞ AND TUNAY BİLGİN

Abstract. In this study, we give some new $FK$-spaces by means of an infinite matrix as an operator and define some new $\beta$- and $\gamma$-type duality of sequence spaces [4, 6]. We also introduce some new sections and investigate some properties like $AB$-, $FAK$-, $SAK$- and $AK$- in an $FK$-space. By this way, we obtain some new distinguished subspaces of an $FK$-space [7]. Among other results, we prove that the sum of finite numbers of $FK$-spaces and the intersection of a sequence of $FK$-spaces which have these new properties with corresponding paranorms have also these new properties. The reader can refer to [2] and [19] for the main results and related topics in $FK$-space theory.

## 1. Preliminaries and Notation

The space of all scalar valued sequences is given by $\omega$ and a $K$ space is a locally convex sequence space (lcss) $\lambda$ containing $\phi$ and a subspace of $\omega$ on which coordinate functionals $\pi_k(x) = x_k$ are continuous for every $k \in \mathbb{N}$. Here $\phi$ is the space of finitely non-zero sequences spanned by $\{(\delta^k) : k \in \mathbb{N}\}$ which is the space of sequences whose $k$th position is 1 and all the others are 0. A complete linear metric (or complete normed linear) $K$ space is called an $FK$ (or $BK$) space.

The multipliers from $\lambda$ into $\mu$ are given by $\lambda^\mu = \{y \in \omega | xy \in \mu, \forall x \in \lambda\}$ for $\lambda, \mu \subset \omega$, where $xy$ is the coordinatewise product, i.e., $xy = \{x_k y_k\}_{k\in\mathbb{N}}$. We notates $(\lambda^\mu)^\nu = \lambda^{\mu\nu} = \{\ y \in \omega | xy \in \nu, \forall x \in \lambda^\mu\}$ for $\lambda, \mu, \nu \subset \omega$. A sequence space $\lambda$ is called $\mu-$perfect if $\lambda = \lambda^{\mu\mu}$. Classical $\alpha-, \beta-$ and $\gamma-$ duals of $\lambda$ are given by $\lambda^\ell$, $\lambda^{cs}$ and $\lambda^{bs}$, respectively, where $\ell = \{(x_k) \in \omega : ||x||_1 = \sum_k |x_k| < \infty\}$, $cs = \{(x_k) \in \omega : \sum_k x_k \ is\ convergent\ \}$ and $bs = \{(x_k) \in \omega : ||x||_{bs} = \sup_n |\sum_{k=1}^n x_k| < \infty\}$. These are Banach spaces with their natural norms and also $cs$ is Banach spaces with $||.||_{bs}$. We know, $\phi \subset \lambda^\alpha \subset \lambda^\beta \subset \lambda^\gamma$. If $\lambda \subset \mu$ then $\mu^\zeta \subset \lambda^\zeta$ and for every $\lambda$ we have $\lambda^\zeta = \lambda^{\zeta\zeta\zeta}$, $\lambda \subset \lambda^{\zeta\zeta}$, where $\zeta$ is one of the $\alpha-$, $\beta-$ or $\gamma-$ duals. Let us note, Fleming and Magee showed that, whenever $\lambda \supset \phi$ is a sequence space (not required be a $BK$) and $\mu \supset \phi$ is a $BK$ space then $\lambda^\mu$ is a $BK$ space iff there exist a norm $||.||$ on $\lambda$ such that for every $y \in \lambda^\mu$ the diagonal map $T_y : \lambda \to \mu$, $T_y(x) = xy$ is continuous with respect to this norm [9]. We denote $f$-dual of a $BK$ space $\lambda \supset \phi$ with $\lambda^f = \{(f(\delta^k))_{k\in\mathbb{N}} | \exists f \in \lambda'\}$. Here $\lambda^f$ is also a $BK$ space with $||f||_{\lambda'} = ||(f(\delta^k))_{k\in\mathbb{N}}||_{\lambda^f}$. A $K$ space $\lambda \supset \phi$ is called a sum space

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if $\lambda^\lambda = \lambda^f$. For example, $\ell$, $cs$ and $bs$ are $BK$ sum spaces. If $\lambda \supset \phi$ is a $K$ space then $S \in \lambda'$ is called a sum on $\lambda$ if $S(\delta^k) = 1, \forall k \in \mathbb{N}$ or equivalently $S$ is a sum on $\lambda$ if $S(x) = \sum x, \forall x \in \phi$, where $S \in \lambda'$. A $K$ space $\lambda$ is called $AD$ space if $\lambda = \overline{\phi}$, where $\overline{\phi}$ is closure of $\phi$ in $\lambda$. Via Hahn-Banach theorem, $\lambda^f = \overline{\phi}^f$.

Let $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive real numbers tending to infinity, that is,

$$0 < \lambda_1 < \lambda_2 < \cdots \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty. \tag{1.1}$$

Then the sequence $x = (x_k) \in \omega$ is said to be $\lambda$-convergent to the number $a \in \mathbb{C}$, if $(\Lambda x)_n \to a$, as $n \to \infty$; where

$$(\Lambda x)_n = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\Delta\lambda)_k x_k$$

for all $n \in \mathbb{N}$. Throughout the text we shall assume that $(\Delta\lambda)_k = \lambda_k - \lambda_{k-1}$ for all $k \in \mathbb{N}$ and $\lambda_0 = 0$. The set $c^\lambda$ of all $\lambda$ convergent sequences is a $BK$ space with the norm $\|x\|_{\ell_\infty^\lambda} = \|\Lambda x\|_\infty = \sup_{n\in\mathbb{N}} |(\Lambda x)_n|$, where $\Lambda x = \{(\Lambda x)_n\}$; [15]. The matrix $\Lambda = (\lambda_{nk})$ is also defined by

$$\lambda_{nk} := \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} & , \quad (1 \le k \le n), \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. In the special case $\lambda_n = n$ for all $n \in \mathbb{N}$, the $\Lambda$-matrix is reduced to the Cesàro matrix $C$ of order one. We also note that $\Lambda$-summability is the special case of the $\overline{N}q$-summability; [15], (see also [2]).

**Lemma 1.1.** *$c_0 \subset c_0^\lambda, c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ strictly hold if and only if*

$$\liminf_n \frac{\lambda_{n+1}}{\lambda_n} = 1$$

[15].

**Lemma 1.2.** *$c_0 = c_0^\lambda, c = c^\lambda$ and $\ell_\infty = \ell_\infty^\lambda$ hold if and only if*

$$\liminf_n \frac{\lambda_{n+1}}{\lambda_n} > 1 \tag{1.2}$$

[15].

## 2. Some New Sections and Distinguished Subspaces of an $FK$-spaces

Let $x = (x_k) \in \omega$ be a sequence, then by using $\Lambda = (\lambda_{nk})$, we have $\Lambda$ $n^{th}$ section of $x$ as;

$$x_\lambda^{[n]} = \Lambda(x^{[k]}) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\Delta\lambda_k) x^{[k]}.$$

Here, $x^{[k]} = \sum_{j=1}^{k} x_j \delta^j$ and $\Delta\lambda_k = \lambda_k - \lambda_{k-1}$,($k \in \mathbf{N}$ and $\lambda_0 = 0$).

A sequence $x$ in any $K$ space $X \supset \phi$ has $\lambda AK$ property if $x_\lambda^{[n]} \to x,\ (n \to \infty)$ in $X$ and we say $X$ is an $\lambda AK$- space if all elements of $\lambda$ have this property. Similarly we can define the properties, $S\lambda AK$, $F\lambda AK$ and $\lambda AB$. So, we define the following sets as:

$$\begin{aligned}
\Lambda S_X &= \left\{x \in X \middle| x = \lim_n x_\lambda^{[n]}\right\}, \\
\Lambda W_X &= \left\{x \in X \middle| x_\lambda^{[n]} \rightharpoonup x \text{ in } \lambda\right\} (\text{"}\rightharpoonup\text{" means } \textit{weakly}) \\
&= \left\{x \in X \middle| f(x) = \lim_n f(x_\lambda^{[n]}),\ \forall f \in X'\right\}, \\
\Lambda F_X^+ &= \left\{x \in \omega \middle| (x_\lambda^{[n]})_{n\in\mathbb{N}} \text{ weakly Cauchy in } X\ \right\} \\
&= \left\{x \in \omega \middle| (f(x_\lambda^{[n]}))_{n\in\mathbb{N}} \in c,\ \forall f \in \lambda'\right\}, \\
\Lambda B_X^+ &= \left\{x \in \omega \middle| \left(x_\lambda^{[n]}\right)_{n\in\mathbb{N}} \text{ is bounded in } X\ \right\} \\
&= \left\{x \in \omega \middle| \left(f(x_\lambda^{[n]})\right)_{n\in\mathbb{N}} \in \ell_\infty,\ \forall f \in X'\right\}.
\end{aligned}$$

One should keep in mind that $\Lambda B_X = \Lambda B_X^+ \cap X$ and $\Lambda F_X = \Lambda F_X^+ \cap \lambda$. These are the spaces of the sequences which have $\lambda AB$ and $F\lambda AK$, respectively. Now for example, if the normed sequence space $X$ is an $\lambda AB$ space (or $\lambda AK$ space), then $\sup_n \| x^{[n]_\lambda} \|_X < \infty$ (or $\lim_n \| x^{[n]_\lambda} - x \|_X = 0$). Further, since the boundedness and weak boundedness are equal in normed spaces, one can easily see that $\sup_n |f(x^{[n]_\lambda})| < \infty$ holds, for every $f \in X'$, $x \in \Lambda B_X$.

For all $x \in\ \omega$, since $\{x^{[n]} | n \in \mathbf{N}\} \supset \{x_\lambda^{[n]} | n \in \mathbf{N}\}$, we have

$$\Lambda \mathcal{P}_X \supset \mathcal{P}_X$$

for the properties $\mathcal{P} = B,\ F,\ W,\ S$ and $\Lambda\mathcal{P} = \Lambda B,\ \Lambda F,\ \Lambda W,\ \Lambda S$.

In the other hand, let us define the operator $\Lambda = (\lambda_{nk})$ associated with the sum operator

$$s_{nk} = \begin{cases} 1 & , \quad (1 \le k \le n) \\ 0 & , \quad (k > n) \end{cases} ,$$

then we obtain the spaces

$$\begin{aligned}
\lambda(B) &= \left\{x = (x_j) \in \omega : \sum_{j=1}^{k} x_j \in \ell_\infty^\lambda\right\} \\
&= \left\{x = (x_j) \in \omega : \sup_n \frac{1}{\lambda_n}\left|\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \sum_{j=1}^{k} x_j\right| < \infty\right\}
\end{aligned}$$

and

$$\begin{aligned}
\lambda(S) &= \left\{x = (x_j) \in \omega : \sum_{j=1}^{k} x_j \in c^\lambda\right\} \\
&= \left\{x = (x_j) \in \omega : \lim_n \frac{1}{\lambda_n}\left(\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \sum_{j=1}^{k} x_j\right) \text{ exists }\right\}
\end{aligned}$$

with the norm

$$\|x\|_{\lambda(B)} = \|x\|_{\lambda(S)} = \sup_n \frac{1}{\lambda_n}\left|\sum_{k=1}^{n}(\lambda_k - \lambda_{k-1})\sum_{j=1}^{k} x_j\right|.$$

We define the $\lambda(B)$ and $\lambda(S)$ duals of a sequence space $X$ as

$$\begin{aligned} X^{\lambda(B)} &= \left\{x = (x_j) \in \omega : \sup_n \frac{1}{\lambda_n}\left|\sum_{k=1}^{n}(\Delta\lambda_k)\sum_{j=1}^{k} x_j y_j\right| < \infty,\ \forall y = (y_j) \in X\right\} \\ &= \{x = (x_j) \in w : xy \in \lambda(B),\ \forall y = (y_j) \in X\} \end{aligned}$$

and

$$\begin{aligned} X^{\lambda(S)} &= \left\{x = (x_j) \in \omega : \lim_n \frac{1}{\lambda_n}\left(\sum_{k=1}^{n}(\Delta\lambda_k)\sum_{j=1}^{k} x_j y_j\right) \text{ exists },\ \forall y = (y_j) \in X\right\} \\ &= \{x = (x_j) \in \omega : xy \in \lambda(S),\ \forall y = (y_j) \in X\}, \end{aligned}$$

respectively. We have $X^{\lambda(S)} \subset X^{\lambda(B)}$ and if $\varsigma = \lambda(S),\ \lambda(B)$, then the inclusion $X \subset Y$ yields $Y^{\varsigma} \subset X^{\varsigma}$. We also have $X^{\varsigma} = X^{\varsigma\varsigma\varsigma}$ and $X \subset X^{\varsigma\varsigma}$. If $X = X^{\varsigma\varsigma}$, then $X$ is said to be a $\varsigma-$ space. In the sake of shortness, we use the notation $X^{\lambda(S)\lambda(S)} = X^{\lambda^2(S)}$ and $X^{\lambda(B)\lambda(B)} = X^{\lambda^2(B)}$. It can be easily seen that, if $\lambda_k = k$, one can obtain the spaces $\sigma s$ and $\sigma b$ from the spaces $\lambda(S)$ and $\lambda(B)$, respectively [4].

**Proposition 2.1.** *The inclusions $cs \subset \lambda(S)$ and $bs \subset \lambda(B)$ hold. $\lambda(B) \subset \sigma b$ and $\lambda(S) \subset \sigma s$ if and only if the condition (1.2) holds.*

*Proof.* It is clear. □

**Theorem 2.2.** *If $X$ is an $AK$-space, then it is an $\lambda AK$-space.*

*Proof.* It is clear with Stolz-Cesàro theorem. □

Let $X \supset \phi$ be a $BK$-space. If the following conditions hold then it is said to be $X$ has a monotone norm [19]:
**i.** For $n < m$ $||x^{[n]}|| \leq ||x^{[m]}||$,
**ii.** $||x|| = \sup_m ||x^{[m]}||$.

**Theorem 2.3.** *$c^{\lambda}$ has monoton norm.*

*Proof.* Since $\Lambda$ is a triangle $c^{\lambda}$ is a $BK$-space. Let $x$ be fixed and $\Lambda(m,n) = \left|\frac{1}{\lambda_n}\sum_{k=1}^{m}(\Delta\lambda_k)x_k\right|$. So, from $||x||_{\lambda^{\infty}} = \sup_n\left|\frac{1}{\lambda_n}\sum_{k=1}^{n}(\Delta\lambda_k)x_k\right|$, we have $||x||_{\lambda^{\infty}} = \lambda(n,n)$. Since, for $x^{[m]} = \{x_1, ..., x_m, 0, 0, ...\}$

$$\left|\Lambda(x^{[m]})_n\right| = \begin{cases} \Lambda(n,n) & , \quad n \leq m \\ \Lambda(m,n) & , \quad n \geq m \end{cases},$$

we also have $\Lambda(m,n)$ is decreasing for $n$. Therefore, the first condition holds. In the other hand, from $||x^{[m]}||_{\lambda^{\infty}} = \sup_n\{\Lambda(n,n) : \ n \leq m\}$, one can easily see that, the second condition also holds. □

**Theorem 2.4.** *Let $X \supset \phi$ be an $FK$-space. Then, the inclusions*

$$\phi \subset \Lambda S_X \subseteq \Lambda W_X \subset \Lambda F_X \subset \Lambda B_X \subset X$$

*and*

$$\phi \subset \Lambda S_X \subseteq \Lambda W_X \subset \overline{\phi}$$

*hold.*

*Proof.* From the definitions of the spaces $\phi,\ \Lambda S_X,\ \Lambda W_X,\ \Lambda F_X,\ \Lambda B_X$, we have $x_\lambda^{[n]} \to x,$( by the norm of $X$) $\Rightarrow f(x_\lambda^{[n]}) \to f(x) \Rightarrow (f(x_\lambda^{[n]})) \in c \Rightarrow (f(x_\lambda^{[n]})) \in \ell_\infty$, for every $f \in X'$.

Now, we shall prove that $\Lambda W_X \subset \overline{\phi}$. Let us suppose that $x \in \Lambda W_X$. So,

$$f(x) = \lim_n \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta\lambda_k f(x^{[k]}),$$

holds for every $f \in X'$. Therefore, we have the result from Hahn - Banach theorem[19]. □

**Theorem 2.5.** *Distinguished subspaces of an $FK$- space are monotone. That is,*

$$X \subset Y \Rightarrow\ \Omega_X \subset\ \Omega_Y$$

*holds for every* $\Omega = \Lambda S,\ \Lambda W,\ \Lambda F,\ \Lambda B.$

*Proof.* Since the others are similar, we only give the proof for $\lambda AK$ property. By bearing in mind that the inclusion map is continuous, let us suppose that $X \subset Y$ and $x \in \Lambda S_X$. Therefore, the convergence $\frac{1}{\lambda_n}\sum_{k=1}^{n}(\Delta\lambda_k)x^{[k]} \to x,$ in $X$ yields that the convergence $\frac{1}{\lambda_n}\sum_{k=1}^{n}(\Delta\lambda_k)x^{[k]} \to x,$ in $Y$ . This completes the proof. □

**Theorem 2.6.** *Let each $X_i \supset \phi,\ (i = 1, 2, ..., m)$ be $FK$ spaces with paranorms $p^{(i)},\ (i = 1, 2, ..., m)$ and $X = \sum_{i=1}^{m} X_i$. If $\Omega = \Lambda S, \Lambda W, \Lambda F, \Lambda B$, then $\sum_{i=1}^{m}\ \Omega_{X_i} \subseteq\ \Omega_X$ holds.*

*Proof.* Since the others are similar, we only give proof for $\lambda AK$ property. Let us suppose that $x^{(i)} \in \Lambda S_{X_i}$ $(i = 1, 2, ..., m)$. Then,

$$p^{(1)}[(x^{(1)})_\lambda^{[n]} - x^{(1)}] \to 0, ..., p^{(m)}[(x^{(m)})_\lambda^{[n]} - x^{(m)}] \to 0,$$

that is, by taking $\{p^{(i)}[(x^{(i)})_\lambda^{[n]} - x^{(i)}] \to 0\}_{i=1}^{m}$, we have

$$\begin{aligned}
q\Big[(\sum_{i=1}^{m} x^{(i)})_\lambda^{[n]} - (\sum_{i=1}^{m} x^{(i)})\Big] &= q\Big[\sum_{i=1}^{m}\big((x^{(i)})_\lambda^{[n]} - x^{(i)}\big)\Big] \\
&= q\Big\{\sum_{i=1}^{m}\big(\frac{1}{\lambda_n}\sum_{k=1}^{n}(\Delta\lambda_k)(x^{(i)})^{[k]}\big) - (x^{(i)})\Big\} \\
&\le p^{(1)}\Big\{\big(\frac{1}{\lambda_n}\sum_{k=1}^{n}(\Delta\lambda_k)(x^{(1)})^{[k]}\big) - (x^{(1)})\Big\} + \\
&+ \ ... + \\
&+ p^{(m)}\Big\{\big(\frac{1}{\lambda_n}\sum_{k=1}^{n}(\Delta\lambda_k)(x^{(m)})^{[k]}\big) - (x^{(m)})\Big\} \\
&= p^{(1)}[(x^{(1)})_\lambda^{[n]} - x^{(1)}] + ... + p^{(m)}[(x^{(m)})_\lambda^{[n]} - x^{(m)}] \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore, we have $\sum_{i=1}^{m} x^{(i)} \in \Lambda S_X$. This completes the proof (see also [8]). □

**Theorem 2.7.** *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $FK$-spaces and $X = \bigcap_n X_n$. Then we have, $\Omega_X = \bigcap_n \ \Omega_{X_n}$ for $\Omega = \Lambda S, \Lambda W, \Lambda F, \Lambda B$.*

*Proof.* By monotonicity, $\Omega_X \subseteq \Omega_{X_n}$, and so for $\Omega = \Lambda S, \Lambda W, \Lambda F, \Lambda B$ we have $\Omega_X \subseteq \bigcap_n \Omega_{X_n}$, for all $n \in \mathbb{N}$. Since the others are similar, we shall only prove that $\bigcap_n \Omega_{X_n} \subseteq \Omega_X$ holds for $\Omega = \Lambda S$. Let us suppose that $x \in \bigcap_n \Lambda S_{X_n}$. Then for all $n, k \in \mathbf{N}$, $q_{nk}(x_\lambda^{[n]} - x) \to 0$ and also $x_\lambda^{[n]} \to x$ in $X$, we have $x \in \Lambda S_X$. This completes the proof (see also [8]). □

## 3. Duals

In the following, we give some relationship between the distinguished subspaces and $f-$, $\lambda(S)$ ve $\lambda(B)$ duals for an $FK$-space $X \supset \phi$.

**Theorem 3.1.** *Let $X \supset \phi$ be an $FK$-space. Then we have*

$$\Lambda B_X^+ = X^{f\lambda(B)} \quad \text{and } \Lambda F_X^+ = X^{f\lambda(S)}.$$

*Proof.* We know that $z \in \Lambda B_X^+$ if and only if $\left(z_n f(\delta^n)\right)_{n\in\mathbf{N}} \in \lambda(B)$, for all $f \in X'$. Let us take $(f(\delta^n))_{n\in\mathbf{N}} \in X^f$, for some $f \in X'$, then from the definition of $\lambda(B)$, we have $z \in X^{f\lambda(B)}$. The other one is similar. □

**Corollary 3.2.** *Let $X \supset \phi$ be an $FK$-space. Then, the spaces $\Lambda B_X^+$ and $\Lambda F_X^+$ are $\lambda(B)$ and $\lambda(S)$ spaces, respectively.*

**Theorem 3.3.** *Let $X \supset \phi$ be an $FK$-space and $\overline{\phi}$ is the closure of $\phi$ in $X$. If $\overline{\phi} \subset Y \subset X$ , then*

$$\Lambda B_X^+ = \Lambda B_Y^+ \quad \text{and } \Lambda F_X^+ = \Lambda F_Y^+.$$

*Proof.* Since $\overline{\phi} \subset Y \subset X$ holds, we have $\Lambda B_{\overline{\phi}}^+ \subset \Lambda B_Y^+ \subset \Lambda B_X^+$. Therefore, for an arbitrary $FK$-space $X \supset \phi$, we have $(\overline{\phi})^f = X^f$, and so $\overline{\phi}^f \subset Y^f \subset X^f = \overline{\phi}^f$. Anymore, we get desired result by taking $\lambda(B)$ dual in both sides. Similarly, we can prove that $\Lambda F_X^+ = \Lambda F_Y^+$. □

**Theorem 3.4.** *Let $X \supset \phi$ be an $FK$-space. Then,*

$$X \text{ is an } \Lambda B \text{ space } \Leftrightarrow X^f \subset X^{\lambda(B)}$$

*and*

$$X \text{ is an } \Lambda F \text{ space } \Leftrightarrow X^f \subset X^{\lambda(S)}.$$

*Proof.* $\{\Rightarrow\}$: By hypothesis and previous result, for $\lambda(B)$ and $\lambda(S)$ duals of an $FK$-space, we have $X \subset \Lambda B_X^+ = X^{f\lambda(B)}$ and $X \subset \Lambda F_X^+ = X^{f\lambda(S)}$. From taking $\lambda(B)$ and $\lambda(S)$ duals,

$$X^{f\lambda^2(B)} \subset X^{\lambda(B)}$$

and

$$X^{f\lambda^2(S)} \subset X^{\lambda(S)}$$

hold. Now, we have also $X^f \subset X^{f\lambda^2(B)}$ and $X^f \subset X^{f\lambda^2(S)}$, then

$$X^f \subset X^{\lambda(B)}$$

and

$$X^f \subset X^{\lambda(S)}$$

hold.

{$\Leftarrow$}: In hypothesis, by using the inclusions $X \supset \phi$ $X^f \subset X^{\lambda(B)}$ and $X^f \subset X^{\lambda(S)}$, let us take $\lambda(B)$ and $\lambda(S)$ duals. From the properties of $\lambda(B)$ and $\lambda(S)$ duals, we have

$$X \subset X^{\lambda^2(S)} \subset X^{f\lambda(B)} = \Lambda B_X^+$$

and

$$X \subset X^{\lambda^2(S)} \subset X^{f\lambda(S)} = \Lambda F_X^+,$$

respectively. This means that, $X$ is a $\lambda AB$ and $F\lambda AK$ space, respectively. □

As a result of this theorem, we have following since $X^{\lambda(S)}$ is a closed subspace of $X^{\lambda(B)}$ space (see also [19]).

**Corollary 3.5.** *Let $X \supset \phi$ be a $BK-\lambda AB$-space, then $X^{\lambda(S)}$ is closed in $\lambda^f$.*

**Theorem 3.6.** *The following assertions for an $FK$-space $X \supset \phi$ are true:*

*(i) If $X$ is an $F\lambda AK$-space, then $X^f = X^{\lambda(S)}$,*

*(ii) If $X$ is an AD-space, then $X^{\lambda(S)} = X^{\lambda(B)}$,*

*(iii) The inclusions $X^\beta \subset X^{\lambda(S)} \subset X^{\lambda(B)} \subset X^f$ are hold.*

*Proof.* *(i)* Let us suppose that $y \in X^{\lambda(S)}$ and

$$f(x) = \lim_n \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta\lambda_k \sum_{j=1}^{k} x_j y_j$$

holds, for every $x \in X$. By Banach-Steinhaus theorem, we have$f \in X'$. Since we have

$$\begin{aligned} f(x) &= \lim_n \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta\lambda_k \sum_{j=1}^{k} x_j y_j \\ &= \lim_n \frac{1}{\lambda_n} \left( \lambda_n \sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} \lambda_{k-1} x_k y_k \right) \\ &= \lim_n \left( \sum_{k=1}^{n} x_k y_k - \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1} x_k y_k \right), \end{aligned}$$

by taking $x = \delta^m$, we have

$$\begin{aligned} f(\delta^m) &= \lim_n \left( y_m - \frac{\lambda_{m-1}}{\lambda_n} y_m \right) \\ &= \lim_n \left( y_m (1 - \frac{\lambda_{m-1}}{\lambda_n}) \right) \\ &= y_m, \ m < n. \end{aligned}$$

and then $y = (y_m) \in X^f$. This means that, $X^{\lambda(S)} \subseteq X^f$.

In the other hand, let us take $y \in X^f$. Since $X$ is an $F\lambda AK$-space,

$$\lim_n \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta\lambda_k f(x^{[k]})$$

exists, and so $y = (y_j) = (f(\delta^j)) \in X^{\lambda(S)}$, for all $x \in X$. Therefore, $X^f \subseteq X^{\lambda(S)}$.

*(ii)* It is enough to show that, if $X$ is an $AD$-space, then $X^{\lambda(B)} \subset X^{\lambda(S)}$ holds.

Let us suppose that $y \in X^{\lambda(B)}$ and define $\{f_n\}$ as,

$$f_n(x) = \lim_n \frac{1}{\lambda_n} \sum_{k=1}^{n} \Delta\lambda_k \sum_{j=1}^{k} x_j y_j,$$

for all $x \in X$. Then $\{f_n\}$ is point-wise bounded and so is equicontinuous [19].

For all $m \leq n$,

$$\lim_n f_n(\delta^m) = y_m$$

and so is $\phi \subset \{x : \lim_n f_n(x) \text{ mevcut}\}$. By convergence lemma [19], $\{x : \lim_n f_n(x) \text{ mevcut }\}$ is a closed subspace of $X$. Since $X$ is an $AD$-space, we have

$$\phi \subset \{x : \lim_n f_n(x) \text{ exists }\} = \overline{\phi} = X.$$

That is, $y \in X^{\lambda(S)}$. Hence, $X^{\lambda(S)} = X^{\lambda(B)}$.

*(iii)* It is enough that, the inclusion $X^{\lambda(B)} \subset \lambda^f$ holds. For $\overline{\phi} \subset X$,

$$\begin{array}{rcl} X^{\lambda(B)} & \subset & (\overline{\phi})^{\lambda(B)} \\ & = & (\overline{\phi})^{\lambda(S)} \\ & \subset & (\overline{\phi})^{f} \\ & = & X^f, \end{array}$$

since $\overline{\phi}$ is an $AD$-space. □

We have the following corollary by using previous theorems.

**Corollary 3.7.** *Let $X \supset \phi$ be an $FK$-space. Then,*

$$X \text{ is a } \lambda AB \text{ space } \Leftrightarrow X^f = X^{\lambda(B)}$$

*and*

$$X \text{ is a } F\lambda AK \text{ space } \Leftrightarrow X^f = X^{\lambda(S)}.$$

**Theorem 3.8.** *Let $X \supset \phi$ be an $FK - \lambda AB$-space. Then, $\overline{\phi}$ is a $\lambda AK$-space and the equality*

$$\Lambda S_X = \Lambda W_X = \overline{\phi}$$

*holds.*

*Proof.* Since we get the proof by the similar way used in the proof of given theorem in [7], we omit the details. □

**Theorem 3.9.** *Let $X \supset \phi$ be an $FK$-space such that $\overline{\phi}$ is a $\lambda AK$-space. Then,*

$$\Lambda F_X^+ = \overline{\phi}^{\ \lambda^2(S)}.$$

*Proof.* We know that $\Lambda F_X^+ = X^{f\lambda(S)}$ and $X^f = (\overline{\phi})^f$ for an $FK$-space $X \supset \phi$. Now, by taking $\lambda(S)$ dual in both sides, we have $X^{f\lambda(S)} = (\overline{\phi})^{f\lambda(S)}$. □

In this theorem, we can replace $\overline{\phi}$' s $\lambda AK$ property with the weaker property $F\lambda AK$. Because, if $X \supset \phi$ is an $F\lambda AK$-space, then $X^f = X^{\lambda(S)}$.

**Corollary 3.10.** *Let $X \supset \phi$ be an $FK$-space. Then,*

$$X \text{ is an } F\lambda AK \text{ space } \Leftrightarrow \overline{\phi} \text{ is a } \lambda AK \text{ space and } X \subset \overline{\phi}^{\lambda^2(S)}.$$

**Theorem 3.11.** *Let $X \supset \phi$ be an $FK$-space. Then the following are equivalent:*

| | |
|---|---|
| ***(i)*** | *$X$ is an $F\lambda AK$ space ,* |
| ***(ii)*** | $X \subset \Lambda F_X^{\lambda^2(S)}$, |
| ***(iii)*** | $X \subset \Lambda W_X^{\lambda^2(S)}$, |
| ***(iv)*** | $X \subset \Lambda S_X^{\lambda^2(S)}$, |
| ***(v)*** | $X^{\lambda(S)} = \Lambda F_X^{\lambda(S)} = \Lambda W_X^{\lambda(S)} = \Lambda S_X^{\lambda(S)}$. |

*Proof.* $(iv) \Rightarrow (iii) \Rightarrow (ii)$ are clear from the definitions of these spaces.

$(ii) \Rightarrow (i)$: Let us suppose that $X \subset \Lambda F_X^{\lambda^2(S)}$. Then,

$$X^f \subset X^{f\lambda^2(S)} = \Lambda F_X^{+\lambda(S)} \subset \Lambda F_X^{\lambda(S)} \subset X^{\lambda(S)}$$

hold.

$(i) \Rightarrow (iv)$: It is clear from previous results.

$(iv) \Rightarrow (v)$: We have for an $FK$-space $X \supset \phi$;

$$\Lambda S_X \subset \Lambda W_X \subset \Lambda F_X \subset X.$$

By taking $\lambda(S)$ dual in every side, we have

$$X^{\lambda(S)} \subset \Lambda F_X^{\lambda(S)} \subset \Lambda W_X^{\lambda(S)} \subset \Lambda S_X^{\lambda(S)}.$$

By bearing in mind the hypothesis with the previous results, we get the proof.

$(v) \Rightarrow (iv)$: It is clear. □

Let $X$ be an $FK$-space has $\lambda AK$ property. The following theorem tells us that there is a closed relationship between the spaces $X^{\lambda(S)}$ and $X'$.

**Theorem 3.12.** *Let $X \supset \phi$ be an $FK$-space. Then the following are equivalent:*

| | |
|---|---|
| ***(i)*** | *$X$ is an $S\lambda AK$,* |
| ***(ii)*** | *$X$ is a $\lambda AK$,* |
| ***(iii)*** | $X^{\lambda(S)} \widehat{=} X'$, $(f \to f(\delta^k))$. |

*Proof.* $(i) \Rightarrow (ii)$: If $X$ is an $S\lambda AK$ space, then it is an $AD$-space and also is a $\lambda AB$-space. Therefore, $X$ is a $\lambda AK$-space.

$(ii) \Rightarrow (iii)$: Since $X$ is a $\lambda AK$-space, then it is $AD$-space, and so $X^f = X'$ holds.

$(iii) \Rightarrow (i)$: Let us suppose that $u \in X^{\lambda(S)}$. Then, for all $f \in X'$ and $x \in X$, we have

$$f(x) = \lim_n \frac{1}{\lambda_n} \sum_{k=1}^{n} (\Delta\lambda_k) \sum_{j=1}^{k} u_j x_j.$$

Therefore, we have $x \in \Lambda W_X$. □

**Theorem 3.13.** *Let $X \supset \phi$ be an $FK$-space. Then, the following assertions are equivalent:*

$$\begin{aligned} &\textit{(i)} && \Lambda W_X \text{ is closed in } X, \\ &\textit{(ii)} && \overline{\phi} \subset \Lambda B_X, \\ &\textit{(iii)} && \overline{\phi} \subset \Lambda F_X, \\ &\textit{(iv)} && \overline{\phi} = \Lambda W_X, \\ &\textit{(v)} && \overline{\phi} = \Lambda S_X, \\ &\textit{(vi)} && \Lambda S_X \text{ is closed in } X. \end{aligned}$$

*Proof.* $(v) \Rightarrow (iv)$, $(iv) \Rightarrow (iii)$, $(v) \Rightarrow (ii)$ and $(iii) \Rightarrow (ii)$ are clear. Since $\overline{\phi}$ is a $\lambda AK$-space, we have $\overline{\phi} \subset \Lambda S_X$, and so $(ii) \Rightarrow (v)$ holds.

In the other hand, from $\phi \subset \Lambda S_X \subset \Lambda W_X \subset \overline{\phi}$, we have $(i) \Rightarrow (iv)$ and $(vi) \Rightarrow (v)$. $(iv) \Rightarrow (i)$ and $(v) \Rightarrow (vi)$ are also clear from the previous theorems and corollaries. □

## References


[1] B. Altay and F. Başar, *Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space*, J. Math. Anal. Appl. **336**(1)(2007), 632–645.

[2] F. Başar, Summability Theory and Its Applications, Bentham Science Publishers, İstanbul (2012).

[3] J. Boos, *Classical and Modern Methods in Summability*, Oxford University Press. New York, Oxford, 2000.

[4] M. Buntinas, *Convergent and bounded Cesàro sections in FK-spaces*, Math. Z., 121 (1971), 191-200.

[5] M. Buntinas, *On sectionally dense summability fields*, Math. Zeitschr., 132 (1973), 141-149.

[6] M. Buntinas, *On Toeplitz sections in sequence spaces*, Math. Proc. Cambridge Philos. Soc., 78 (1975), 451-460.

[7] İ. Dağadur, On some subspaces of an $FK$-space, Math. Commun., Vol.7, (2002), 15-20.

[8] R. Devos, *Combinations of distinguished subsets and conullity*, Math. Z., 192 (1986), 447-451.

[9] D. J. Fleming and J. C. Magee, *FK-multiplier spaces*, Proc. Amer. Math. Soc., 125(1)(1997), 175-181.

[10] D. J. H. Garling, *The $\beta$- and $\gamma$-duality of sequence spaces*, Math. Proc. Cambridge Philos. Soc., 63 (Jan. 1967), 963-981.

[11] D. J. H. Garling, *On topological sequence spaces*, Math. Proc. Cambridge Philos. Soc., 63 (1967), 997-1019.

[12] G. Goes and S. Goes, *Sequences of bounded variation and sequences of fourier coefficients. I*, Math. Z.,**118**(1970), 93-102.

[13] G. Goes, *Summen von FK-räumen funktionale abschnittskonvergenz und umkehrsatz*, Tohoku. Math. J., 26(1974), 487-504.

[14] K-G., Grosse-Erdmann, *On $l^1$-Invariant Sequence Spaces*, J. Math. Anal. Appl., **262**(2001), 112-132.

[15] M. Mursaleen A. K. Noman, *On the spaces of $\lambda$-convergent and bounded sequences*, Thai J. Math. **8** (2) (2010), 311-329.

[16] E. Malkowsky, *Recent results in the theory of matrix transformations in sequence spaces*, Mat. Vesnik **49**(1997), 187–196.

[17] A. Wilansky, *Functional Analysis*, Blaisdell Press, 1964.

[18] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw Hill, New York, 1978.

[19] A. Wilansky, *Summability Through Functional Analysis*, North-Holland, Amsterdam, 1984.

[20] K. Zeller, *Allgemeine eigenschaften von limitierungsverfahren*, Math. Z., 53 (1951), 463-487.

Van Yüzüncü Yil University, Faculty of Science, Department of Mathematics, 65080, Van, Turkey

Van Yüzüncü Yil University, Faculty of Education, Department of Mathematics Education, 65080, Van, Turkey