Journal of Universal Mathematics
Vol.3 No.1 pp.33-45 (2020)
ISSN-2618-5660

INTUITIONISTIC FUZZY HEAT-LIKE EQUATIONS

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Résumé. In this paper, the variational iteration method (VIM) is used for finding exact intuitionistic fuzzy solution of the intuitionistic fuzzy heat-like equations with variable coefficients in one and two dimensions. Several examples are given to show the new theorem of the solution. The results obtained in all cases show the reliability and the efficiency of this methods.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [12] has showed successful applications in various fields. In fuzzy set theory, the membership of an element to a fuzzy set is a single value between zero and one. But in reality, it may not always be certain that the degree of nonmembership of an element in a fuzzy set is just equal to 1 minus the degree of membership. That is to say, there may be some hesitation degree. So, as a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1].
In this work, our idea is solving heat-like equations with intuitionistic fuzzy parameters via the same strategy as α-cuts using Variational Iteration Method VIM.
The VIM proposed by He in [7], is a method of solving linear or nonlinear problems [11] and gives rapidly convergent successive approximations of the exact solution if that last exists.
In comparison with the paper [5, 6], we investigate problems with intuitionistic fuzzy initial value and intuitionistic fuzzy forcing functions, we propose a new theorem for finding the exact intuitionistic fuzzy solutions, witch extended to the solution for the proposed models.

2. Basic concept of intuitionistic fuzzy sets

Let a set X be fixed. An intuitionistic fuzzy set $\tilde{A}^i$ in X is an object having the form $\tilde{A}^i = \left\{ \left\langle x, \mu_{\tilde{A}^i}(x), \nu_{\tilde{A}^i}(x) \right\rangle \right\}$, where $\mu_{\tilde{A}^i}(x) : X \to [0,1]$ and $\nu_{\tilde{A}^i}(x) : X \to [0,1]$ define the degree of memberschip and degree of non-membership respectively, of the element $x \in X$ to the set $\tilde{A}^i$, which is subset of X, for every element of $x \in X$, $0 \leq \mu_{\tilde{A}^i}(x) + \nu_{\tilde{A}^i}(x) \leq 1$. Let $X = \mathbb{R}$

Definition 2.1. Let $\mathbb{IF} = \left\{ \tilde{A}^i | \tilde{A}^i : \mathbb{R} \to [0,1]^2, \text{ satisfies } (1)-(5) \right\}$:
An intuitionistic fuzzy number $\tilde{A}^i$ is

(1) normal i.e there is any $x_0, x_1 \in \mathbb{R}$ such that $\mu_{\tilde{A}^i}(x_0) = 1$ and $\nu_{\tilde{A}^i}(x_1) = 1$
(2) convex for the membership function $\mu_{\tilde{A}^i}(x)$ i.e

$$\mu_{\tilde{A}^i}(\lambda x_1 + (1-\lambda)x_2) \geq \min\Big(\mu_{\tilde{A}^i}(x_1), \mu_{\tilde{A}^i}(x_2)\Big) \ \forall x_1, x_2 \in \mathbb{R}, \lambda \in [0,1]$$

2000 *Mathematics Subject Classification.* intuitionistic fuzzy solution.

Key words and phrases. Variational Iteration Method, intuitionistic fuzzy solution, the intuitionistic fuzzy heat-like equations.

(3) concave for non-membership function $\nu_{\tilde{A}^i}(x)$ i.e

$$\nu_{\tilde{A}^i}\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \max\Big(\nu_{\tilde{A}^i}(x_1), \nu_{\tilde{A}^i}(x_2)\Big) \ \forall x_1, x_2 \in \mathbb{R}, \lambda \in [0,1]$$

(4) $\mu_{\tilde{A}^i}(x)$ is upper semi-continuous and $\nu_{\tilde{A}^i}(x)$ is lower semi-continuous and

(5) $\text{supp}(\mu_{\tilde{A}^i}, \nu_{\tilde{A}^i}) = cl\{x \in \mathbb{R} : \nu_{\tilde{A}^i}(x) < 1\}$ is bounded.

Then $\mathbb{IF}$ is called intuitionistic fuzzy space.

Definition 2.2. If $\tilde{A}^i$ is an intuitionistic fuzzy number α-cut is given by

$$\tilde{A}^i[\alpha] = \Big\{A^+[\alpha], A^-[\alpha];\ \alpha \in [0,1]\Big\}$$

where $A^-[\alpha] = \{x \in \mathbb{R} : \nu_{\tilde{A}^i}(x) \leq 1-\alpha\}, \quad A^+[\alpha] = \{x \in \mathbb{R} : \mu_{\tilde{A}^i}(x) \geq \alpha\}$.
It is expressed as $\tilde{A}^i[\alpha] = \Big\{\big[A_1^+(\alpha), A_2^+(\alpha)\big], \big[A_1^-(\alpha), A_2^-(\alpha)\big];\ \alpha \in [0,1]\Big\}$

(i) $A_1^+(\alpha)$ and $A_2^-(\alpha)$ will be continuous, monotonic increasing function of α
(ii) $A_2^+(\alpha)$ and $A_1^-(\alpha)$ will be continuous, monotonic decreasing function of α
(iii) $A_1^+(1) = A_2^+(1)\,;\ A_1^-(0) = A_2^-(0)$.

Definition 2.3. A Triangular Intuitionistic Fuzzy Number (TIFN) $\tilde{A}^i$ is an intuitionistic fuzzy in $\mathbb{R}$ with following membership function $\Big(\mu_{\tilde{A}^i}(x)\Big)$ and non-membership function $\Big(\nu_{\tilde{A}^i}(x)\Big)$,

$$\mu_{\tilde{A}^i}(x) = \begin{cases} \frac{x-a_1}{b_1-a_1}, & a_1 \leq x \leq b_1 \\ \\ \frac{c_1-x}{c_1-b_1}, & b_1 \leq x \leq c_1 \\ \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } \nu_{\tilde{A}^i}(x) = \begin{cases} \frac{b_1-x}{b_1-a_1^{'}}, & a_1^{'} \leq x \leq b_1 \\ \\ \frac{x-b_1}{c_1^{'}-b_1}, & b_1 \leq x \leq c_1^{'} \\ \\ 1, & \text{otherwise.} \end{cases}$$

Where $a_1^{'} < a_1 < b_1 < c_1 < c_1^{'}$ and $\mu_{\tilde{A}^i}(x),\ \nu_{\tilde{A}^i}(x) \leq 0.5$ for $\mu_{\tilde{A}^i}(x) = \nu_{\tilde{A}^i}(x)\ \forall x \in \mathbb{R}$. This TIFN is denoted by $\tilde{A}^i = (a_1^{'}, a_1, b_1, c_1, c_1^{'})$

We will write : (1) $\tilde{A}^i > 0$ if $a_1^{'} > 0$, (2)$\tilde{A}^i \geq 0$ if $a_1^{'} \geq 0$, (3) $\tilde{A}^i < 0$ if $c_1^{'} < 0$ and (4) $\tilde{A}^i \leq 0$ if $c_1^{'} \leq 0$. and

$$A^+[\alpha] = \big[a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)\big] \text{ and } A^-[\alpha] = \big[a_1^{'} + \alpha(a_2 - a_1^{'}), a_3^{'} - \alpha(a_3^{'} - a_2)\big]$$

For $\tilde{A}^i$, $\tilde{B}^i \in \mathbb{IF}$ and $\lambda \in \mathbb{R}$, the addition and scaler-multiplication are defined as follows

$$\Big(\tilde{A}^i + \tilde{B}^i\Big)[\alpha] = \Big(A^+[\alpha] + B^+[\alpha], A^-[\alpha] + B^-[\alpha]\Big)$$

$$\lambda \tilde{A}^i[\alpha] = \begin{cases} \Big(\big[\lambda A_1^+(\alpha), \lambda A_2^+(\alpha)\big], \big[\lambda A_1^-(\alpha), \lambda A_2^-(\alpha)\big]\Big) & \lambda \geq 0 \\ \Big(\big[\lambda A_2^+(\alpha), \lambda A_1^+(\alpha)\big], \big[\lambda A_2^-(\alpha), \lambda A_1^-(\alpha)\big]\Big), & \lambda < 0 \end{cases}$$

Theorem 2.4. *Let $F : I \to \mathbb{IF}$ be differentiable. Denote $F^+[\alpha] = \Big[f_1^+(t,\alpha), f_2^+(t,\alpha)\Big]$, $F^-[\alpha] = \Big[f_1^-(t,\alpha), f_2^-(t,\alpha)\Big]$. Then $f_1^+(t,\alpha), f_2^+(t,\alpha), f_1^-(t,\alpha)$ and $f_2^-(t,\alpha)$ are differentiable and*

$$\begin{aligned} F^{+\prime}[\alpha] &= \Big[f_1^{+\prime}(t,\alpha), f_2^{+\prime}(t,\alpha)\Big] \\ F^{-\prime}[\alpha] &= \Big[f_1^{-\prime}(t,\alpha), f_2^{-\prime}(t,\alpha)\Big] \end{aligned}$$

We adopt the general definition of a intuitionistic fuzzy number given in [2, 3, 4, 10, 9].

3. Intuitionistic fuzzy heat-like equations

We consider the heat-like equations in one and two dimensional cases which can be written in the forms

– One-dimensional [5] :

$$U_t(t,x) + P(x)U_{xx}(t,x) = F(t,x,k) \tag{3.1}$$

– Two-dimensional [5] :

$$U_t(t,x,y) + P(x)U_{xx}(t,x,y) + Q(y)U_{yy}(t,x,y) = F(t,x,y,k) \tag{3.2}$$

$$\text{or } U_t(t,x,y) + Q(y)U_{xx}(t,x,y) + P(x)U_{yy}(t,x,y) = F(t,x,y,k) \tag{3.3}$$

subject to certain initial and boundary conditions.
These initial and boundary conditions, in state two-dimensional, can come in a variety of forms such as $U(0,x,y) = c_1$ or $U(0,x,y) = g_1(x,y,c_2)$ or $U(M_1,x,y) = g_2(x,y,c_3,c_4)$,...
In this paper the method is applied for the heat-like equation (3.2). For (3.1) and (3.3), the same discussion can be made. In following lines, the components of (3.2) are enumerated :

- $I_1 = [0, M_1]$, $I_2 = [M_2, M_3]$ and $I_3 = [M_4, M_5]$ are three intervals, which $M_{n_1}(n_1 = 2,3,4,5)$ is negative or positive and $M_1 > 0$.
- $F(t,x,y,k)$, $U(t,x,y)$, $P(x)$ and $Q(y)$ will be continuous functions for $(t,x,y) \in \prod_{j=1}^3 I_j$.
- $P(x)$ and $Q(y)$ have a finite number of roots for each $(x,y) \in I_2 \times I_3$
- $k = (k_1,...,k_n)$ and $c = (c_1,...,c_m)$ are vectors of constants with $k_j \in J_j$ and $c_r \in L_r$.

Assume that (3.2) has a solution

$$U(t,x,y) = G(t,x,y,k,c) \tag{3.4}$$

for G and

$$((G(t,x,y,k,c))_t + P(x)(G(t,x,y,k,c))_{xx} + Q(y)(G(t,x,y,k,c))_{yy}$$

are continuous with $(t,x,y) \in \prod_{j=1}^3 I_j, k \in J = \prod_{j=1}^n J_j,$ and $c \in L = \prod_{r=1}^m L_r$.
The constants k_j and c_r are not known exactly so there will be uncertainty in their values. We will model this uncertainty using intuitionistic fuzzy numbers. So, we will substitute intuitionistic fuzzy numbers $\tilde{K}_j^i$ for k_j, $\tilde{K}_j^i$ in $J_j,\ 1 \le j \le n$, and substitute intuitionistic fuzzy numbers $\tilde{C}_r^i$ for c_r, $\tilde{C}_r^i$ in $L_r,\ 1 \le r \le m$.
The intuitionistic fuzzy heat-like equation is

$$\tilde{U}_t^i(t,x,y) + P(x)\tilde{U}_{xx}^i(t,x,y) + Q(y)\tilde{U}_{yy}^i(t,x,y) = \tilde{F}^i(t,x,y,\tilde{K}^i) \tag{3.5}$$

where $\tilde{K}^i = \tilde{K}_1^i, \cdots \tilde{K}_n^i$ for $\tilde{K}_j^i$ an intuitionistic fuzzy number in $J_j,\ 1 \le j \le n$. The function U where $\tilde{U}^i$ maps $\prod_{j=1}^3 I_j$ into intuitionistic fuzzy numbers.
That is, $\tilde{U}^i(t,x,y) = \tilde{Z}^i$ where $\tilde{Z}^i$ is an intuitionistic fuzzy number.
subject to certain initial and boundary conditions. The initial and boundary conditions can be of the form

$$\tilde{U}^i(0,x,y) = \tilde{C}_1^i \textit{ or } \tilde{U}^i(0,x,y) = \tilde{g}_1^i(x,y,\tilde{C}_2^i) \textit{ or } \tilde{U}^i(M_1,x,y) = \tilde{g}_2^i(x,y,\tilde{C}_3^i,\tilde{C}_4^i)$$

The $\tilde{g}_j^i$ is the model intuitionistic fuzzy function of g_j. Let $\tilde{C}^i = \tilde{C}_1^i, \cdots \tilde{C}_n^i$ with $\tilde{C}^i$ triangular intuitionistic fuzzy number in $L_r,\ 1 \le r \le m$. We wish to solve the problem given in (3.5). Let $\tilde{Z}^i(t,x,y) = \tilde{G}^i\big(t,x,y,\tilde{K}^i,\tilde{C}^i\big)$ where $\tilde{Z}^i$ is a intuitionistic fuzzy solution. Let $\tilde{K}^i[\alpha] = \prod_{j=1}^n \tilde{K}_j^i[\alpha]$ and $\tilde{C}^i[\alpha] = \prod_{r=1}^m \tilde{C}_r^i[\alpha]$.

Of cours, we mean that the G in Eq. (3.2) is not defined in terms of a series. That is, there are no Fourier series used to define G. Since in this paper we will interested for G with intuitionistic fuzzy parameters we do not wish to consider Fourier series in intuitionistic fuzzy sets concept. We need the solution G to be fairly simple. So, we also assume that Bessel functions and Legendre functions are not used in G.

4. The Variational iteration method

To illustrate the basic idea of the VIM we consider the following PDE model

$$L_tU + L_xU + L_yU + NU = F(t, x, y, k) \tag{4.1}$$

where L_t, L_x and L_y are linear operators of t, x and y, respectively, and N is a nonlinear operator, also $F(t, x, y, k)$ is the source non-homogeneous term. According to the VIM [11], we can express the following correction function (4.1) in t, x and y directions can be written as

$$U_{n+1}(t, x, y) = U_n(t, x, y) + \int_0^t \lambda\{L_sU_n + (L_x + L_y + N)\widetilde{U}_n - F(s, x, y, k)\}ds \tag{4.2}$$

where λ is general Lagrange multiplier [8], which can be identified optimally via the variational theory [11], and $\widetilde{U}_n$ is a restricted variation which means $\delta\widetilde{U}_n = 0$. It is required first to determine the Lagrange multipliers λ that will be identified optimally via integration by parts. The approximations $U_{n+1}, n \geq 0$, of the solution $U(t, x, y)$ will immediately follow upon using any selective function U_0.
The initial values $U(0, x, y)$ is usually used for the selected zeroth approximations U_0. With the Lagrange multiplies λ determined, then several approximation $u_i(t, x, y)$, $i \geq 0$ can be determined. Consequently, the solution is given as

$$U(t, x, y) = \lim_{n\to\infty} U_n(t, x, y) \tag{4.3}$$

According to the VIM, we construct a correction functional for (3.2) in t-direction as follows

$$U_{n+1}(t, x, y) = U_n(t, x, y) + \int_0^t \lambda(s)\{(U_n)_s + P(x)(\widetilde{U}_n)_{xx} + Q(y)(\widetilde{U}_n)_{yy} - F(s, x, y, k)\}ds \tag{4.4}$$

where $n \geq 0$ and λ is a Lagrange multiplier. We now determine the Lagrange multiplier

$$\delta U_{n+1}(t, x, y) = \delta U_n(t, x, y) + \delta \int_0^t \lambda(s)\Big\{(U_n)_s + P(x)(\widetilde{U}_n)_{xx} + Q(y)(\widetilde{U}_n)_{yy} - F(s, x, y, k)\Big\}ds$$

Therefore, the stationary conditions are : $\lambda^{'}(s) = 0$ and $1 + \lambda(s)|_{s=t} = 0$.
So, the Lagrange multiplier is $\lambda = -1$. Submitting the results into (4.4) leads to the following iteration formula

$$U_{n+1}(t, x, y) = U_n(t, x, y) - \int_0^t \{(U_n)_s + P(x)(\widetilde{U}_n)_{xx} + Q(y)(\widetilde{U}_n)_{yy} - F(s, x, y, k)\}ds \tag{4.5}$$

Iteration formula start with initial approximation, for example $U_0(t, x, y) = U(0, x, y)$. Also the VIM used for system of linear and nonlinear partial differential equation [8] which handled in obtain the solution.

5. Solution concept

The united extension of f for $\alpha \in [0, 1]$ is $\Omega[\alpha] = \big(\Omega^+[\alpha], \Omega^-[\alpha]\big)$

$$\Omega^+[\alpha] = \{y|y = f(x), x \in A^+[\alpha]\} \text{ and } \Omega^-[\alpha] = \{y|y = f(x), x \in A^-[\alpha]\}$$

Define $\tilde{C}^i$, a intuitionistic fuzzy subset of the real numbers, by its membership function

$$\mu_{\tilde{C}^i}(y) = \sup\{\alpha|y \in \Omega^+[\alpha]\}, \quad \nu_{\tilde{C}^i}(y) = 1 - \sup\{\alpha|y \in \Omega^-[\alpha]\}$$

Theorem 5.1. *If f is continuous, then $\tilde{C}^i[\alpha] = \Omega[\alpha]$ where $\alpha \in]0, 1]$*

Proof. All we need to show is that $\tilde{C}^i[\alpha] = \Omega[\alpha]$ since,
– Let $y \in \Omega[\alpha]$. Then $\mu_{\tilde{C}^i}(y) \geq \alpha$ and $\nu_{\tilde{C}^i}(y) \leq 1 - \alpha$ so that $y \in \tilde{C}^i[\alpha]$.

– Let $y \in \tilde{C}^i[\alpha]$ so that $\mu_{\tilde{C}^i}(y) = \alpha_0 \geq \alpha$ and $\nu_{\tilde{C}^i}(y) = 1-\alpha_0 \leq 1-\alpha$. Choose $\{\alpha_n\}$ such that $0 \leq \alpha_n \uparrow \alpha_0$ and $\alpha_n < \alpha_0$. We know that $y \in \Omega[\alpha_n]$ from the definition of supremum used to define $\tilde{C}^i$. Choose $z_n \in \tilde{A}^i[\alpha_n]$ so that $f(z_n) = y$. Since $z_n \in \tilde{A}^i[0]$, which is compact, there is a convergent subsequence $z_{n_k} \to z^* \in \tilde{A}^i[\alpha_0]$. Hence $f(z_{n_k}) \to f(z^*) = y$ and $y \in \Omega[\alpha_0]$. Therefore, $y \in \Omega[\alpha]$ because $\alpha \leq \alpha_0$.

□

Using Theorem 5.1 we may now discuss the first solution.

5.1. **The first solution.** They define for all t, x, y and α,

$$\begin{aligned}\tilde{Z}^i(t,x,y)[\alpha] &= \Big\{\big[z_1^+(t,x,y,\alpha), z_2^+(t,x,y,\alpha)\big], \big[z_1^-(t,x,y,\alpha), z_2^-(t,x,y,\alpha)\big]\Big\} \text{ and} \\ \tilde{F}^i(t,x,y,\tilde{K}^i)[\alpha] &= \Big\{\big[F_1^+(t,x,y,\alpha), F_2^+(t,x,y,\alpha)\big], \big[F_1^-(t,x,y,\alpha), F_2^-(t,x,y,\alpha)\big]\Big\}\end{aligned}$$

Let $W = K^+[\alpha] \times C^+[\alpha]$ and $P = K^-[\alpha] \times C^-[\alpha]$. By definition

$$z_1^+(t,x,y,\alpha) = \min\Big\{G(t,x,y,k,c) : (k,c) \in W\Big\} \tag{5.1}$$

$$z_2^+(t,x,y,\alpha) = \max\Big\{G(t,x,k,y,c) : (k,c) \in W\Big\} \tag{5.2}$$

$$z_1^-(t,x,y,\alpha) = \min\Big\{G(t,x,y,k,c) : (k,c) \in P\Big\} \tag{5.3}$$

$$z_2^-(t,x,y,\alpha) = \max\Big\{G(t,x,k,y,c) : (k,c) \in P\Big\} \tag{5.4}$$

$\forall (t,x,y) \in \prod_{j=1}^3 I_j$ and $\alpha \in [0,1]$, and

$$F_1^+(t,x,y,\alpha) = \min\Big\{F(t,x,y,k) : k \in K^+[\alpha]\Big\} \tag{5.5}$$

$$F_2^+(t,x,y,\alpha) = \max\Big\{F(t,x,y,k) : k \in K^+[\alpha]\Big\} \tag{5.6}$$

$$F_1^-(t,x,y,\alpha) = \min\Big\{F(t,x,y,k) : k \in K^-[\alpha]\Big\} \tag{5.7}$$

$$F_2^-(t,x,y,\alpha) = \max\Big\{F(t,x,y,k) : k \in K^-[\alpha]\Big\} \tag{5.8}$$

$\forall (t,x,y) \in \prod_{j=1}^3 I_j$ and $\alpha \in [0,1]$
Assume that $P(x) > 0$, $Q(y) > 0$ and the $z_i^{\triangle}(t,x,y,\alpha)$ $i = 1,2$ and $\triangle \in \{+,-\}$, has continuous partial derivatives so $(z_i^+)_t + P(x)(z_i^+)_{xx} + Q(y)(z_i^+)_{yy}$ and $(z_i^-)_t + P(x)(z_i^-)_{xx} + Q(y)(z_i^-)_{yy}$ are continuous for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all $\alpha \in [0,1]$. Define

$$\begin{aligned}\Gamma(t,x,y,\alpha) = \bigg\{ &\Big[(z_1^+)_t + P(x)(z_1^+)_{xx} + Q(y)(z_1^+)_{yy}, (z_2^+)_t + P(x)(z_2^+)_{xx} + Q(y)(z_2^+)_{yy}\Big], \\ &\Big[(z_1^-)_t + P(x)(z_1^-)_{xx} + Q(y)(z_1^-)_{yy}, (z_2^-)_t + P(x)(z_2^-)_{xx} + Q(y)(z_2^-)_{yy}\Big]\bigg\}\end{aligned} \tag{5.9}$$

for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all α.
If, for each fixed $t,x,y \in \prod_{j=1}^3 I_j$, $\Gamma(t,x,y,\alpha)$ defines the α-cut of a intuitionistic fuzzy number, then will be said that $\tilde{Z}^i(t,x,y)$ is differentiable and is written

$$\tilde{Z}_t^i[\alpha] + P(x)\tilde{Z}_{xx}^i[\alpha] + Q(y)\tilde{Z}_{yy}^i[\alpha] = \Gamma(t,x,y,\alpha) \tag{5.10}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all α
Sufficient conditions for $\Gamma(t,x,y,\alpha)$ to define α-cut of a intuitionistic fuzzy number are [10] :

(i) $(z_1^+)_t(t,x,y,\alpha) + P(x)(z_1^+)_{xx}(t,x,y,\alpha) + Q(y)(z_1^+)_{yy}(t,x,y,\alpha)$ and

$$(z_2^-)_t(t,x,y,\alpha) + P(x)(z_2^-)_{xx}(t,x,y,\alpha) + Q(y)(z_2^-)_{yy}(t,x,y,\alpha)$$

are an increasing function of α for each $(t,x,y) \in \prod_{j=1}^{3} I_j$

(ii) $(z_2^+)_t(t,x,y,\alpha) + P(x)(z_2^+)_{xx}(t,x,y,\alpha) + Q(y)(z_2^+)_{yy}(t,x,y,\alpha)$ and

$$(z_1^-)_t(t,x,y,\alpha) + P(x)(z_1^-)_{xx}(t,x,y,\alpha) + Q(y)(z_1^-)_{yy}(t,x,y,\alpha)$$

are an decreasing function of α for each $(t,x,y) \in \prod_{j=1}^{3} I_j$ and

(iii) for $(t,x,y) \in \prod_{j=1}^{3} I_j$
$(z_1^+)_t(t,x,y,1) + P(x)(z_1^+)_{xx}(t,x,y,1) + Q(y)(z_1^+)_{yy}(t,x,y,1)$

$$\leq (z_2^+)_t(t,x,y,1) + P(x)(z_2^+)_{xx}(t,x,y,1) + Q(y)(z_2^+)_{yy}(t,x,y,1)$$

(iv) for $(t,x,y) \in \prod_{j=1}^{3} I_j$
$(z_1^-)_t(t,x,y,0) + P(x)(z_1^-)_{xx}(t,x,y,0) + Q(y)(z_1^-)_{yy}(t,x,y,0)$

$$\leq (z_2^-)_t(t,x,y,0) + P(x)(z_2^-)_{xx}(t,x,y,0) + Q(y)(z_2^-)_{yy}(t,x,y,0)$$

Now we assume that the $z_i^\triangle(t,x,y,\alpha)$ for $\triangle \in \{+,-\}$ has continuous partial derivatives, so $(z_i^+)_t + P(x)(z_i^+)_{xx} + Q(y)(z_i^+)_{yy}$ and $(z_i^-)_t + P(x)(z_i^-)_{xx} + Q(y)(z_i^-)_{yy}$ are continuous on $\prod_{j=1}^{3} I_j \times [0,1]$ $i = 1,2$. Hence, if conditions $(i) - (iv)$ above hold, $\tilde{Z}^i(t,x,y)$ is differentiable.
For $\tilde{Z}^i(t,x,y)$ to be a first solution (1-S) of the intuitionistic fuzzy heat-like equation we need

(a) $\tilde{Z}^i(t,x,y)$ differentiable
(b) (3.5) holds for $\tilde{U}^i(t,x,y) = \tilde{Z}^i(t,x,y)$,
(c) $\tilde{Z}^i(t,x,y)$ satisfies the initial and boundary conditions. Since no exist specified any particular initial and boundary conditions, then only is checked if (3.5) hold.

$\tilde{Z}^i(t,x,y)$ is a (1-S) (without the initial and boundary conditions) if $\tilde{Z}^i(t,x,y)$ is differentiable and $(\tilde{Z}^i)_t + P(x)(\tilde{Z}^i)_{xx} + Q(y)(\tilde{Z}^i)_{yy} = \tilde{F}^i(t,x,y,\tilde{K}^i)$ or the following equations must hold

$$(z_1^+)_t + P(x)(z_1^+)_{xx} + Q(y)(z_1^+)_{yy} = F_1^+(t,x,y,\alpha) \tag{5.11}$$

$$(z_2^+)_t + P(x)(z_2^+)_{xx} + Q(y)(z_2^+)_{yy} = F_2^+(t,x,y,\alpha) \tag{5.12}$$

$$(z_1^-)_t + P(x)(z_1^-)_{xx} + Q(y)(z_1^-)_{yy} = F_1^-(t,x,y,\alpha) \tag{5.13}$$

$$(z_2^-)_t + P(x)(z_2^-)_{xx} + Q(y)(z_2^-)_{yy} = F_2^-(t,x,y,\alpha) \tag{5.14}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$.
Now we will present a sufficient condition for the (1-S) to exist. Since there are such a variety of possible initial and boundary conditions, so we will omit them from the following theorem. One must separately check out the initial and boundary conditions. So, we will omit the constants c_i, $1 \leq i \leq m$, from the problem. Therefore, (3.4) becomes $U(t,x,y) = G(t,x,y,k,)$, so $\tilde{Z}^i(t,x,y) = \tilde{G}^i(t,x,y,\tilde{K}^i)$.

Theorem 5.2. *Assume $\tilde{Z}^i(t,x,y)$ is differentiable.*

(a) If

$$P(x) > 0 \text{ and } Q(y) > 0 \qquad (x,y) \in I_2 \times I_3 \tag{5.15}$$

and if

$$\frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0 \ for\ j = 1, 2, ..., n \tag{5.16}$$

Then $\tilde{Z}^i(t,x,y)$ *is* (1-S)

(b) If relations (5.15) does not hold or relation (5.16) does not hold for some j, then $\tilde{Z}^i(t,x,y)$ *is not a* (1-S).

Proof. (a) For simplicity assume $k_j = k$ and $\frac{\partial G}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$. The proof for $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, is similar and omitted.
Since $\frac{\partial G}{\partial k} < 0$ then from (5.1) and (5.2) we have

$$\begin{aligned} z_1^+(t,x,y,\alpha) &= G\Big(t,x,y,k_2^+(\alpha)\Big), \quad z_2^+(t,x,y,\alpha) = G\Big(t,x,y,k_1^+(\alpha)\Big) \\ z_1^-(t,x,y,\alpha) &= G\Big(t,x,y,k_2^-(\alpha)\Big), \quad z_2^-(t,x,y,\alpha) = G\Big(t,x,y,k_1^-(\alpha)\Big) \end{aligned}$$

from (5.5), (5.6) and $\frac{\partial F}{\partial k} < 0$ we have

$$\begin{aligned} F_1^+(t,x,y,\alpha) &= F\Big(t,x,y,k_2^+(\alpha)\Big), \quad F_2^+(t,x,y,\alpha) = F\Big(t,x,y,k_1^+(\alpha)\Big) \\ F_1^-(t,x,y,\alpha) &= F\Big(t,x,y,k_2^-(\alpha)\Big), \quad F_2^-(t,x,y,\alpha) = F\Big(t,x,y,k_1^-(\alpha)\Big) \end{aligned}$$

for all $\alpha \in [0,1]$ where $\tilde{K}^i[\alpha] = \Big(\big[k_1^+(\alpha), k_2^+(\alpha)\big], \big[k_1^-(\alpha), k_2^-(\alpha)\big]\Big)$.
Now $G(t,x,y,k)$ solves (3.2), which means

$$G_t + P(x)G_{xx} + Q(y)G_{yy} = F(t,x,y,k) \tag{5.17}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j,\ k \in J$.
Suppose $\overline{Z}(t,x,y)$ is differentiable and $P(x) > 0$ and $Q(y) > 0$ so

$$\begin{aligned} \partial_t z_1^+(t,x,y,\alpha) + P(x)\partial_{xx} z_1^+(t,x,y,\alpha) + Q(y)\partial_{yy} z_1^+(t,x,y,\alpha) &= F_1^+(t,x,y,\alpha) \\ \partial_t z_2^+(t,x,y,\alpha) + P(x)\partial_{xx} z_2^+(t,x,y,\alpha) + Q(y)\partial_{yy} z_2^+(t,x,y,\alpha) &= F_2^+(t,x,y,\alpha) \\ \partial_t z_1^-(t,x,y,\alpha) + P(x)\partial_{xx} z_1^-(t,x,y,\alpha) + Q(y)\partial_{yy} z_1^-(t,x,y,\alpha) &= F_1^-(t,x,y,\alpha) \\ \partial_t z_2^-(t,x,y,\alpha) + P(x)\partial_{xx} z_2^-(t,x,y,\alpha) + Q(y)\partial_{yy} z_2^-(t,x,y,\alpha) &= F_2^-(t,x,y,\alpha) \end{aligned}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$
Hence, (5.11), (5.12), (5.13), (5.14) hold and $\tilde{Z}^i(t,x,y)$ is a (1-S).
(b) Now consider the situation where (5.15) or (5.16) does not hold.

Let us only look at one case where $\frac{\partial F}{\partial k} < 0$ $\Big($ assume $\frac{\partial G}{\partial k} > 0$, $P(x) > 0$ and $Q(y) > 0\Big)$. Then we have

$$\begin{aligned} z_1^+(t,x,y,\alpha) &= G\Big(t,x,y,k_1^+(\alpha)\Big) \quad z_2^+(t,x,y,\alpha) = G\Big(t,x,y,k_2^+(\alpha)\Big) \\ z_1^-(t,x,y,\alpha) &= G\Big(t,x,y,k_1^-(\alpha)\Big) \quad z_2^-(t,x,y,\alpha) = G\Big(t,x,y,k_2^-(\alpha)\Big) \\ F_1^+(t,x,y,\alpha) &= F\Big(t,x,y,k_2^+(\alpha)\Big) \quad F_2^+(t,x,y,\alpha) = F\Big(t,x,y,k_1^+(\alpha)\Big) \\ F_1^-(t,x,y,\alpha) &= F\Big(t,x,y,k_2^-(\alpha)\Big) \quad F_2^-(t,x,y,\alpha) = F\Big(t,x,y,k_1^-(\alpha)\Big) \end{aligned}$$

then we have

$$\begin{aligned}
\partial_t z_1^+(t,x,y,\alpha)+P(x)\partial_{xx}z_1^+(t,x,y,\alpha)+Q(y)\partial_{yy}z_1^+(t,x,y,\alpha) &= F_1^+(t,x,y,\alpha)\\
\partial_t z_2^+(t,x,y,\alpha)+P(x)\partial_{xx}z_2^+(t,x,\alpha)+Q(y)\partial_{yy}z_2^+(t,x,y,\alpha) &= F_2^+(t,x,y,\alpha)\\
\partial_t z_1^-(t,x,y,\alpha)+P(x)\partial_{xx}z_1^-(t,x,y,\alpha)+Q(y)\partial_{yy}z_1^-(t,x,y,\alpha) &= F_1^-(t,x,y,\alpha)\\
\partial_t z_2^-(t,x,y,\alpha)+P(x)\partial_{xx}z_2^-(t,x,\alpha)+Q(y)\partial_{yy}z_2^-(t,x,y,\alpha) &= F_2^-(t,x,y,\alpha)
\end{aligned}$$

which are not true. because

$$\begin{aligned}
G_t\Big(t,x,y,k_1^+(\alpha)\Big)+P(x)G_{xx}\Big(t,x,y,k_1^+(\alpha)\Big)+Q(y)G_{yy}\Big(t,x,y,k_1^+(\alpha)\Big) &= F\big(t,x,y,k_2^+(\alpha)\big)\\
G_t\Big(t,x,y,k_2^+(\alpha)\Big)+P(x)G_{xx}\Big(t,x,y,k_2^+(\alpha)\Big)+Q(y)G_{yy}\Big(t,x,k_2^+(\alpha)\Big) &= F\big(t,x,y,k_1^+(\alpha)\big)\\
G_t\Big(t,x,y,k_1^-(\alpha)\Big)+P(x)G_{xx}\Big(t,x,y,k_1^-(\alpha)\Big)+Q(y)G_{yy}\Big(t,x,y,k_1^-(\alpha)\Big) &= F\big(t,x,y,k_2^-(\alpha)\big)\\
G_t\Big(t,x,y,k_2^-(\alpha)\Big)+P(x)G_{xx}\Big(t,x,y,k_2^-(\alpha)\Big)+Q(y)G_{yy}\Big(t,x,k_2^-(\alpha)\Big) &= F\big(t,x,y,k_1^-(\alpha)\big)
\end{aligned}$$

□

Therefore, if $\tilde{Z}^i(t,x,y)$ is a (1-S) and it satisfies the initial and boundary conditions we will say that $\tilde{Z}^i(t,x,y)$ is a (1-S) satisfying the initial and boundary conditions. If $\tilde{Z}^i(t,x,y)$ is not a (1-S), then we will consider the second solution (2-S).

5.2. **The second solution.** Now let us define the second solution (2-S). Let

$$\tilde{U}^i(t,x,y)[\alpha]=\Big\{\big[u_1^+(t,x,y,\alpha),u_2^+(t,x,y,\alpha)\big],\big[u_1^-(t,x,y,\alpha),u_2^-(t,x,y,\alpha)\big]\Big\}$$

For example suppose $P(x)<0$ and $Q(y)>0$, so consider the system of heat-like equations

$$(u_1^+)_t+P(x)(u_2^+)_{xx}+Q(y)(u_1^+)_{yy}=F_1^+(t,x,y,\alpha) \tag{5.18}$$

$$(u_2^+)_t+P(x)(u_1^+)_{xx}+Q(y)(u_2^+)_{yy}=F_2^+(t,x,y,\alpha) \tag{5.19}$$

$$(u_1^-)_t+P(x)(u_2^-)_{xx}+Q(y)(u_1^-)_{yy}=F_1^-(t,x,y,\alpha) \tag{5.20}$$

$$(u_2^-)_t+P(x)(u_1^-)_{xx}+Q(y)(u_2^-)_{yy}=F_2^-(t,x,y,\alpha) \tag{5.21}$$

or if $P(x)>0$, $Q(y)>0$, $\frac{\partial F}{\partial k}>0$, $\frac{\partial G}{\partial k}<0$,

$$(u_1^+)_t+P(x)(u_1^+)_{xx}+Q(y)(u_1^+)_{yy}=F_1^+(t,x,y,\alpha) \tag{5.22}$$

$$(u_2^+)_t+P(x)(u_2^+)_{xx}+Q(y)(u_2^+)_{yy}=F_2^+(t,x,y,\alpha)) \tag{5.23}$$

$$(u_1^-)_t+P(x)(u_1^-)_{xx}+Q(y)(u_1^-)_{yy}=F_1^-(t,x,y,\alpha) \tag{5.24}$$

$$(u_2^-)_t+P(x)(u_2^-)_{xx}+Q(y)(u_2^-)_{yy}=F_2^-(t,x,y,\alpha)) \tag{5.25}$$

for all $(t,x,y)\in\prod_{j=1}^{3}I_j$ and $\alpha\in[0,1]$. We append to (5.20) and (5.21) any initial and boundary conditions. For example, if it was $\tilde{U}^i(0,x,y)=\tilde{C}^i$ then we add

$$u_1^+(0,x,y,\alpha)=c_1^+(\alpha)\quad u_1^-(0,x,y,\alpha)=c_1^-(\alpha) \tag{5.26}$$

$$u_2^+(0,x,y,\alpha)=c_2^+(\alpha)\quad u_2^-(0,x,y,\alpha)=c_2^-(\alpha) \tag{5.27}$$

where $\tilde{C}^i[\alpha] = \Big\{ \big[c_1^+(\alpha), c_2^+(\alpha)\big], \big[c_1^-(\alpha), c_2^-(\alpha)\big] \Big\}$.
Let $u_i^{\triangle}(t,x,y,\alpha)$ for $i = 1,2$ and $\triangle \in \{+,-\}$ solve (5.20) and (5.21) plus initial and boundary conditions.
If

$$\Big[u_1^+(t,x,y,\alpha), u_2^+(t,x,y,\alpha)\Big] \quad \text{and} \quad \Big[u_1^-(t,x,y,\alpha), u_2^-(t,x,y,\alpha)\Big] \tag{5.28}$$

defines the α-cut of a intuitionistic fuzzy number, for all $(t,x,y) \in \prod_{j=1}^3 I_j$, then $\tilde{U}^i(t,x,y)$ is the (2-S).
We will say that derivative condition holds for intuitionistic fuzzy heat-like equation when Eqs.(5.15) and (5.16) are true.

Theorem 5.3. *(1) If* (1-S)=$\tilde{Z}^i(t,x,y)$*, then* (2-S)=$\tilde{Z}^i(t,x,y)$

(2) If (2-S)=$\tilde{Z}^i(t,x,y)$ *and the derivative condition holds, then* (1-S)=$\tilde{U}^i(t,x,y)$

Proof. (1) Follows from the definition of (1-S) and (2-S).

(2) If (2-S)=$\tilde{U}^i(t,x,y)$ then the derivative exists and since the derivative condition holds, therefore, Eqs. following holds

$$\begin{aligned}(u_1^+)_t + P(x)(u_1^+)_{xx} + Q(y)(u_1^+)_{yy} &= F_1^+(t,x,y,\alpha)\\ (u_2^+)_t + P(x)(u_2^+)_{xx} + Q(y)(u_2^+)_{yy} &= F_2^+(t,x,y,\alpha)\end{aligned} \tag{5.29}$$

$$\begin{aligned}(u_1^-)_t + P(x)(u_1^-)_{xx} + Q(y)(u_1^-)_{yy} &= F_1^-(t,x,y,\alpha)\\ (u_2^-)_t + P(x)(u_2^-)_{xx} + Q(y)(u_2^-)_{yy} &= F_2^-(t,x,y,\alpha)\end{aligned} \tag{5.30}$$

Also suppose one $k_j = k$, $\frac{\partial G}{\partial k} < 0$ and $\frac{\partial F}{\partial k} < 0$ (the other cases are similar and are omitted). We see

$$\begin{aligned}
z_1^+(t,x,y,\alpha) &= G\Big(t,x,y,k_2^+(\alpha)\Big) \quad z_2^+(t,x,y,\alpha) = G\Big(t,x,y,k_1^+(\alpha)\Big)\\
z_1^-(t,x,y,\alpha) &= G\Big(t,x,y,k_2^-(\alpha)\Big) \quad z_2^-(t,x,y,\alpha) = G\Big(t,x,y,k_1^-(\alpha)\Big)\\
F_1^+(t,x,y,\alpha) &= F\Big(t,x,y,k_2^+(\alpha)\Big), \quad F_2^+(t,x,y,\alpha) = F\Big(t,x,y,k_1^+(\alpha)\Big)\\
F_1^-(t,x,y,\alpha) &= F\Big(t,x,y,k_2^-(\alpha)\Big), \quad F_2^-(t,x,y,\alpha) = F\Big(t,x,y,k_1^-(\alpha)\Big)
\end{aligned}$$

Now look at Eqs. (5.11), (5.12), (5.13), (5.14) also Eqs. (5.1), (5.2), implies that

$$\begin{aligned}
u_1^+(t,x,y,\alpha) &= G\Big((t,x,y,k_2^+(\alpha)\Big) = z_1^+(t,x,y,\alpha)\\
u_2^+(t,x,y,\alpha) &= G\Big(t,x,y,k_1^+(\alpha)\Big) = z_2^+(t,x,y,\alpha)\\
u_1^-(t,x,y,\alpha) &= G\Big((t,x,y,k_2^-(\alpha)\Big) = z_1^-(t,x,y,\alpha)\\
u_2^-(t,x,y,\alpha) &= G\Big(t,x,y,k_1^-(\alpha)\Big) = z_2^-(t,x,y,\alpha)
\end{aligned}$$

Therefore (1-S)=$\tilde{U}^i(t,x,y)$

□

Lemma 5.4. *Consider (3.1) suppose $\tilde{Z}^i(t,x)$ is differentiable.*

(a) If

$$P(x) > 0 \quad x \in I_2 \tag{5.31}$$

and if

$$\frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0 \ for \ j = 1,2,...,n \tag{5.32}$$

Then (1-S)=$\tilde{Z}^i(t,x)$

(b) If relations (5.31) does not hold or relation (5.32) does not hold for some j, then $\tilde{Z}^i(t,x)$ *is not a* (1-S).

Proof. It is similar to theorem 5.2 □

6. Examples

We consider the following examples ([5]) and we added intuitionistic fuzzy parameters to these reference.

Example 6.1. We first consider the one-dimensional heat-like equation with variable coefficients as

$$U_t + \frac{1}{2}x^2 U_{xx} = k \tag{6.1}$$

with the initial condition

$$U(0,x) = cx^2$$

where $t \in (0, M_1], x \in (0, M_2]$, $k \in [0, J]$ is constant.
According to the VIM, a correct functional for (6.1) from (4.5) can be constructed as follows

$$U_{n+1}(t,x) = U_n(t,x) - \int_0^t \{(U_n)_s(s,x) + \frac{1}{2}x^2(\widetilde{U}_n)_{xx}(s,x) - F(s,x,k)\}ds$$

Beginning with an initial approximation $U_0(t,x) = U(0,x) = cx^2$, we can obtain the following successive approximations $U_1(t,x) = cx^2(1-t) + kt$
$U_2(t,x) = cx^2(1 - t + \frac{t^2}{2!}) + kt$
$U_3(t,x) = cx^2(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}) + kt$
and $U_n(t,x) = cx^2(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ... + (-1)^n \frac{t^n}{n!}) + kt$, $n \geq 1$
The VIM admits the use of $U(t,x) = \lim_{n\to\infty} U_n(t,x)$, which gives the exact solution

$$U(t,x) = cx^2 \exp(-t) + kt$$

and extension for intuitionistic fuzzy sets of $F(t,x,k)$ and
$G(t,x,k,c) = cx^2 \exp(-t) + kt$. Clearly $\tilde{F}^i(t,x,\tilde{K}^i) = \tilde{K}^i$ so that

$$\begin{aligned} F_1^+(t,x,\alpha) &= k_1^+(\alpha), \quad F_2^+(t,x,\alpha) = k_2^+(\alpha) \\ F_1^-(t,x,\alpha) &= k_1^-(\alpha), \quad F_2^-(t,x,\alpha) = k_2^-(\alpha) \end{aligned}$$

Also $\tilde{G}^i(t,x,\tilde{K}^i,\tilde{C}^i) = \tilde{C}^i x^2 \exp(-t) + \tilde{K}^i t$, therefore for $i = 1,2$

$$\begin{aligned} z_i^+(t,x,\alpha) &= c_i^+(\alpha)x^2 \exp(-t) + k_i^+(\alpha)t \\ z_i^-(t,x,\alpha) &= c_i^-(\alpha)x^2 \exp(-t) + k_i^-(\alpha)t \end{aligned}$$

$$\tilde{K}^i[\alpha] = \Big(\big[k_1^+(\alpha), k_2^+(\alpha)\big], \big[k_1^-(\alpha), k_2^-(\alpha)\big]\Big) \quad \text{and} \quad \tilde{C}^i[\alpha] = \Big(\big[c_1^+(\alpha), c_2^+(\alpha)\big], \big[c_1^+(\alpha), c_2^+(\alpha)\big]\Big).$$

$\tilde{Z}^i(t,x)$ is differentiable because

$$\begin{aligned} (z_i^+(t,x,\alpha))_t + \frac{1}{2}x^2(z_i^+(t,x,\alpha))_{xx} &= k_i^+(\alpha) \\ (z_i^-(t,x,\alpha))_t + \frac{1}{2}x^2(z_i^-(t,x,\alpha))_{xx} &= k_i^-(\alpha) \end{aligned}$$

for i=1,2 are α-cuts of $\tilde{K}^i$ i.e α-cuts of a intuitionistic fuzzy number. Due to

$$P(x) > 0, \quad \frac{\partial G}{\partial k} > 0, \quad \frac{\partial F}{\partial k} > 0$$

That is, $(\tilde{Z}^i)_t + \frac{1}{2}x^2(\tilde{Z}^i)_{xx} = \tilde{K}^i$, a intuitionistic fuzzy number.

So Lemma 5.4 implies the result that $\tilde{Z}^i(t,x)$ is a (1-S). We easily see that

$$z_i^+(0,x,\alpha) = c_i^+(\alpha)x^2 \quad \text{and} \quad z_i^-(0,x,\alpha) = c_i^-(\alpha)x^2$$

for i=1,2 , so $\tilde{Z}^i(t,x)$ also satisfies the initial condition. The (1-S) that satisfies the initial condition may be written as

$$\tilde{Z}^i(t,x) = \tilde{C}^i x^2 \exp(-t) + \tilde{K}^i t \quad \text{for all} \quad (t,x) \in (0,M_1] \times (0,M_2]$$

Example 6.2. We consider the one-dimensional heat-like model

$$\begin{aligned} U_t(t,x) &- U_{xx}(t,x) = -K\cos(x) \\ U(0,x) &= C\sin(x) \end{aligned} \tag{6.2}$$

which $x \in (0, \frac{\pi}{2})$, $t \in [0,M]$ and the value of parameters K and C are in intervals $[0,J]$ and $[0,L]$, respectively. According to the VIM, a correct functional for Eq. (6.2) from Eq. (4.4) which gives the exact solution

$$U(t,x) = G(t,x,K,C) = C\exp(-t)\sin(x) + K\cos(x)\big(\exp(-t)-1\big)$$

which is the exact solution. There is no (1-S) because $p(x) = -1 < 0$ (Lemme5.4).

We proceed to look for a (2-S). We must solve

$$\begin{aligned} (u_1^+)_t - (u_2^+)_{xx} &= -k_2^+(\alpha)\cos(x) \quad (u_2^+)_t - (u_1^+)_{xx} = -k_1^+(\alpha)\cos(x) \\ (u_1^-)_t - (u_2^-)_{xx} &= -k_2^-(\alpha)\cos(x) \quad (u_2^-)_t - (u_1^-)_{xx} = -k_1^-(\alpha)\cos(x) \end{aligned}$$

subject to

$$\begin{aligned} u_1^+(0,x,\alpha) &= c_1^+(\alpha)\sin(x), \quad u_2^+(0,x,\alpha) = c_2^+(\alpha)\sin(x) \\ u_1^-(0,x,\alpha) &= c_1^-(\alpha)\sin(x), \quad u_2^-(0,x,\alpha) = c_2^-(\alpha)\sin(x) \end{aligned}$$

If the intervals $\Big(\big[u_1^+(t,x,\alpha), u_2^+(t,x,\alpha)\big], \big[u_1^-(t,x,\alpha), u_2^-(t,x,\alpha)\big]\Big)$ define α-cuts of a intuitionistic fuzzy number $\tilde{U}^i(t,x)$; then (2-S) $=\tilde{U}^i(t,x)$. By VIM, the solution is

$$u_1^+(t,x,\alpha) = c_1^+(\alpha)\cosh(t)\sin(x) - c_2^+(\alpha)\sinh(t)\sin(x) + k_1^+(\alpha)\cos(x)\big(\cosh(t)-1\big) - k_2^+(\alpha)\cos(x)\sinh(t)$$

$$u_2^+(t,x,\alpha) = c_2^+(\alpha)\cosh(t)\sin(x) - c_1^+(\alpha)\sinh(t)\sin(x) + k_2^+(\alpha)\cos(x)\big(\cosh(t)-1\big) - k_1^+(\alpha)\cos(x)\sinh(t)$$

$$u_1^-(t,x,\alpha) = c_1^-(\alpha)\cosh(t)\sin(x) - c_2^-(\alpha)\sinh(t)\sin(x) + k_1^-(\alpha)\cos(x)\big(\cosh(t)-1\big) - k_2^-(\alpha)\cos(x)\sinh(t)$$

$$u_2^-(t,x,\alpha) = c_2^-(\alpha)\cosh(t)\sin(x) - c_1^-(\alpha)\sinh(t)\sin(x) + k_2^-(\alpha)\cos(x)\big(\cosh(t)-1\big) - k_1^-(\alpha)\cos(x)\sinh(t)$$

Now we show

$$\Big(\big[u_1^+(t,x,\alpha), u_2^+(t,x,\alpha)\big], \big[u_1^-(t,x,\alpha), u_2^-(t,x,\alpha)\big]\Big)$$

defines α-cut of a intuitionistic fuzzy number.
Thus we only need to check if $\frac{\partial u_1^+}{\partial \alpha} > 0$, $\frac{\partial u_2^+}{\partial \alpha} < 0$ and $\frac{\partial u_1^-}{\partial \alpha} < 0$, $\frac{\partial u_2^-}{\partial \alpha} > 0$.

Since $u_i^{\triangle}(t,x,\alpha)$ for $\triangle=\{+,-\}$ are continuous and

$$u_1^+(t,x,1)=u_2^+(t,x,1) \quad \text{and } u_1^-(t,x,0)=u_2^-(t,x,0).$$

There is a region $\mathfrak{R}$ contained in $(0,M]\times(0,\frac{\pi}{2})$ for which the (2-S) exists and $(0,M]\times(0,\frac{\pi}{2})-\mathfrak{R}$ there may be no (2-S).
Since $\tilde{K}^i$ and $\tilde{C}^i$ are triangular intuitionistic fuzzy numbers, hence, we pick simple intuitionistic fuzzy parameter so that $(k_1^+(\alpha))'$, $(c_1^+(\alpha))'$, $(k_2^-(\alpha))'$ and $(c_2^-(\alpha))'$ are all positive numbers while $(k_2^+(\alpha))'$, $(c_2^+(\alpha))'$, $(k_1^-(\alpha))'$ and $(c_1^-(\alpha))'$ are negative numbers. The "prime" denotes differentiation with respect to α. Then there is a $\lambda>0$ so that $(k_1^+(\alpha))'=(c_1^+(\alpha))'=(k_2^-(\alpha))'=(c_2^-(\alpha))'=\lambda$ and $(k_2^+(\alpha))'=(c_2^+(\alpha))'=(k_1^-(\alpha))'=(c_1^-(\alpha))'=-\lambda$. Then, for the (2-$S$) exist we need

$$\begin{aligned}
\frac{\partial u_1^+}{\partial\alpha} &= \frac{\partial u_2^-}{\partial\alpha}=\lambda\Big(\sin(x)\big(\cosh(t)+\sinh(t)\big)+\cos(x)\big(\cosh(t)-1+\sinh(t)\big)\Big)>0\\
\frac{\partial u_2^+}{\partial\alpha} &= \frac{\partial u_1^-}{\partial\alpha}=-\lambda\Big(\sin(x)\big(\cosh(t)+\sinh(t)\big)+\cos(x)\big(\cosh(t)-1+\sinh(t)\big)\Big)<0
\end{aligned} \tag{6.3}$$

Since (6.3) holds for each $t\in[0,M]$ and $x\in(0,\frac{\pi}{2})$ therefore, $\tilde{U}^i(t,x)$ is (2-S) and

$$\tilde{U}^i(t,x)=\tilde{C}^i\cosh(t)\sin(x)-\tilde{C}^i\sinh(t)\sin(x)+\tilde{K}^i\cos(x)(\cosh(t)-1)-\tilde{K}^i\sinh(t)\cos(x)$$

for all $t\in[0,M]$ and $x\in(0,\frac{\pi}{2})$.

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